Topic 6 Part 3 *[713 marks]*

1. *[7 marks]*

Markscheme

(a)

$2^{\frac{1}{x}} = 4 - 2^{\frac{1}{x}}$
attempt to solve the equation ***M1***

$x = 1$ ***A1***

so P is (1, 2) , as

A1 N1
$f(1) = 2$

(b)

A1
$f'(x) = -\frac{1}{x^2}2^{\frac{1}{x}}\ln 2$
attempt to substitute x-value found in part (a) into their

M1
$f'(x)$

$f'(1) = -2\ln 2$
M1A1 N0
$y - 2 = -2\ln 2(x - 1) \quad (\text{or equivalent})$
[7 marks]

Examiners report

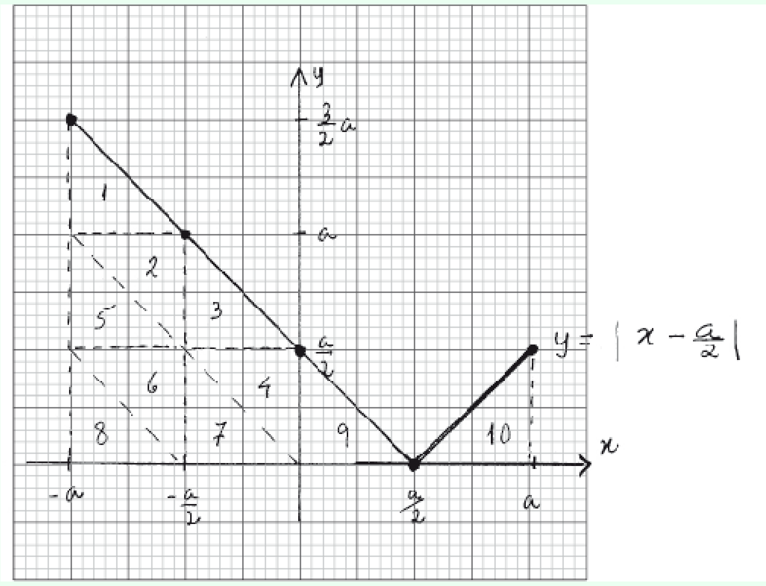
Most candidates answered part (a) correctly although some candidates showed difficulty solving the equation using valid methods.

Part (b) was less successful with many candidates failing to apply chain rule to obtain the derivative of the exponential function.

2. [7 marks]

Markscheme

(a)



A1A1

Note: Award *A1* for the correct *x*-intercept,

A1 for completely correct graph.

(b) **METHOD 1**

the area under the graph of $y = \left|x - \frac{a}{2}\right|$ for $-a \leqslant x \leqslant a$, can be divided into ten congruent triangles; *M1A1*

the area of eight of these triangles is given by $\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x$ and the areas of the other two by $\int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x$ *M1A1*

so, $\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x = 4\int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x \Rightarrow k = 4$ *A1* *N0*

METHOD 2

use area of trapezium to calculate $\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x = a \times \frac{1}{2}\left(\frac{3a}{2} + \frac{a}{2}\right) = a^2$ *M1* *A1*

and area of two triangles to obtain $\int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x = 2 \times \frac{1}{2}\left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$ *M1* *A1*

so, $k = 4$ *A1* *N0*

METHOD 3

use integration to find the area under the curve

$\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x = \int_{-a}^{0} -x + \frac{a}{2} \mathrm{d}x$ *M1*

$= \left[-\frac{x^2}{2} + \frac{a}{2}x\right]_{-a}^{0} = \frac{a^2}{2} + \frac{a^2}{2} = a^2$ *A1*

and

$\int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x = \int_{0}^{\frac{a}{2}} -x + \frac{a}{2} \mathrm{d}x + \int_{\frac{a}{2}}^{a} x - \frac{a}{2} \mathrm{d}x$ *M1*

A1

$= \left[-\frac{\ldots}{2} + \frac{\ldots}{2}x\right]_{0} + \left[\frac{\ldots}{2} - \frac{\ldots}{2}x\right]_{\frac{a}{2}} = \frac{\ldots}{8} + \frac{\ldots}{4} + \frac{\ldots}{2} - \frac{\ldots}{2} - \frac{\ldots}{8} + \frac{\ldots}{4} = \frac{\ldots}{4}$

so, $k = 4$ ***A1*** ***N0***

[7 marks]

Examiners report

Most candidates attempted this question but very often produced sketches lacking labels on axes and intercepts or ignored the domain of the function. For part (b) many candidates attempted to use integration to find the areas but seldom considered the absolute value. A small number of candidates used geometrical methods to determine the areas, showing good understanding of the problem.

3.

[20 marks]

Markscheme

(a)

$f(1) = 1 - \arctan 1 = 1 - \frac{\pi}{4}$ ***A1***

$f(-\sqrt{3}) = -\sqrt{3} - \arctan(-\sqrt{3}) = -\sqrt{3} + \frac{\pi}{3}$ ***A1***

[2 marks]

(b)

$f(-x) = -x - \arctan(-x)$ ***M1***

$= -x + \arctan x$ ***A1***

$= -(x - \arctan x)$

$= -f(x)$ ***AG N0***

[2 marks]

(c) as $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$, for any $x \in \mathbb{R}$

$\Rightarrow -\frac{\pi}{2} < -\arctan x < \frac{\pi}{2}$, for any $x \in \mathbb{R}$ ***A1***

then by adding x (or equivalent) ***R1***

we have

$x - \frac{\pi}{2} < x - \arctan x < x + \frac{\pi}{2}$ ***AG N0***

[2 marks]

(d)

$f'(x) = 1 - \frac{1}{1+x^2}$ or $\frac{x^2}{1+x^2}$ ***A1A1***

$f''(x) = \frac{2x(1+x^2)-2x^3}{(1+x^2)^2}$ or $\frac{2x}{(1+x^2)^2}$ ***M1A1***

$f'(0) = f''(0) = 0$ ***A1A1***

EITHER

as $f'(x) \geqslant 0$ for all values of $x \in \mathbb{R}$

($(0, 0)$ is not an extreme of the graph of f (or equivalent)) ***R1***

OR

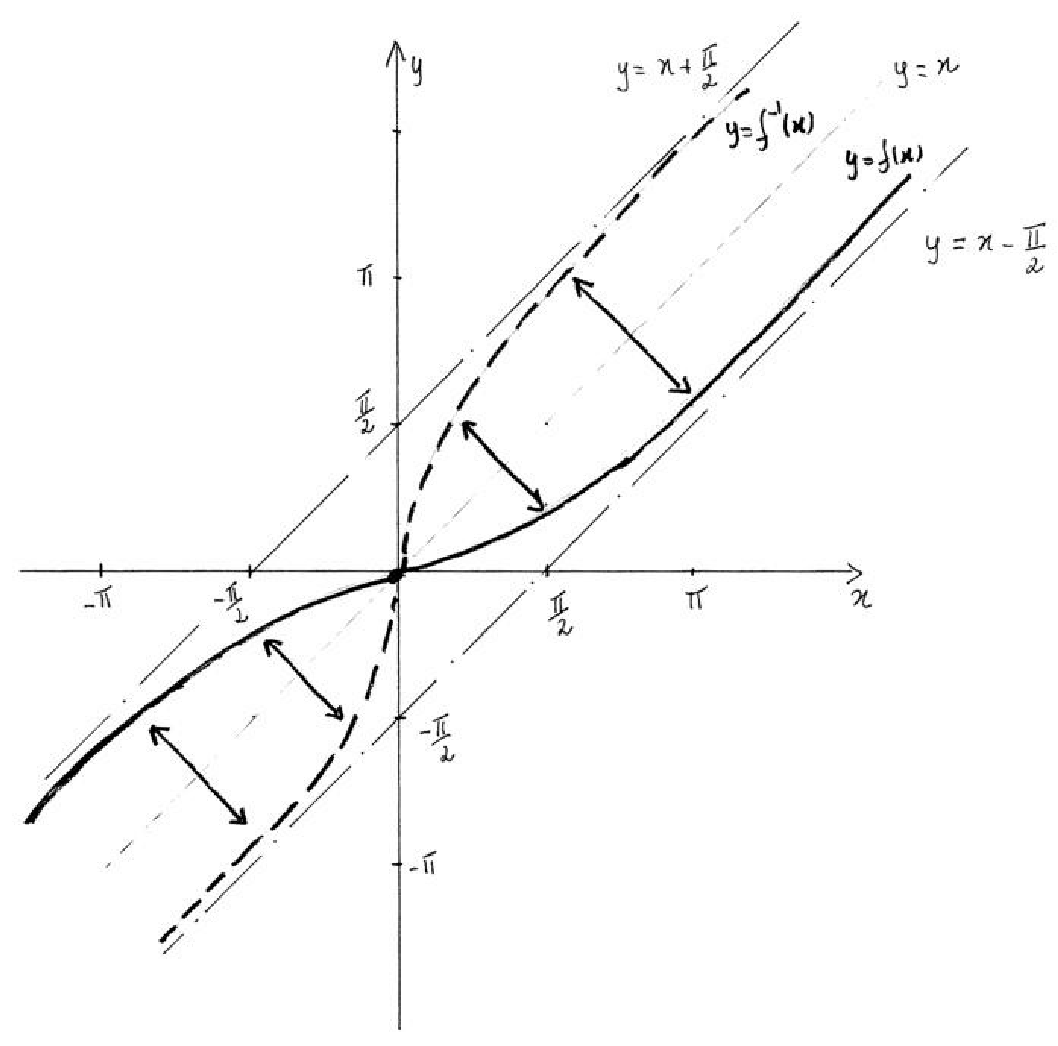
as $f''(x) > 0$ for positive values of x and $f''(x) < 0$ for negative values of x ***R1***

THEN

(0, 0) is a point of inflexion of the graph of f (with zero gradient) ***A1 N2***

[8 marks]

(e)



A1A1A1

Note: Award *A1* for both asymptotes.

