Topic 10 Part 2 *[474 marks]*

The complete graph H has the following cost adjacency matrix.

	A	B	C	D	E
A	–	19	17	10	15
B	19	–	11	16	13
C	17	11	–	14	13
D	10	16	14	–	18
E	15	13	13	18	–

Consider the travelling salesman problem for H .

1a. By first finding a minimum spanning tree on the subgraph of H formed by deleting vertex A and all edges connected to A, find a lower bound for this problem. *[5 marks]*

1b. Find the total weight of the cycle ADCBEA. *[1 mark]*

1c. What do you conclude from your results? *[1 mark]*

2a. Given that a , *[2 marks]*

$b \in \mathbb{N}$ and

$c \in \mathbb{Z}^+$, show that if

$a \equiv 1(\bmod c)$, then

$ab \equiv b(\bmod c)$.

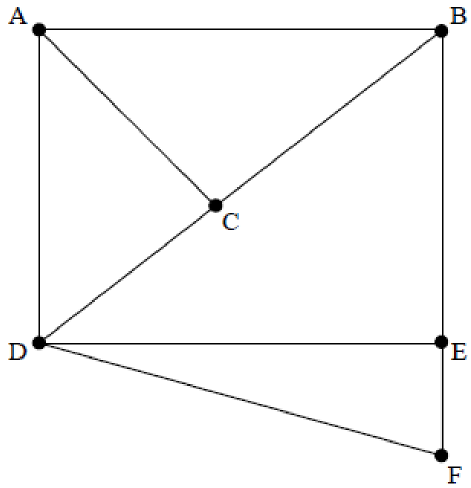
2b. Using mathematical induction, show that *[6 marks]*

$9^n \equiv 1(\bmod 4)$, for

$n \in \mathbb{N}$.

2c. The positive integer M is expressed in base 9. Show that M is divisible by 4 if the sum of its digits is divisible by 4. *[4 marks]*

The diagram below shows the graph G with vertices A, B, C, D, E and F.



3. (i) Determine if any Hamiltonian cycles exist in G . If so, write one down. *[4 marks]*

Otherwise, explain what feature of G makes it impossible for a Hamiltonian cycle to exist.

(ii) Determine if any Eulerian circuits exist in G . If so, write one down.

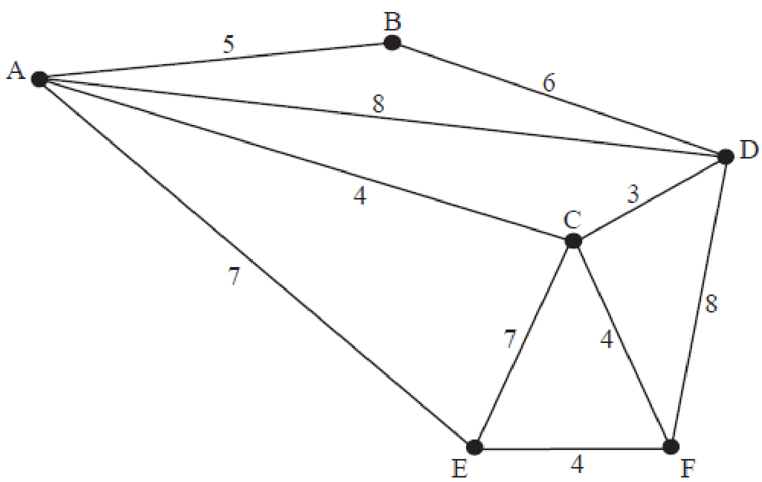
Otherwise, explain what feature of G makes it impossible for an Eulerian circuit to exist.

4a. Explaining your method fully, determine whether or not 1189 is a prime number. *[4 marks]*

4b. (i) State the fundamental theorem of arithmetic. *[6 marks]*

(ii) The positive integers M and N have greatest common divisor G and least common multiple L . Show that $GL = MN$.

5. Sameer is trying to design a road system to connect six towns, A, B, C, D, E and F. The possible roads and the costs of building them are shown in the graph below. Each vertex represents a town, each edge represents a road and the weight of each edge is the cost of building that road. He needs to design the lowest cost road system that will connect the six towns. *[8 marks]*



(a) Name an algorithm which will allow Sameer to find the lowest cost road system.

(b) Find the lowest cost road system and state the cost of building it. Show clearly the steps of the algorithm.

6. (a) Use the Euclidean algorithm to find gcd(*[14 marks]*

$12\,306, 2976)$.

(b) Hence give the general solution to the diophantine equation

$12\,306x + 2976y = 996$.

7. The adjacency table of the graph G , with vertices P, Q, R, S, T is given by: *[11 marks]*

$$\begin{array}{c} \\ \mathrm{P} \\ \mathrm{Q} \\ \mathrm{R} \\ \mathrm{S} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{P} & \mathrm{Q} & \mathrm{R} & \mathrm{S} & \mathrm{T} \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

(a) Draw the graph G .

(b) (i) Define an Eulerian circuit.

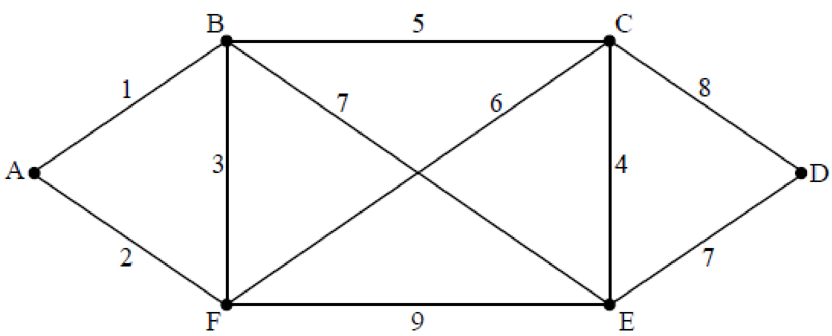
(ii) Write down an Eulerian circuit in G starting at P.

(c) (i) Define a Hamiltonian cycle.

(ii) Explain why it is not possible to have a Hamiltonian cycle in G .

8. An arithmetic sequence has first term 2 and common difference 4. Another arithmetic sequence has first term 7 and common difference 5. Find the set of all numbers which are members of both sequences. *[9 marks]*

The diagram below shows the weighted graph G .



9. (a) (i) What feature of the graph enables you to deduce that G contains an Eulerian circuit? *[9 marks]*

(ii) Find an Eulerian circuit.

(c) Use Kruskal's Algorithm to find the minimum spanning tree for G , showing the order in which the edges are added.

10. The planar graph G and its complement *[5 marks]*

G' are both simple and connected.

Given that G has 6 vertices and 10 edges, show that

G' is a tree.

11a. Show that a positive integer, written in base 10, is divisible by 9 if the sum of its digits is divisible by 9. *[7 marks]*

11b. The representation of the positive integer N in base p is denoted by *[9 marks]*

$(N)_p$.

If

$(5^{(126)_7})_7 = (a_n a_{n-1} \dots a_1 a_0)_7$, find

a_0 .

12. Show that a graph is bipartite if and only if it contains only cycles of even length. *[8 marks]*

13a. (i) One version of Fermat's little theorem states that, under certain conditions, *[8 marks]*

$$a^{p-1} \equiv 1(\bmod p).$$

Show that this result is not valid when $a = 4$, $p = 9$ and state which

condition is not satisfied.

(ii) Given that

$5^{64} \equiv n(\bmod 7)$, where

$0 \leqslant n \leqslant 6$, find the value of n.

13b. Find the general solution to the simultaneous congruences *[6 marks]*

$$x \equiv 3(\bmod 4)$$

$$3x \equiv 2(\bmod 5).$$

14. A graph G with vertices A, B, C, D, E has the following cost adjacency table. *[9 marks]*

	A	B	C	D	E
A	–	12	10	17	19
B	12	–	13	20	11
C	10	13	–	16	14
D	17	20	16	–	15
E	19	11	14	15	–

(a) (i) Use Kruskal's algorithm to find and draw the minimum spanning tree for G.