A1 for correct shape (concavities)

$x < 0$

A1 for correct shape (concavities)

$x > 0$

[3 marks]

(f) (see sketch above)

as *f* is increasing (and therefore one-to-one) and its range is $\mathbb{R}$, f^{-1} is defined for all $x \in \mathbb{R}$ ***R1***

use the result that the graph of $y = f^{-1}(x)$ is the reflection in the line *y* = *x* of the graph of $y = f(x)$ to draw the graph of f^{-1} ***(M1)A1***

[3 marks]

Total [20 marks]

Examiners report

Parts of this question were answered quite well by many candidates. A few candidates had difficulties with domain of arctan in part (a) and in justifying their reasoning in parts (b) and (c). In part (d) although most candidates were successful in finding the expressions of the derivatives and their values at $x = 0$, many were unable to use the results to find the nature of the curve at the origin. Very few candidates were successful in answering parts (e) and (f).

4.

[6 marks]

Markscheme

(a)

$\frac{3}{x+1} + \frac{2}{x+3} = \frac{3(x+3)+2(x+1)}{(x+1)(x+3)}$ ***M1***

$= \frac{3x+9+2x+2}{x^2+4x+3}$ ***A1***

$= \frac{5x+11}{x^2+4x+3}$ ***AG***

(b)

$\int_0^2 \frac{5x+11}{x^2+4x+3}\mathrm{d}x = \int_0^2 \left(\frac{3}{x+1} + \frac{2}{x+3}\right)\mathrm{d}x$ ***M1***

$= [3\ln(x+1) + 2\ln(x+3)]_0^2$ ***A1***

$= 3\ln 3 + 2\ln 5 - 3\ln 1 - 2\ln 3 \quad (= 3\ln 3 + 2\ln 5 - 2\ln 3)$ ***A1***

$= \ln 3 + 2\ln 5$

$= \ln 75 \quad (k = 75)$ ***A1***

[6 marks]

Examiners report

Many students did not ‘Show’ enough in a) in order to be convincing. The need for the steps of the simplification to be shown was not clear. Too many did not link a) to b) and seemed to not be aware of the Command Term ‘hence’ and its implication for marking (no marks will be awarded to alternative methods). The simplifications of the log expressions were done poorly by many and the fact that

$3^3 = 9$ was noted by too many. There were very few elegant solutions to this question.

5. *[8 marks]*

Markscheme

(a)

$8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1A1***

Note: Award ***M1A0*** for

$8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{y}$ ***A1***

(b) -4 ***A1***

(c)

$V = \int \pi y^2 \mathrm{d}x$ or equivalent ***M1***

$V = \pi \int_0^1 (4 - 4x^2)\mathrm{d}x$ ***A1***

$= \pi\left[4x - \frac{4}{3}x^3\right]_0^1$ ***A1***

$= \frac{8\pi}{3}$ ***A1***

Note: If it is correct except for the omission of π, award 2 marks.

[8 marks]

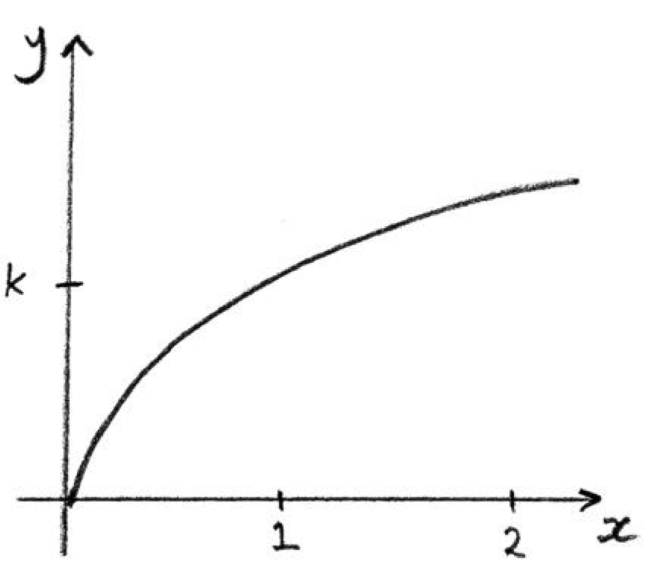
Examiners report

The first part of this question was done well by many, the only concern being the number that did not simplify the result from $-\frac{8x}{2y}$. There were many variations on the formula for the volume in part c), the most common error being a multiple of 2π rather than π. On the whole this question was done well by many.

6. *[16 marks]*

Markscheme

(a)



A1

Note: Award ***A1*** for correct concavity, passing through (0, 0) and increasing.

Scales need not be there.

[1 mark]

(b) a statement involving the application of the Horizontal Line Test or equivalent ***A1***

[1 mark]

(c)

$y = k\sqrt{x}$
for either

or
$x = k\sqrt{y}$
A1
$x = \frac{y^2}{k^2}$
A1
$f^{-1}(x) = \frac{x^2}{k^2}$
A1
$\text{dom}\left(f^{-1}(x)\right) = [0, \infty[$
[3 marks]

(d)

or equivalent method ***M1***
$\frac{x^2}{k^2} = k\sqrt{x}$

$k = \sqrt{x}$
A1
$k = 2$
[2 marks]

(e) (i)

(M1)
$A = \int_a^b (y_1 - y_2)\mathrm{d}x$
A1
$A = \int_0^4 \left(2x^{\frac{1}{2}} - \frac{1}{4}x^2\right)\mathrm{d}x$
A1

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{12}x^3\right]_0$

A1

$= \frac{16}{3}$

(ii) attempt to find either

or

$f'(x)$

M1

$(f^{-1})'(x)$

A1A1

$f'(x) = \frac{1}{\sqrt{x}}, \ \left((f^{-1})'(x) = \frac{x}{2}\right)$

M1

$\frac{1}{\sqrt{c}} = \frac{c}{2}$

A1

$c = 2^{\frac{2}{3}}$

[9 marks]

Total [16 marks]

Examiners report

Many students could not sketch the function. There was confusion between the vertical and horizontal line test for one-to-one functions. A significant number of students gave long and inaccurate explanations for a one-to-one function. Finding the inverse was done very well by most students although the notation used was generally poor. The domain of the inverse was ignored by many or done incorrectly even if the sketch was correct. Many did not make the connections between the parts of the question. An example of this was the number of students who spent time finding the point of intersection in part e) even though it was given in d).

7a. *[6 marks]*

Markscheme

EITHER

let

(M1)

$u = \tan x; \ \mathrm{d}u = \sec^2 x \mathrm{d}x$

consideration of change of limits ***(M1)***

(A1)

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{u^{\frac{1}{3}}} \mathrm{d}u$

Note: Do not penalize lack of limits.

A1

$= \left[\frac{3u^{\frac{2}{3}}}{2}\right]_1^{\sqrt{3}}$

A1A1 ***N0***

$= \frac{3 \times \sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3} - 3}{2}\right)$

OR

M2A2

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x = \left[\frac{3(\tan x)^{\frac{2}{3}}}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

A1A1 ***N0***

$= \frac{3 \times \sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3} - 3}{2}\right)$

[6 marks]

Examiners report

Quite a variety of methods were successfully employed to solve part (a).

7b. *[3 marks]*

Markscheme

M1
$\int \tan^3 x \mathrm{d}x = \int \tan x(\sec^2 x - 1)\mathrm{d}x$

$= \int (\tan x \times \sec^2 x - \tan x)\mathrm{d}x$

A1A1
$= \frac{1}{2}\tan^2 x - \ln|\sec x| + C$

Note: Do not penalize the absence of absolute value or C.

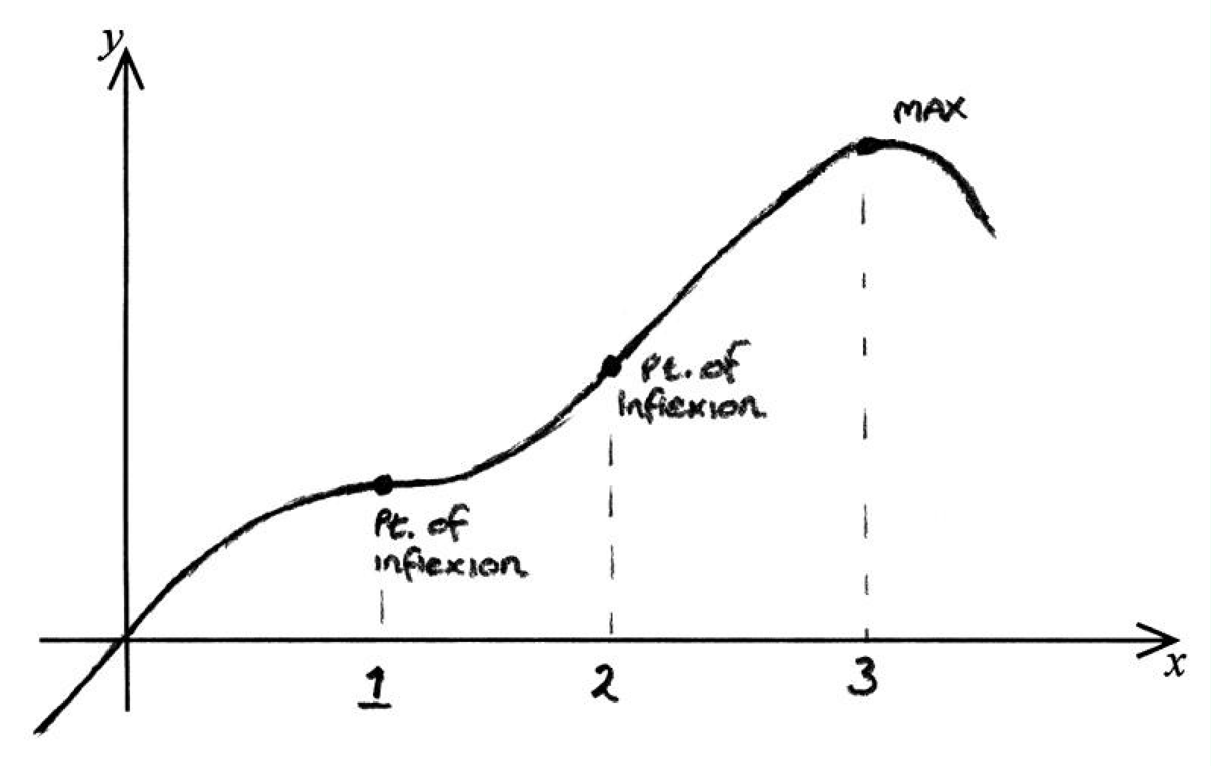
[3 marks]

Examiners report

Many candidates did not attempt part (b).

8. *[5 marks]*

Markscheme



A5

Note: Award ***A1*** for origin

A1 for shape

A1 for maximum

A1 for each point of inflexion.

[5 marks]

Examiners report

A reasonable number of candidates answered this correctly, although some omitted the

2nd point of inflection.

9. *[8 marks]*

Markscheme

$y = \mathrm{e}^x \Rightarrow x = \ln y$

volume

(M1)A1

$= \pi \int_1^5 (\ln y)^2 \mathrm{d}y$

using integration by parts ***(M1)***

A1A1

$\pi \int_1^5 (\ln y)^2 \mathrm{d}y = \pi \left[y(\ln y)^2 \right]_1^5 - 2\int_1^5 \ln y \mathrm{d}y$

A1A1

$= \pi \left[y(\ln y)^2 - 2y\ln y + 2y \right]_1^5$

Note: Award ***A1*** marks if π is present in at least one of the above lines.

A1

$\Rightarrow \pi \int_1^5 (\ln y)^2 \mathrm{d}y = \pi\, 5(\ln 5)^2 - 10\ln 5 + 8$

[8 marks]

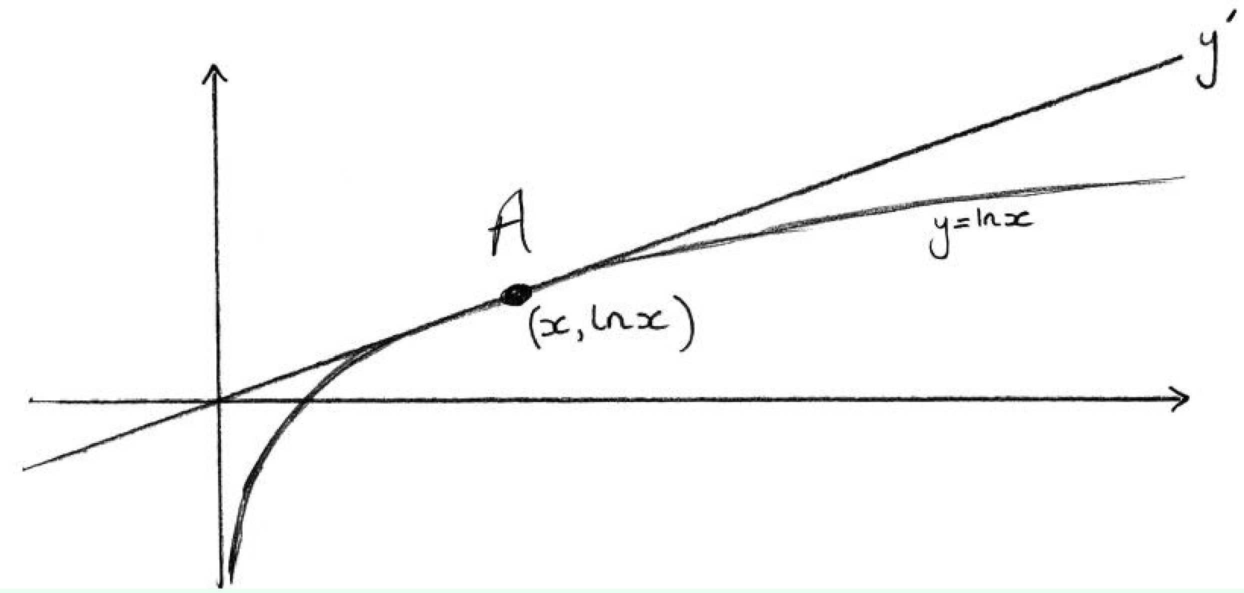
Examiners report

Only the best candidates were able to make significant progress with this question. Quite a few did not consider rotation about the *y*-axis. Others wrote the correct expression, but seemed daunted by needing to integrate by parts twice.

10. *[17 marks]*

Markscheme

(a)



A3

Note: Award ***A1*** for each graph

A1 for the point of tangency.

point on curve and line is

$(a,\ \ln a)$ ***(M1)***

$y = \ln(x)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a}\ \ (\text{when } x = a)$ ***(M1)A1***

EITHER

gradient of line, m, through (0, 0) and $(a,\ \ln a)$ is $\frac{\ln a}{a}$ ***(M1)A1***

$\Rightarrow \frac{\ln a}{a} = \frac{1}{a} \Rightarrow \ln a = 1 \Rightarrow a = \mathrm{e} \Rightarrow m = \frac{1}{\mathrm{e}}$ ***M1A1***

OR

$y - \ln a = \frac{1}{a}(x - a)$ ***(M1)A1***

passes through 0 if

$\ln a - 1 = 0$ ***M1***

$a = e \Rightarrow m = \frac{1}{\mathrm{e}}$ ***A1***

THEN

$\therefore y = \frac{1}{\mathrm{e}}x$ ***A1***

[11 marks]

(b) the graph of $y = \frac{1}{\mathrm{e}}x$ never goes above the graph of $\ln x$, hence $\ln x \leqslant \frac{x}{\mathrm{e}}$ ***R1***

[1 mark]

(c)

M1A1
$\ln x \leqslant \frac{x}{e} \Rightarrow e\ln x \leqslant x \Rightarrow \ln x^{e} \leqslant x$
exponentiate both sides of

R1AG
$\ln x^{e} \leqslant x \Rightarrow x^{e} \leqslant e^{x}$
[3 marks]

(d) equality holds when

R1
$x = e$
letting

A1 N0
$x = \pi \Rightarrow \pi^{e} < e^{\pi}$
[2 marks]

Total [17 marks]

Examiners report

This was the least accessible question in the entire paper, with very few candidates achieving high marks. Sketches were generally done poorly, and candidates failed to label the point of intersection. A ‘dummy’ variable was seldom used in part (a), hence in most cases it was not possible to get more than 3 marks. There was a lot of good guesswork as to the coordinates of the point of intersection, but no reasoning showed. Many candidates started with the conclusion in part (c). In part (d) most candidates did not distinguish between the inequality and strict inequality.

11. *[7 marks]*

Markscheme

(A1)
$\sqrt{x}e^{x} = e\sqrt{x} \Rightarrow x = 0 \text{ or } 1$
attempt to find

M1
$\int y^{2}\mathrm{d}x$

$V_1 = \pi\int_0^1 e^{2}x\mathrm{d}x$

$= \pi\left[\frac{1}{2}e^{2}x^{2}\right]_0^1$
A1
$= \frac{\pi e^{2}}{2}$

$V_2 = \pi\int_0^1 xe^{2x}\mathrm{d}x$
M1A1
$= \pi\left(\left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}\mathrm{d}x\right)$
Note: Award ***M1*** for attempt to integrate by parts.

$= \frac{\pi e^{2}}{2} - \pi\left[\frac{1}{4}e^{2x}\right]_0^1$
finding difference of volumes ***M1***

volume

$= V_1 - V_2$

$= \pi\left[\frac{1}{4}e^{2x}\right]_0^1$
A1
$= \frac{1}{4}\pi(e^{2} - 1)$
[7 marks]

Examiners report

While only a minority of candidates achieved full marks in this question, many candidates made good attempts. Quite a few candidates obtained the limits correctly and many realized a square was needed in the integral, though a number of them subtracted then squared rather than squaring and then subtracting. There was evidence that quite a few knew about integration by parts. One common mistake was to have 2π, rather than π in the integral.

12. *[7 marks]*

Markscheme

(a)

$u = \frac{1}{x} \Rightarrow \mathrm{d}u = -\frac{1}{x^2}\mathrm{d}x$ ***M1***

$\Rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{u^2}$ ***A1***

$\int_1^{\alpha} \frac{1}{1+x^2}\mathrm{d}x = -\int_1^{\frac{1}{\alpha}} \frac{1}{1+\left(\frac{1}{u}\right)^2}\frac{\mathrm{d}u}{u^2}$ ***A1M1A1***

Note: Award ***A1*** for correct integrand and ***M1A1*** for correct limits.