(ii) The graph H is formed from G by removing the vertex D and all the edges connected to D. Draw the minimum spanning tree for H and use it to find a lower bound for the travelling salesman problem for G.

(b) Show that 80 is an upper bound for this travelling salesman problem.

15. The positive integer N is expressed in base 9 as [12 marks]

$(a_n a_{n-1} \ldots a_0)_9$.

(a) Show that N is divisible by 3 if the least significant digit,

a_0, is divisible by 3.

(b) Show that N is divisible by 2 if the sum of its digits is even.

(c) Without using a conversion to base 10, determine whether or not

$(464860583)_9$ is divisible by 12.

16. (a) Show that, for a connected planar graph, [18 marks]

$$v + f - e = 2.$$

(b) Assuming that

$v \geqslant 3$, explain why, for a simple connected planar graph,

$3f \leqslant 2e$ and hence deduce that

$e \leqslant 3v - 6$.

(c) The graph G and its complement

G' are simple connected graphs, each having 12 vertices. Show that

G and

G' cannot both be planar.

17. Given that [7 marks]

$a,\ b,\ c,\ d \in \mathbb{Z}$, show that

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \equiv 0(\bmod 3).$$

18. (i) A graph is simple, planar and connected. Write down the inequality connecting v and e, and give the condition on v for this inequality to hold. [8 marks]

(ii) Sketch a simple, connected, planar graph with $v = 2$ where the inequality from part (b)(i) is not true.

(iii) Sketch a simple, connected, planar graph with $v = 1$ where the inequality from part (b)(i) is not true.

(iv) Given a connected, planar graph with v vertices,

v^2 edges and 8 faces, find v. Sketch a graph that fulfils all of these conditions.

19. (a) Find the general solution for the following system of congruences. [11 marks]

$N \equiv 3(\bmod 11)$

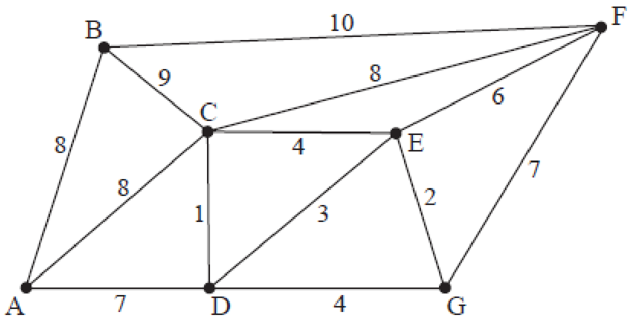
$N \equiv 4(\bmod 9)$

$N \equiv 0(\bmod 7)$

(b) Find all values of N such that

$2000 \leqslant N \leqslant 4000$.

20. Consider the following weighted graph. *[12 marks]*



(a) (i) Use Kruskal's algorithm to find the minimum spanning tree. Indicate the order in which you select the edges and draw the final spanning tree.

(ii) Write down the total weight of this minimum spanning tree.

(b) Sketch a spanning tree of maximum total weight and write down its weight.

21. *[11 marks]*

(a) Write down Fermat's little theorem.

(b) In base 5 the representation of a natural number X is

$(k00013(5-k))_5$.

This means that

$X = k \times 5^6 + 1 \times 5^2 + 3 \times 5 + (5-k)$.

In base 7 the representation of X is

$(a_n a_{n-1} \ldots a_2 a_1 a_0)_7$.

Find

a_0.

(c) Given that $k = 2$, find X in base 7.

22a. A graph has n vertices with degrees 1, 2, 3, …, n. Prove that *[6 marks]*

$n \equiv 0 (\bmod 4)$ or

$n \equiv 3 (\bmod 4)$.

22b. Let G be a simple graph with n vertices, *[8 marks]*

$n \geqslant 2$. Prove, by contradiction, that at least two of the vertices of G must have the same degree.

23. Use the Euclidean Algorithm to find the greatest common divisor of 7854 and 3315. *[7 marks]*

Hence state the number of solutions to the diophantine equation $7854x + 3315y = 41$ and justify your answer.

24a. Define what is meant by the statement [1 mark]

$x \equiv y(\bmod n)$ where $x, y, n \in \mathbb{Z}^+$.