$= \int_{\frac{1}{\alpha}}^{1} \frac{1}{1+u^2}\mathrm{d}u$ (upon interchanging the two limits) ***AG***

(b)

$\arctan x_1^{\alpha} = \arctan u_{\frac{1}{\alpha}}^{1}$ ***A1***

$\arctan \alpha - \frac{\pi}{4} = \frac{\pi}{4} - \arctan \frac{1}{\alpha}$ ***A1***

$\arctan \alpha + \arctan \frac{1}{\alpha} = \frac{\pi}{2}$ ***AG***

[7 marks]

Examiners report

This question was successfully answered by few candidates. Both parts of the question prescribed the approach which was required – “use the substitution” and “hence”. Many candidates ignored these. The majority of the candidates failed to use substitution properly to change the integration variables and in many cases the limits were fudged. The logic of part (b) was missing in many cases.

13. *[20 marks]*

Markscheme

(a)

(M1)
$x^2 + 5x + 4 = 0 \Rightarrow x = -1$ or $x = -4$
so vertical asymptotes are $x = -1$ and $x = -4$ ***A1***

as $x \to \infty$ then $y \to 1$ so horizontal asymptote is $y = 1$ ***(M1)A1***

[4 marks]

(b)

A1
$x^2 - 5x + 4 = 0 \Rightarrow x = 1$ or $x = 4$
A1
$x = 0 \Rightarrow y = 1$
so intercepts are (1, 0), (4, 0) and (0,1)

[2 marks]

(c) (i)

M1A1A1
$f'(x) = \frac{(x^2+5x+4)(2x-5)-(x^2-5x+4)(2x+5)}{(x^2+5x+4)^2}$
A1
$= \frac{10x^2-40}{(x^2+5x+4)^2} \left(= \frac{10(x-2)(x+2)}{(x^2+5x+4)^2}\right)$
M1
$f'(x) = 0 \Rightarrow x = \pm 2$
so the points under consideration are (–2, –9) and $\left(2, -\frac{1}{9}\right)$ ***A1A1***

looking at the sign either side of the points (or attempt to find $f''(x)$) ***M1***

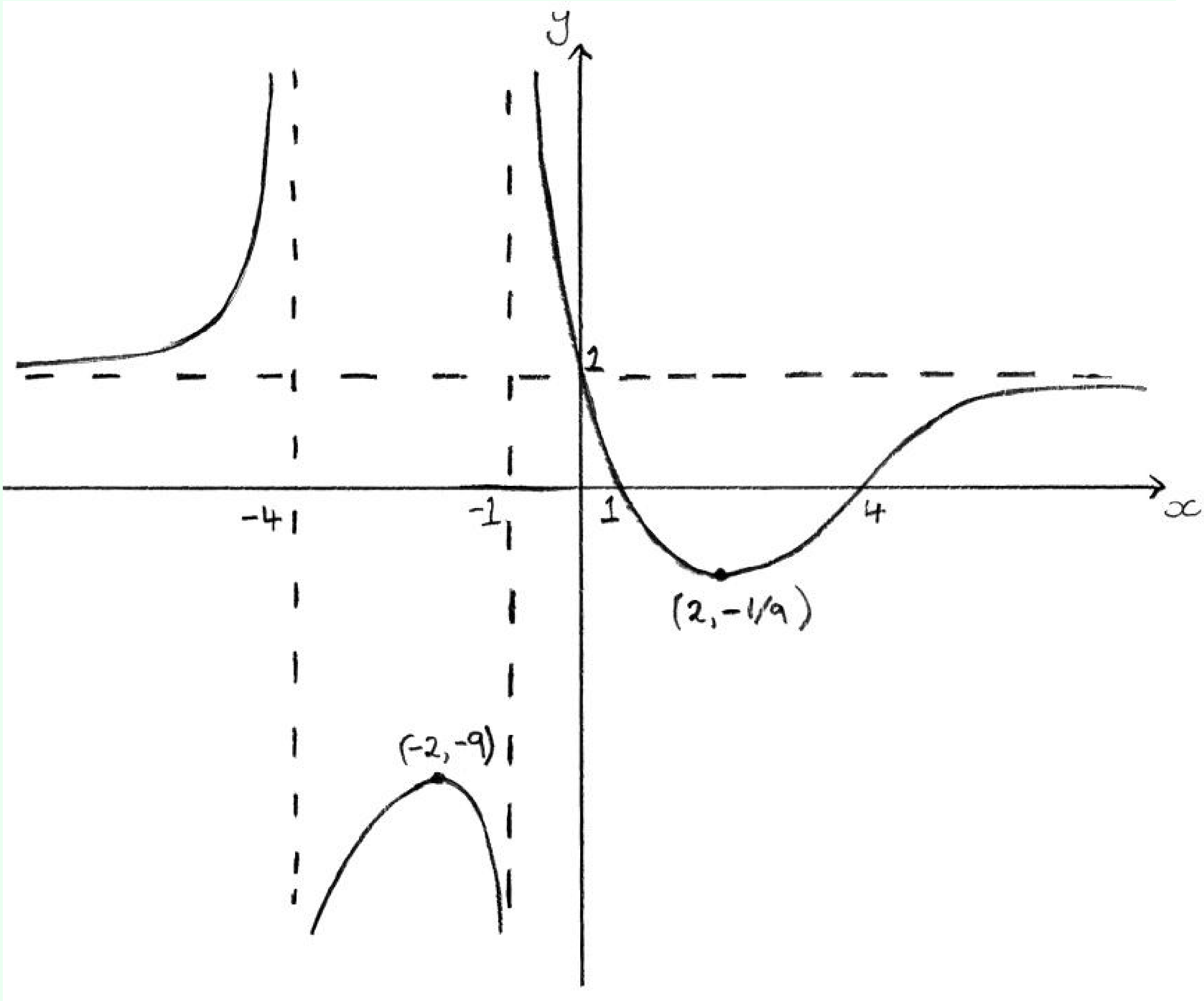
e.g. if $x = -2^-$ then $(x-2)(x+2) > 0$ and if $x = -2^+$ then $(x-2)(x+2) < 0$, therefore (–2, –9) is a maximum ***A1***

(ii) *e.g.* if $x = 2^-$ then $(x-2)(x+2) < 0$ and if $x = 2^+$ then $(x-2)(x+2) > 0$, therefore $\left(2, -\frac{1}{9}\right)$ is a minimum ***A1***

Note: Candidates may find the minimum first.

[10 marks]

(d)



A3

Note: Award *A1* for each branch consistent with and including the features found in previous parts.

[3 marks]

(e) one *A1*

[1 mark]

Total [20 marks]

Examiners report

This was the most successfully answered question in part B, in particular parts (a), (b) and (c). In part (a) the horizontal asymptote was often missing (or $x = 4$, $x = 1$ given). Part (b) was well done. Use of the quotient rule was well done in part (c) and many simplified correctly. There was knowledge of max/min and how to justify their answer, usually with a sign diagram but also with the second derivative. A common misconception was that, as $-9 < -\frac{1}{9}$, the minimum is at (–2, –9). In part (d) many candidates were unable to sketch the graph consistent with the main features that they had determined before. Very few candidates answered part (e) correctly.

14. *[8 marks]*

Markscheme

when $x = 2$ ***(A1)***

$\left|\frac{3}{2}x - 3\right| = 0$

the equation of the parabola is

(M1)

$y = p(x-2)^2 - 3$

through

(M1)

$(0,\ 3) \Rightarrow 3 = 4p - 3 \Rightarrow p = \frac{3}{2}$

the equation of the parabola is

A1

$y = \frac{3}{2}(x-2)^2 - 3\ \left(= \frac{3}{2}x^2 - 6x + 3\right)$

area

M1M1A1

$= 2\int_0^2 \left(3 - \frac{3}{2}x\right) - \left(\frac{3}{2}x^2 - 6x + 3\right) \mathrm{d}x$

Note: Award ***M1*** for recognizing symmetry to obtain

$2\int_0^2$,

M1 for the difference,

A1 for getting all parts correct.

A1

$= \int_0^2 (-3x^2 + 9x)\mathrm{d}x$

[8 marks]

Examiners report

This was a difficult question and, although many students obtained partial marks, there were few completely correct solutions.

15. *[14 marks]*

Markscheme

(a)

(M1)
$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v^2}{200} - 32 \left(= \frac{-v^2-6400}{200}\right)$
M1A1A1
$\int_0^T \mathrm{d}t = \int_{40}^V -\frac{200}{v^2+80^2}\mathrm{d}v$
AG
$T = 200\int_V^{40} \frac{1}{v^2+80^2}\mathrm{d}v$
[4 marks]

(b) (i)

R1
$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}t}$
AG
$= v\frac{\mathrm{d}v}{\mathrm{d}s}$

(ii)

(M1)
$v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{-v^2-80^2}{200}$
M1A1A1
$\int_0^S \mathrm{d}s = \int_{40}^V -\frac{200v}{v^2+80^2}\mathrm{d}v$
M1
$\int_0^S \mathrm{d}s = \int_V^{40} \frac{200v}{v^2+80^2}\mathrm{d}v$
A1
$S = 200\int_V^{40} \frac{v}{v^2+80^2}\mathrm{d}v$
[7 marks]

(c) letting $V = 0$ ***(M1)***

distance

A1
$= 200\int_0^{40} \frac{v}{v^2+80^2}\mathrm{d}v = 22.3 \text{ metres}$
time

A1
$= 200\int_0^{40} \frac{1}{v^2+80^2}\mathrm{d}v = 1.16 \text{ seconds}$
[3 marks]

Total [14 marks]

Examiners report

Many students failed to understand the problem as one of solving differential equations. In addition there were many problems seen in finding the end points for the definite integrals. Part (b) (i) should have been a simple point having used the chain rule, but it seemed that many students had not seen this, even though it is clearly in the syllabus.

16. *[8 marks]*

Markscheme

(a)

$\left(u = x^2 - 2x - 1.5; \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 2\right)$ ***(M1)***

$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^u(2x - 2)$ ***A1***

$= 2(x-1)\mathrm{e}^{x^2-2x-1.5}$

(b)

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-1)\times 2(x-1)\mathrm{e}^{x^2-2x-1.5} - 1\times\mathrm{e}^{x^2-2x-1.5}}{(x-1)^2}$ ***M1A1***

$= \frac{2x^2-4x+1}{(x-1)^2}\mathrm{e}^{x^2-2x-1.5}$ ***(A1)***

minimum occurs when

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***(M1)***

$x = 1 \pm \sqrt{\frac{1}{2}} \quad \left(\text{accept } x = \frac{4\pm\sqrt{8}}{4}\right)$ ***A1***

$a = 1 + \sqrt{\frac{1}{2}} \quad \left(\text{accept } a = \frac{4+\sqrt{8}}{4}\right)$ ***R1***

[8 marks]

Examiners report

Part (a) was successfully answered by most candidates. Most candidates were able to make significant progress with part (b) but were then let down by being unable to simplify the expression or by not understanding the significance of being told that $a > 1$.

17. *[7 marks]*

Markscheme

EITHER

differentiating implicitly:

M1A1

$1 \times \mathrm{e}^{-y} - x\mathrm{e}^{-y}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$

at the point (c,

)

$\ln c$

M1

$\frac{1}{c} - c \times \frac{1}{c}\frac{\mathrm{d}y}{\mathrm{d}x} + c\frac{\mathrm{d}y}{\mathrm{d}x} = 1$

(A1)

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{c} \quad (c \neq 1)$

OR

reasonable attempt to make expression explicit ***(M1)***

$x\mathrm{e}^{-y} + \mathrm{e}^{y} = 1 + x$

$x + \mathrm{e}^{2y} = \mathrm{e}^{y}(1 + x)$

$\mathrm{e}^{2y} - \mathrm{e}^{y}(1 + x) + x = 0$

(A1)

$(\mathrm{e}^{y} - 1)(\mathrm{e}^{y} - x) = 0$

$\mathrm{e}^{y} = 1,\ \mathrm{e}^{y} = x$

A1

$y = 0,\ y = \ln x$

Note: Do not penalize if $y = 0$ not stated.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$

gradient of tangent

A1

$= \frac{1}{c}$

Note: If candidate starts with

with no justification, award ***(M0)(A0)A1A1***.

$y = \ln x$

THEN

the equation of the normal is

M1

$y - \ln c = -c(x - c)$

$x = 0,\ y = c^2 + 1$

(A1)

$c^2 + 1 - \ln c = c^2$

$\ln c = 1$

A1

$c = \mathrm{e}$

[7 marks]

Examiners report

This was the first question to cause the majority of candidates a problem and only the better candidates gained full marks. Weaker candidates made errors in the implicit differentiation and those who were able to do this often were unable to simplify the expression they gained for the gradient of the normal in terms of c; a significant number of candidates did not know how to simplify the logarithms appropriately.

18. *[6 marks]*

Markscheme

EITHER

attempt at integration by substitution *(M1)*

using

$u = t + 1,\ \mathrm{d}u = \mathrm{d}t$, the integral becomes

$\int_1^2 (u-1)\ln u\,\mathrm{d}u$ ***A1***

then using integration by parts ***M1***

$\int_1^2 (u-1)\ln u\,\mathrm{d}u = \left[\left(\frac{u^2}{2} - u\right)\ln u\right]_1^2 - \int_1^2 \left(\frac{u^2}{2} - u\right) \times \frac{1}{u}\mathrm{d}u$ ***A1***

$= -\left[\frac{u^2}{4} - u\right]_1^2$ ***(A1)***

$= \frac{1}{4}$ (accept 0.25) ***A1***

OR

attempt to integrate by parts *(M1)*

correct choice of variables to integrate and differentiate ***M1***

$\int_0^1 t\ln(t+1)\mathrm{d}t = \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \int_0^1 \frac{t^2}{2} \times \frac{1}{t+1}\mathrm{d}t$ ***A1***

$= \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \frac{1}{2}\int_0^1 t - 1 + \frac{1}{t+1}\mathrm{d}t$ ***A1***

$= \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right]_0^1$ ***(A1)***

$= \frac{1}{4}$ (accept 0.25) ***A1***

[6 marks]

Examiners report

Again very few candidates gained full marks on this question. The most common approach was to begin by integrating by parts, which was done correctly, but very few candidates then knew how to integrate $\frac{t^2}{t+1}$. Those who began with a substitution often made more progress. Again a number of candidates were let down by their inability to simplify appropriately.

19. *[8 marks]*

Markscheme

(a) the distance of the spot from P is

A1
$x = 500\tan\theta$
the speed of the spot is

M1A1
$\frac{\mathrm{d}x}{\mathrm{d}t} = 500\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\ (= 4000\pi\sec^2\theta)$
when

$x = 2000,\ \sec^2\theta = 17\ (\theta = 1.32581\ldots)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} = 8\pi\right)$
M1A1
$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 500 \times 17 \times 8\pi$
speed is

(metres per minute) ***AG***
214 000
Note: If their displayed answer does not round to

, they lose the final ***A1***.
214 000

(b)

or
$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 8000\pi\sec^2\theta\tan\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}$
M1A1
$500 \times 2\sec^2\theta\tan\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$

$$\left(\text{since } \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = 0\right)$$

A1
$= 43\,000\,000\ (= 4.30 \times 10^7)\ (\text{metres per minute}^2)$

[8 marks]

Examiners report

This was a wordy question with a clear diagram, requiring candidates to state variables and do some calculus. Very few responded appropriately.

20.

[14 marks]

Markscheme

(a) solving to obtain one root: 1, – 2 or – 5 ***A1***

obtain other roots ***A1***

[2 marks]

(b)

(or equivalent) ***M1A1***
$D = x \in [-5,\ -2] \cup [1,\ \infty)$
Note: ***M1*** is for 1 finite and 1 infinite interval.

[2 marks]

(c) coordinates of local maximum

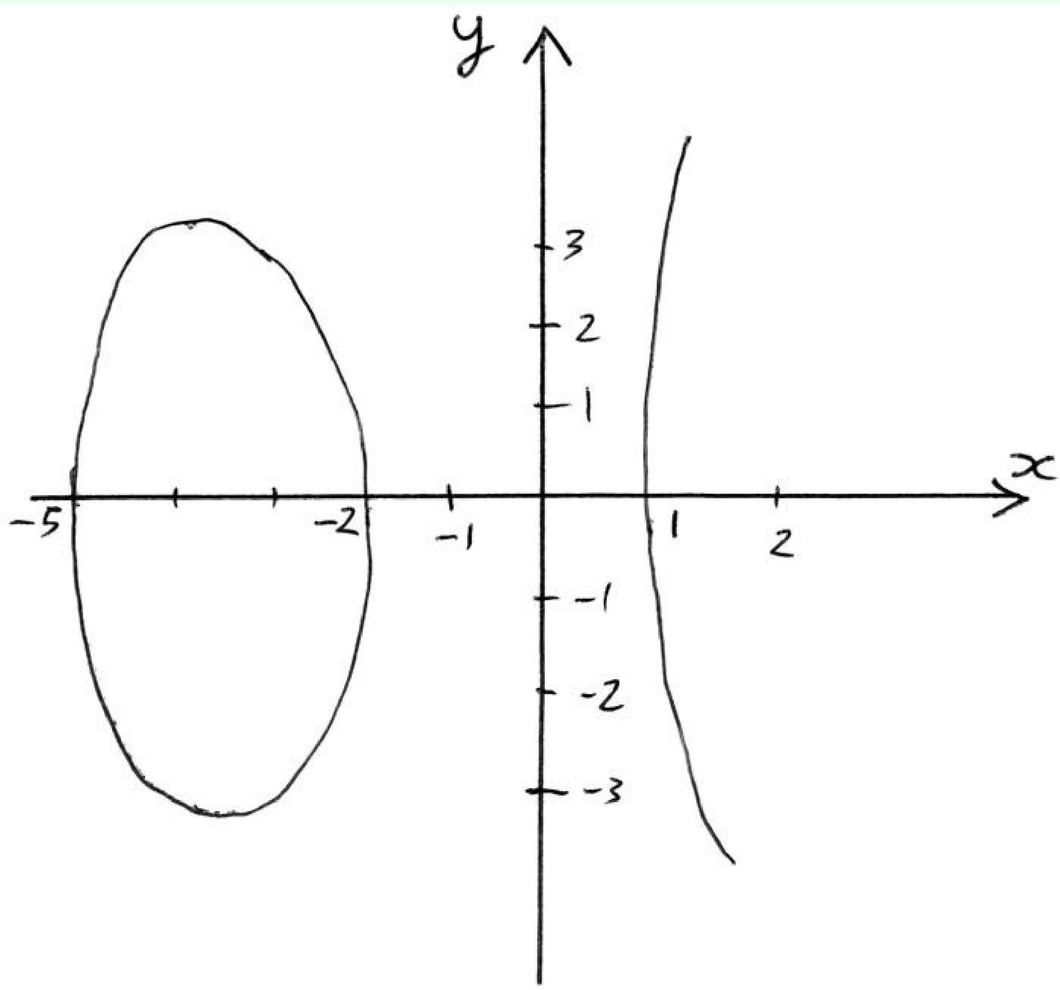
A1A1
$-3.73 - 2 - \sqrt{3},\ 3.22\sqrt{6\sqrt{3}}$
[2 marks]

(d) use GDC to obtain one root: 1.41, – 3.18 or – 4.23 ***A1***

obtain other roots ***A1***

[2 marks]

(e)



A1A1A1

Note: Award ***A1*** for shape, ***A1*** for max and for min clearly in correct places, ***A1*** for all intercepts.