24b. Hence prove that if [4 marks]

$x \equiv y(\bmod n)$ then

$x^2 \equiv y^2(\bmod n)$.

24c. Determine whether or not [4 marks]

$x^2 \equiv y^2(\bmod n)$ implies that

$x \equiv y(\bmod n)$.

The positive integer N is expressed in base p as

$(a_n a_{n-1} \ldots a_1 a_0)_p$.

25a. Show that when $p = 2$, N is even if and only if its least significant digit, a_0 , is 0. [5 marks]

25b. Show that when $p = 3$, N is even if and only if the sum of its digits is even. [6 marks]

The graph G has the following cost adjacency table.

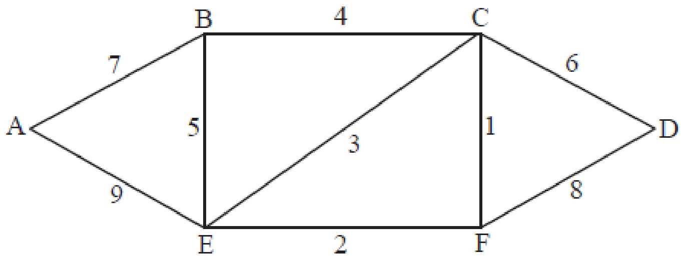
	A	B	C	D	E
A	-	9	-	8	4
B	9	-	7	-	2
C	-	7	-	7	3
D	8	-	7	-	5
E	4	2	3	5	-

26a. Draw G in a planar form. [2 marks]

26b. Giving a reason, determine the maximum number of edges that could be added to G while keeping the graph both simple and planar. [4 marks]

26c. List all the distinct Hamiltonian cycles in G beginning and ending at A, noting that two cycles each of which is the reverse of the other are to be regarded as identical. Hence determine the Hamiltonian cycle of least weight. [10 marks]

27a. The weighted graph H is shown below. *[6 marks]*

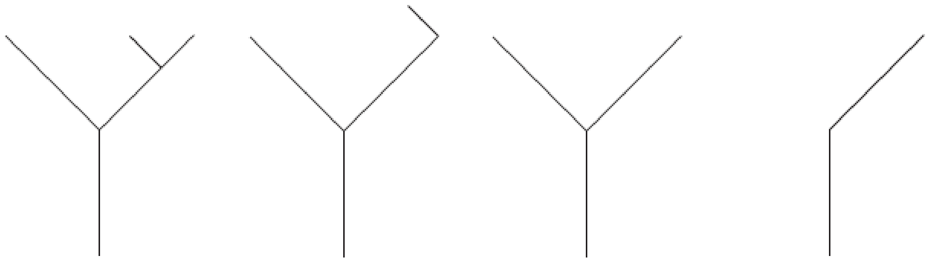


Use Kruskal's Algorithm, indicating the order in which the edges are added, to find and draw the minimum spanning tree for H.

27b. (i) A tree has v vertices. State the number of edges in the tree, justifying your answer. *[5 marks]*

(ii) We will call a graph with v vertices a "forest" if it consists of c components each of which is a tree.

Here is an example of a forest with 4 components.



How many edges will a forest with v vertices and c components have?

28. (a) Use the Euclidean algorithm to find the gcd of 324 and 129. *[11 marks]*

(b) Hence show that

$324x + 129y = 12$ has a solution and find both a particular solution and the general solution.

(c) Show that there are no integers x and y such that

$82x + 140y = 3$.

29a. The table below shows the distances between towns A, B, C, D and E. *[9 marks]*

$$\begin{array}{c c}
 & \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\
\begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} &
\begin{pmatrix} - & 5 & 7 & 10 & 6 \\ 5 & - & 2 & 9 & - \\ 7 & 2 & - & 3 & 8 \\ 10 & 9 & 3 & - & - \\ 6 & - & 8 & - & - \end{pmatrix}
\end{array}$$

(i) Draw the graph, in its planar form, that is represented by the table.

(ii) Write down with reasons whether or not it is possible to find an Eulerian trail in this graph.