Award ***A1A0A0*** if only the complete top half is shown.

[3 marks]

(f) required area is twice that of

between – 5 and – 2 ***M1A1***
$y = f(x)$
answer 14.9 ***A1*** ***N3***

Note: Award ***M1A0A0*** for

or ***N1*** for 7.47.
$\int_{-5}^{-2} f(x)\mathrm{d}x = 7.47\ldots$

[3 marks]

Total [14 marks]

Examiners report

This was a multi-part question that was well answered by many candidates. The main difficulty was sketching the graph and this meant that the last part was not well answered.

21a. *[10 marks]*

Markscheme

(i) the period is 2 ***A1***

(ii)

(M1)A1
$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 2\pi\cos(\pi t) + 2\pi\cos(2\pi t)$
(M1)A1
$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -2\pi^2\sin(\pi t) - 4\pi^2\sin(2\pi t)$

(iii)

$v = 0$

$2\pi\left(\cos(\pi t) + \cos(2\pi t)\right) = 0$
EITHER

M1
$\cos(\pi t) + 2\cos^2(\pi t) - 1 = 0$
(A1)
$(2\cos(\pi t) - 1)(\cos(\pi t) + 1) = 0$
A1
$\cos(\pi t) = \frac{1}{2} \text{ or } \cos(\pi t) = -1$
A1
$t = \frac{1}{3},\ t = 1$
A1
$t = \frac{5}{3},\ t = \frac{7}{3},\ t = \frac{11}{3},\ t = 3$
OR

M1
$2\cos\left(\frac{\pi t}{2}\right)\cos\left(\frac{3\pi t}{2}\right) = 0$
A1A1
$\cos\left(\frac{\pi t}{2}\right) = 0 \text{ or } \cos\left(\frac{3\pi t}{2}\right) = 0$
A1
$t = \frac{1}{3},\ 1$
A1
$t = \frac{5}{3},\ \frac{7}{3},\ 3,\ \frac{11}{3}$
[10 marks]

Examiners report

In (a), only a few candidates gave the correct period but the expressions for velocity and acceleration were correctly obtained by most candidates. In (a)(iii), many candidates manipulated the equation $v = 0$ correctly to give the two possible values for $\cos(\pi t)$ but then failed to find all the possible values of t.

21b. *[8 marks]*

Markscheme

$P(n): f^{(2n)}(x) = (-1)^n \left(Aa^{2n} \sin(ax) + Bb^{2n} \sin(bx)\right)$

M1

$P(1): f''(x) = (Aa\cos(ax) + Bb\cos(bx))\,'$

$= -Aa^2 \sin(ax) - Bb^2 \sin(bx)$

A1

$= -1\left(Aa^2 \sin(ax) + Bb^2 \sin(bx)\right)$

$\therefore P(1)$ true

assume that $P(k): f^{(2k)}(x) = (-1)^k \left(Aa^{2k} \sin(ax) + Bb^{2k} \sin(bx)\right)$ is true ***M1***

consider $P(k+1)$

M1A1

$f^{(2k+1)}(x) = (-1)^k \left(Aa^{2k+1} \cos(ax) + Bb^{2k+1} \cos(bx)\right)$

A1

$f^{(2k+2)}(x) = (-1)^k \left(-Aa^{2k+2} \sin(ax) - Bb^{2k+2} \sin(bx)\right)$

A1

$= (-1)^{k+1} \left(Aa^{2k+2} \sin(ax) + Bb^{2k+2} \sin(bx)\right)$

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true $\forall n \in \mathbb{Z}^+$ ***R1***

Note: Award the final ***R1*** only if the previous three ***M*** marks have been awarded.

[8 marks]

Examiners report

Solutions to (b) were disappointing in general with few candidates giving a correct solution.

22. *[23 marks]*

Markscheme

(a) (i)

A1
$x\mathrm{e}^x = 0 \Rightarrow x = 0$
so, they intersect only once at (0, 0)

(ii)

M1A1
$y' = \mathrm{e}^x + x\mathrm{e}^x = (1 + x)\mathrm{e}^x$
A1
$y'(0) = 1$
A1
$\theta = \arctan 1 = \frac{\pi}{4}\ (\theta = 45^\circ)$
[5 marks]

(b) when

$k = 1,\ y = x$
M1
$x\mathrm{e}^x = x \Rightarrow x(\mathrm{e}^x - 1) = 0$
A1
$\Rightarrow x = 0$
which equals the gradient of the line
$y'(0) = 1$
R1
$y = x$
so, the line is tangent to the curve at origin ***AG***

Note: Award full credit to candidates who note that the equation

has a double root $x = 0$ so $y = x$ is a tangent.
$x(\mathrm{e}^x - 1) = 0$

[3 marks]

(c) (i)

M1
$x\mathrm{e}^x = kx \Rightarrow x(\mathrm{e}^x - k) = 0$
A1
$\Rightarrow x = 0$ or $x = \ln k$
A1
$k > 0$ and $k \neq 1$

(ii) (0, 0) and

A1A1
$(\ln k,\ k\ln k)$

(iii)

M1A1

$A = \left|\int_0^{\ln k} kx - x\mathrm{e}^x \mathrm{d}x\right|$

Note: Do not penalize the omission of absolute value.

(iv) attempt at integration by parts to find

M1
$\int x\mathrm{e}^x \mathrm{d}x$
A1
$\int x\mathrm{e}^x \mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x(x-1)$

$\int xe^x dx = xe^x - \int e^x dx = e^x(x-1)$
as

R1
$0 < k < 1 \Rightarrow \ln k < 0$
A1
$A = \int_{\ln k}^{0} kx - xe^x dx = \left[\frac{k}{2}x^2 - (x-1)e^x\right]_{\ln k}^{0}$
A1
$= 1 - \left(\frac{k}{2}(\ln k)^2 - (\ln k - 1)k\right)$

$= 1 - \frac{k}{2}\left((\ln k)^2 - 2\ln k + 2\right)$
M1A1
$= 1 - \frac{k}{2}\left((\ln k - 1)^2 + 1\right)$
since

R1
$\frac{k}{2}\left((\ln k - 1)^2 + 1\right) > 0$
AG
$A < 1$
[15 marks]

Total [23 marks]

Examiners report

Many candidates solved (a) and (b) correctly but in (c), many failed to realise that the equation

has two roots under certain conditions and that the point of the question was to identify those conditions. Most candidates made a
$xe^x = kx$
reasonable attempt to write down the appropriate integral in (c)(iii) with the modulus signs and limits often omitted but no correct

solution has yet been seen to (c)(iv).

23.

[7 marks]

Markscheme

$x^3y^3 - xy = 0$

$3x^2y^3 + 3x^3y^2y' - y - xy' = 0$
Note: Award ***A1*** for correctly differentiating each term.

$x = 1,\ y = 1$

$3 + 3y' - 1 - y' = 0$

$2y' = -2$

(M1)A1
$y' = -1$
gradient of normal = 1 ***(A1)***

equation of the normal

A1 ***N2***
$y - 1 = x - 1$

$y = x$
Note: Award ***A2R5*** for correct answer and correct justification.

[7 marks]

Examiners report

This implicit differentiation question was well answered by most candidates with many achieving full marks. Some candidates made algebraic errors which prevented them from scoring well in this question.

Other candidates realised that the equation of the curve could be simplified although the simplification was seldom justified.

24. *[7 marks]*

Markscheme

EITHER

M1A1

$y = \frac{1}{1-x} \Rightarrow y' = \frac{1}{(1-x)^2}$

solve simultaneously ***M1***

$\frac{1}{1-x} = m(x-m)$ and $\frac{1}{(1-x)^2} = m$

A1

$\frac{1}{1-x} = \frac{1}{(1-x)^2}\left(x - \frac{1}{(1-x)^2}\right)$

Note: Accept equivalent forms.

$(1-x)^3 - x(1-x)^2 + 1 = 0,\ x \neq 1$

$x = 1.65729\ldots \Rightarrow y = \frac{1}{1-1.65729\ldots} = -1.521379\ldots$

tangency point (1.66, –1.52) ***A1A1***

A1

$m = (-1.52137\ldots)^2 = 2.31$

OR

$(1-x)y = 1$

M1

$m(1-x)(x-m) = 1$

$m(x - x^2 - m + mx) = 1$

A1

$mx^2 - x(m + m^2) + (m^2 + 1) = 0$

(M1)

$b^2 - 4ac = 0$

$(m + m^2)^2 - 4m(m^2 + 1) = 0$

A1

$m = 2.31$

substituting

(M1)

$m = 2.31\ldots$ into $mx^2 - x(m + m^2) + (m^2 + 1) = 0$

A1

$x = 1.66$

A1

$y = \frac{1}{1-1.65729} = -1.52$

tangency point (1.66, –1.52)

[7 marks]

Examiners report

Very few candidates answered this question well but among those a variety of nice approaches were seen. This question required some organized thinking and good understanding of the concepts involved and therefore just strong candidates were able to go beyond the first steps. Sadly a few good answers were spoiled due to early rounding.

25. *[19 marks]*

Markscheme

(a)

$f'(x) = \frac{be^x(ae^x+b) - ae^x(a+be^x)}{(ae^x+b)^2}$ ***M1A1***

$= \frac{abe^{2x} + b^2e^x - a^2e^x - abe^{2x}}{(ae^x+b)^2}$ ***A1***

$= \frac{(b^2-a^2)e^x}{(ae^x+b)^2}$ ***AG***

[3 marks]

(b) ***EITHER***

$f'(x) = 0 \Rightarrow (b^2 - a^2)e^x = 0 \Rightarrow b = \pm a$ or $e^x = 0$ ***A1***

which is impossible as $0 < b < a$ and $e^x > 0$ for all $x \in \mathbb{R}$ ***R1***

OR

$f'(x) < 0$ for all $x \in \mathbb{R}$ since $0 < b < a$ and $e^x > 0$ for all $x \in \mathbb{R}$ ***A1R1***

OR

$f'(x)$ cannot be equal to zero because e^x is never equal to zero ***A1R1***

[2 marks]

(c) **EITHER**

$f''(x) = \frac{(b^2-a^2)e^x(ae^x+b)^2 - 2ae^x(ae^x+b)(b^2-a^2)e^x}{(ae^x+b)^4}$ ***M1A1A1***

Note: Award ***A1*** for each term in the numerator.

$= \frac{(b^2-a^2)e^x(ae^x+b-2ae^x)}{(ae^x+b)^3}$

$= \frac{(b^2-a^2)(b-ae^x)e^x}{(ae^x+b)^3}$

OR

$f'(x) = (b^2 - a^2)e^x(ae^x + b)^{-2}$

$f''(x) = (b^2 - a^2)e^x(ae^x + b)^{-2} + (b^2 - a^2)e^x(-2ae^x)(ae^x + b)^{-3}$ ***M1A1A1***

Note: Award ***A1*** for each term.

$= (b^2 - a^2)e^x(ae^x + b)^{-3}\left((ae^x + b) - 2ae^x\right)$

$= (b^2 - a^2)e^x(ae^x + b)^{-3}(b - ae^x)$

$= (b^2 - a^2)e^x(ae^x + b)^{-3}(b - ae^x)$

THEN

$f''(x) = 0 \Rightarrow b - ae^x = 0 \Rightarrow x = \ln\frac{b}{a}$ ***M1A1***

$f\left(\ln\frac{b}{a}\right) = \frac{a^2+b^2}{2ab}$ ***A1***

coordinates are $\left(\ln\frac{b}{a}, \frac{a^2+b^2}{2ab}\right)$

[6 marks]

(d)

$\lim_{x \to -\infty} f(x) = \frac{a}{b} \Rightarrow y = \frac{a}{b}$ horizontal asymptote ***A1***

$\lim_{x \to +\infty} f(x) = \frac{b}{a} \Rightarrow y = \frac{b}{a}$ horizontal asymptote ***A1***

$0 < b < a \Rightarrow ae^x + b > 0$ for all $x \in \mathbb{R}$ (accept $ae^x + b \neq 0$)

so no vertical asymptotes ***R1***

Note: Statement on vertical asymptote must be seen for ***R1***.

[3 marks]

(e)

$y = \frac{4+e^x}{4e^x+1}$

$y = \frac{1}{2} \Leftrightarrow x = \ln\frac{7}{2}$ (or 1.25 to 3 sf) ***(M1)(A1)***

$V = \pi\int_0^{\ln\frac{7}{2}} \left(\left(\frac{4+e^x}{4e^x+1}\right)^2 - \frac{1}{4}\right)\mathrm{d}x$ ***(M1)A1***

$= 1.09$ (3 sf) ***A1 N4***

[5 marks]

Total [19 marks]

Examiners report

This question was well attempted by many candidates. In some cases, candidates who skipped other questions still answered, with some success, parts of this question. Part (a) was in general well done but in (b) candidates found difficulty in justifying that f'(x) was non-zero. Performance in part (c) was mixed: it was pleasing to see good levels of algebraic ability of good candidates who successfully answered this question; weaker candidates found the simplification required difficult. There were very few good answers to part (d) which showed the weaknesses of most candidates in dealing with the concept of asymptotes. In part (e) there were a large number of good attempts, with many candidates evaluating correctly the limits of the integral and a smaller number scoring full marks in this part.

26.

[7 marks]

Markscheme

(a) shaded area area of triangle area of sector, *i.e.* ***(M1)***

A1A1AG
$\left(\frac{1}{2} \times 4^2 \sin x\right) - \left(\frac{1}{2} 2^2 x\right) = 8 \sin x - 2x$

(b) **EITHER**

any method from GDC gaining

(M1)(A1)
$x \approx 1.32$
maximum value for given domain is

A2
5.11
OR

A1
$\frac{\mathrm{d}A}{\mathrm{d}x} = 8 \cos x - 2$
set
$\frac{\mathrm{d}A}{\mathrm{d}x} = 0$
, hence
M1
$8 \cos x - 2 = 0$
A1
$\cos x = \frac{1}{4} \Rightarrow x \approx 1.32$
hence
A1
$A_{\max} = 5.11$

[7 marks]

Examiners report

Generally a well answered question.

27.

[7 marks]

Markscheme

$\mathrm{e}^{xy} + \ln\left(y^2\right) + \mathrm{e}^y = 1 + \mathrm{e}$
, at (0, 1) ***A1A1A1A1A1***
$\mathrm{e}^{xy}\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
$1\,(1 + 0) + 2\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$1 + 2\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
$\left(\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2+\mathrm{e}}\right)$ ***M1A1*** ***N2***
$= -0.212$
[7 marks]

Examiners report

Implicit differentiation is usually found to be difficult, but on this occasion there were many correct solutions. There were also a number of errors in the differentiation of
, and although these often led to the correct final answer, marks could not be awarded.
e^{xy}

28.

[11 marks]

Markscheme

(a)

(i)

M1

$x - a\sqrt{x}$

(A1)

$\sqrt{x}\sqrt{x} - a = 0$

2

,

$x = 0$

A1 ***N2***

$x = a^2$

(ii)

A1

$f'(x) = 1 - \frac{a}{2\sqrt{x}}$

is decreasing when

f

(M1)

$f' < 0$

A1

$1 - \frac{a}{2\sqrt{x}} < 0 \Rightarrow \frac{2\sqrt{x}-a}{2\sqrt{x}} < 0 \Rightarrow x > \frac{a^2}{4}$

(iii)

is increasing when

f

$f' > 0$

A1

$1 - \frac{a}{2\sqrt{x}} > 0 \Rightarrow \frac{2\sqrt{x}-a}{2\sqrt{x}} > 0 \Rightarrow x > \frac{a^2}{4}$

Note: Award the ***M1*** mark for either (ii) or (iii).

(iv) minimum occurs at

$x = \frac{a^2}{4}$

minimum value is

(M1)A1

$y = -\frac{a^2}{4}$

hence

A1

$y \geqslant -\frac{a^2}{4}$

[10 marks]

(b) concave up for all values of

R1

x

[1 mark]

Total [11 marks]

Examiners report

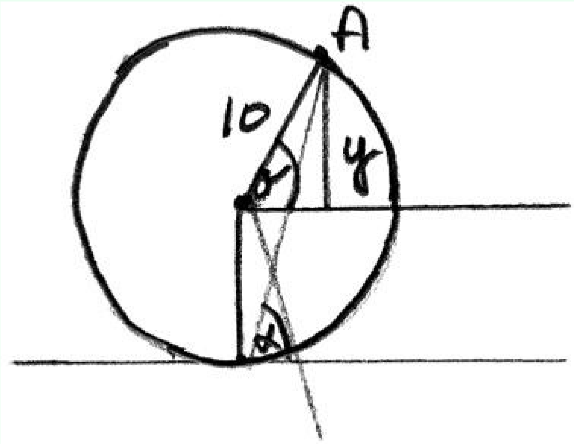
This was generally a well answered question.

29.

[10 marks]

Markscheme

(a)



$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 3$ ***(A1)***

$y = 10\sin\theta$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 10\cos\theta$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = 30\cos\theta$ ***M1***

at $y = 6$, $\cos\theta = \frac{8}{10}$ ***(M1)(A1)***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 24$ (metres per minute) (accept 24.0) ***A1***

[7 marks]

(b)

$\alpha = \frac{\theta}{2} + \frac{\pi}{4}$ ***M1A1***

$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1.5$ ***A1***

[3 marks]

Total [10 marks]

Examiners report

Many students were unable to get started with this question, and those that did were generally very poor at defining their variables at the start.

30. *[5 marks]*

Markscheme

(a)

A1
$f'(x) = 1 - \frac{2}{x^{\frac{1}{3}}}$

A1
$\Rightarrow 1 - \frac{2}{x^{\frac{1}{3}}} = 0 \Rightarrow x^{\frac{1}{3}} = 2 \Rightarrow x = 8$

(b)

A1
$f''(x) = \frac{2}{3x^{\frac{4}{3}}}$

M1A1
$f''(8) > 0 \Rightarrow$ at $x = 8,\ f(x)$ has a minimum.

[5 marks]

Examiners report

Most candidates were able to correctly differentiate the function and find the point where $f'(x) = 0$. They were less successful in determining the nature of the point.

31. *[7 marks]*

Markscheme

M1
$2 + x - x^2 = 2 - 3x + x^2$

$\Rightarrow 2x^2 - 4x = 0$

$\Rightarrow 2x(x - 2) = 0$

A1A1
$\Rightarrow x = 0,\ x = 2$

Note: Accept graphical solution.

Award ***M1*** for correct graph and ***A1A1*** for correctly labelled roots.