(iii) Solve the Chinese postman problem with reference to this graph if A is to be the starting and finishing point. Write down the walk and determine the length of the walk.

29b. Show that a graph cannot have exactly one vertex of odd degree. *[2 marks]*

30a. (i) Given that [11 marks]

$a \equiv d(\bmod n)$ and

$b \equiv c(\bmod n)$ prove that

$$(a+b) \equiv (c+d)(\bmod n)\ .$$

(ii) Hence solve the system

$$\begin{cases} 2x+5y \equiv 1(\bmod 6) \\ x+y \equiv 5(\bmod 6) \end{cases}$$

30b. Show that [3 marks]

$x^{97} - x + 1 \equiv 0(\bmod 97)$ has no solution.

31a. Prove that a graph containing a triangle cannot be bipartite. [3 marks]

31b. Prove that the number of edges in a bipartite graph with n vertices is less than or equal to [5 marks]

$\frac{n^2}{4}$.

Let G be a simple, connected, planar graph.

32. (a) (i) Show that Euler's relation [12 marks]

$f - e + v = 2$ is valid for a spanning tree of G.

(ii) By considering the effect of adding an edge on the values of f, e and v show that Euler's relation remains true.

(b) Show that K_5 is not planar.

33a. Convert the decimal number 51966 to base 16. [4 marks]

33b. (i) Using the Euclidean algorithm, find the greatest common divisor, d , of 901 and 612. [10 marks]

(ii) Find integers p and q such that $901p + 612q = d$.

(iii) Find the least possible positive integers s and t such that $901s - 612t = 85$.

33c. In each of the following cases find the solutions, if any, of the given linear congruence. [5 marks]

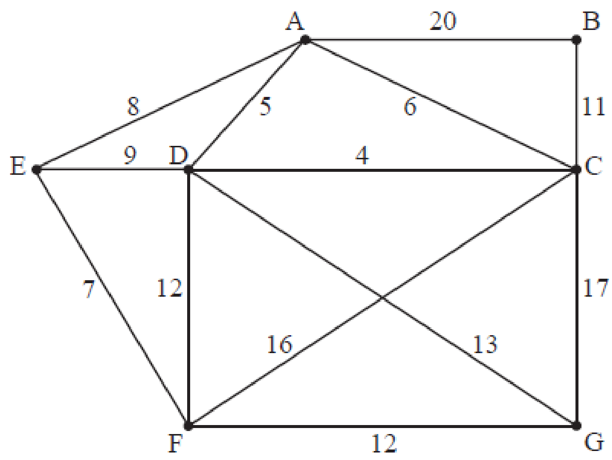
(i)

$9x \equiv 3(\bmod 18)$

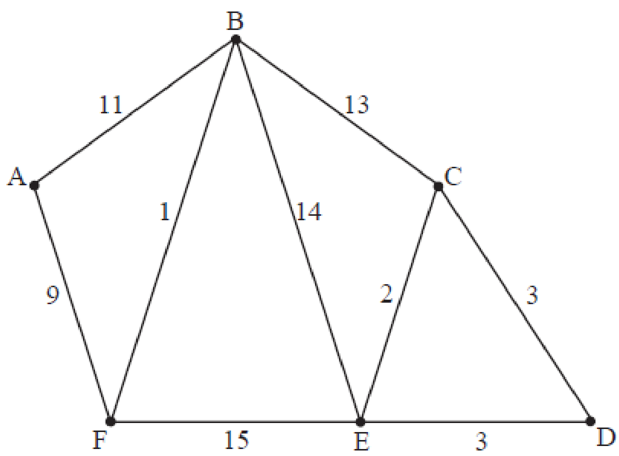
(ii)

$9x \equiv 3(\bmod 15)$

34a. Use Kruskal's algorithm to find the minimum spanning tree for the following weighted graph and state its length. *[5 marks]*



34b. Use Dijkstra's algorithm to find the shortest path from A to D in the following weighted graph and state its length. *[7 marks]*



35a. Write 57128 as a product of primes. *[4 marks]*

35b. Prove that *[4 marks]*

$22 | 5^{11} + 17^{11}$.

36. (a) A connected planar graph G has e edges and v vertices. *[17 marks]*

(i) Prove that

$e \geqslant v - 1$.

(ii) Prove that $e = v - 1$ if and only if G is a tree.

(b) A tree has k vertices of degree 1, two of degree 2, one of degree 3 and one of degree 4. Determine k and hence draw a tree that satisfies these conditions.

(c) The graph H has the adjacency table given below.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(i) Explain why H cannot be a tree.

(ii) Draw the graph of H .

(d) Prove that a tree is a bipartite graph.

37. Two mathematicians are planning their wedding celebration and are trying to arrange the seating plan for the guests. The only restriction is that all tables must seat the same number of guests and each table must have more than one guest. There are fewer than 350 guests, but they have forgotten the exact number. However they remember that when they try to seat them with two at each table there is one guest left over. The same happens with tables of 3, 4, 5 and 6 guests. When there were 7 guests per table there were none left over. Find the number of guests. *[10 marks]*

38. (a) Using Fermat's little theorem, show that, in base 10, the last digit of n is always equal to the last digit of n^5 . *[9 marks]*

(b) Show that this result is also true in base 30.

39a. Use the Euclidean algorithm to find the greatest common divisor of 259 and 581. *[4 marks]*

39b. Hence, or otherwise, find the general solution to the diophantine equation $259x + 581y = 7$. *[5 marks]*

The graph G has vertices P , Q , R , S , T and the following table shows the number of edges joining each pair of vertices.

	P	**Q**	**R**	**S**	**T**
P	0	1	0	1	2
Q	1	0	1	0	0
R	0	1	0	1	1
S	1	0	1	0	0
T	2	0	1	0	0

40a. Draw the graph G as a planar graph. *[2 marks]*

40b. Giving a reason, state whether or not G is [4 marks]

(i) simple;

(ii) connected;

(iii) bipartite.

40c. Explain what feature of G enables you to state that it has an Eulerian trail and write down a trail. [2 marks]

40d. Explain what feature of G enables you to state it does not have an Eulerian circuit. [1 mark]

40e. Find the maximum number of edges that can be added to the graph G (not including any loops or further multiple edges) whilst still keeping it planar. [4 marks]

41a. One version of Fermat's little theorem states that, under certain conditions, [6 marks]

$a^{p-1} \equiv 1(\bmod p)$.

(i) Show that this result is not true when $a = 2$, $p = 9$ and state which of the conditions is not satisfied.

(ii) Find the smallest positive value of k satisfying the congruence

$2^{45} \equiv k(\bmod 9)$.

41b. Find all the integers between 100 and 200 satisfying the simultaneous congruences [6 marks]

$3x \equiv 4(\bmod 5)$ and

$5x \equiv 6(\bmod 7)$.

The weights of the edges of a graph G with vertices A, B, C, D and E are given in the following table.

	A	**B**	**C**	**D**	**E**
A	–	11	18	12	9
B	11	–	17	13	14
C	18	17	–	16	10
D	12	13	16	–	15
E	9	14	10	15	–

42a. Starting at A, use the nearest neighbour algorithm to find an upper bound for the travelling salesman problem for G. [4 marks]

42b. (i) Use Kruskal's algorithm to find and draw a minimum spanning tree for the subgraph obtained by removing the vertex A from G. [8 marks]

(ii) Hence use the deleted vertex algorithm to find a lower bound for the travelling salesman problem for G.

43a. The sequence [6 marks]

$\{u_n\}$,

$n \in \mathbb{Z}^+$, satisfies the recurrence relation

$u_{n+2} = 5u_{n+1} - 6u_n$.

Given that

$u_1 = u_2 = 3$, obtain an expression for

u_n in terms of n .

43b. The sequence [8 marks]

$\{v_n\}$,

$n \in \mathbb{Z}^+$, satisfies the recurrence relation

$v_{n+2} = 4v_{n+1} - 4v_n$.

Given that

$v_1 = 2$ and

$v_2 = 12$, use the principle of strong mathematical induction to show that

$v_n = 2^n(2n - 1)$.