(M1)
$\therefore \text{A} = \int_0^2 \left((2 + x - x^2) - (2 - 3x + x^2)\right) \mathrm{d}x$

A1
$= \int_0^2 (4x - 2x^2)\mathrm{d}x$ or equivalent

A1
$= \left[2x^2 - \frac{2x^3}{3}\right]_0^2$

A1
$= \frac{8}{3} \left(= 2\frac{2}{3}\right)$

[7 marks]

Examiners report

This was the question that gained the most correct responses. A few candidates struggled to find the limits of the integration or found a negative area.

32. *[6 marks]*

Markscheme

METHOD 1

M1
$V = \pi \int_1^{\mathrm{e}} \left(\frac{\ln x}{x}\right)^2 \mathrm{d}x$

Integrating by parts:

(M1)
$u = (\ln x)^2,\ \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\ln x}{x},\ v = -\frac{1}{x}$

A1
$\Rightarrow V = \pi\left(-\frac{(\ln x)^2}{x} + 2\int \frac{\ln x}{x^2}\mathrm{d}x\right)$

(M1)
$u = \ln x,\ \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x},\ v = -\frac{1}{x}$

A1
$\therefore \int \frac{\ln x}{x^2}\mathrm{d}x = -\frac{\ln x}{x} + \int \frac{1}{x^2}\mathrm{d}x = -\frac{\ln x}{x} - \frac{1}{x}$

$\therefore V = \pi\left[-\frac{(\ln x)^2}{x} + 2\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\right]_1^{\mathrm{e}}$

A1
$= 2\pi - \frac{5\pi}{e}$

[6 marks]

METHOD 2

M1
$V = \pi \int_1^{\mathrm{e}} \left(\frac{\ln x}{x}\right)^2 \mathrm{d}x$

Let

(M1)
$\ln x = u \Rightarrow x = \mathrm{e}^u,\ \frac{\mathrm{d}x}{x} = \mathrm{d}u$

A1
$\int \left(\frac{\ln x}{x}\right)^2 \mathrm{d}x = \int \frac{u^2}{\mathrm{e}^u}\mathrm{d}u = \int \mathrm{e}^{-u}u^2\mathrm{d}u = -\mathrm{e}^{-u}u^2 + 2\int \mathrm{e}^{-u}u\mathrm{d}u$

$= -\mathrm{e}^{-u}u^2 + 2\left(-\mathrm{e}^{-u}u + \int \mathrm{e}^{-u}\mathrm{d}u\right) = -\mathrm{e}^{-u}u^2 - 2\mathrm{e}^{-u}u - 2\mathrm{e}^{-u}$

A1
$= -\mathrm{e}^{-u}(u^2 + 2u + 2)$

When $x = \mathrm{e}$, $u = 1$. When $x = 1$, $u = 0$.

M1
$\therefore$ Volume $= \pi\left[-\mathrm{e}^{-u}(u^2 + 2u + 2)\right]_0^1$

A1
$= \pi(-5\mathrm{e}^{-1} + 2)\ \left(= 2\pi - \frac{5\pi}{\mathrm{e}}\right)$

[6 marks]

Examiners report

Only the best candidates were able to make significant progress with this question. It was disappointing to see that many candidates could not state that the formula for the required volume was

$\pi \int_1^e \left(\frac{\ln x}{x}\right)^2 \mathrm{d}x$

. Of those who could, very few either attempted integration by parts or used an appropriate substitution.

33.

[27 marks]

Markscheme

(a)

$f'(x) = (1 + 2x)e^{2x}$ ***A1***

$f'(x) = 0$ ***M1***

$\Rightarrow (1 + 2x)e^{2x} = 0 \Rightarrow x = -\frac{1}{2}$ ***A1***

$f''(x) = (2^2x + 2 \times 2^{2-1})e^{2x} = (4x + 4)e^{2x}$ ***A1***

$f''\left(-\frac{1}{2}\right) = \frac{2}{e}$ ***A1***

$\frac{2}{e} > 0 \Rightarrow$ at $x = -\frac{1}{2}$, $f(x)$ has a minimum. ***R1***

$P\left(-\frac{1}{2}, -\frac{1}{2e}\right)$ ***A1***

[7 marks]

(b)

$f''(x) = 0 \Rightarrow 4x + 4 = 0 \Rightarrow x = -1$ ***M1A1***

Using the 2^{nd} derivative $f''\left(-\frac{1}{2}\right) = \frac{2}{e}$ and $f''(-2) = -\frac{4}{e^4}$, ***M1A1***

the sign change indicates a point of inflexion. ***R1***

[5 marks]

(c) (i) $f(x)$ is concave up for

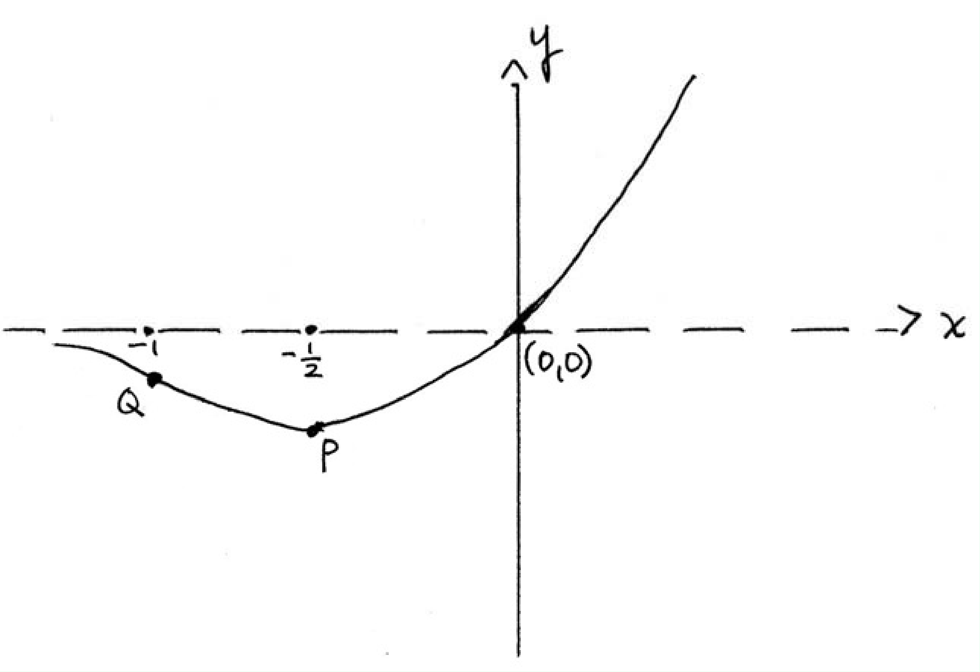
$x > -1$. ***A1***

(ii) $f(x)$ is concave down for

$x < -1$. ***A1***

[2 marks]

(d)



A1A1A1A1

Note: Award ***A1*** for P and Q, with Q above P,

A1 for asymptote at $y = 0$,

A1 for (0, 0),

A1 for shape.

[4 marks]

(e) Show true for $n = 1$ ***(M1)***

A1
$f'(x) = \mathrm{e}^{2x} + 2x\mathrm{e}^{2x}$

$= \mathrm{e}^{2x}(1 + 2x) = (2x + 2^0)\mathrm{e}^{2x}$
Assume true for

M1A1
$n = k, i.e.\ f^{(k)} x = (2^k x + k \times 2^{k-1})\mathrm{e}^{2x}, k \geqslant 1$
Consider

. ***M1***
$n = k + 1$, $i.e.$ an attempt to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(f^k(x)\right)$
A1
$F^{(k+1)}(x) = 2^k\mathrm{e}^{2x} + 2\mathrm{e}^{2x}(2^k x + k \times 2^{k-1})$

$= \left(2^k + 2(2^k x + k \times 2^{k-1})\right)\mathrm{e}^{2x}$

$= (2 \times 2^k x + 2^k + k \times 2 \times 2^{k-1})\mathrm{e}^{2x}$
A1
$= (2^{k+1}x + 2^k + k \times 2^k)\mathrm{e}^{2x}$
A1
$= \left(2^{k+1}x + (k+1)2^k\right)\mathrm{e}^{2x}$
$P(n)$ is true for

is true for $k + 1$, and since true for $n = 1$, result proved by mathematical induction
$k \Rightarrow P(n)$

$\forall n \in \mathbb{Z}^+$
Note: Only award ***R1*** if a reasonable attempt is made to prove the

step.
$(k+1)^{\mathrm{th}}$

[9 marks]

Total [27 marks]

Examiners report

This was the most accessible question in section B for these candidates. A majority of candidates produced partially correct answers to part (a), but a significant number struggled with demonstrating that the point is a minimum, despite the hint being given in the question. Part (b) started quite successfully but many students were unable to prove it is a point of inflexion or, more commonly, did not attempt to justify it. Correct answers were often seen for part (c). Part (d) was dependent on the successful completion of the first three parts. If candidates made errors in earlier parts, this often became obvious when they came to sketch the curve. However, few candidates realised that this part was a good way of checking that the above answers were at least consistent. The quality of curve sketching was rather weak overall, with candidates not marking points appropriately and not making features such as asymptotes clear. It is not possible to tell to what extent this was an effect of candidates not having a calculator, but it should be noted that asking students to sketch curves without a calculator will continue to appear on non-calculator papers. In part (e) the basic idea of proof by induction had clearly been taught with a significant majority of students understanding this. However, many candidates did not understand that they had to differentiate again to find the result for $(k + 1)$.

34.

[6 marks]

Markscheme

METHOD 1

$3x^2y^2 + 2x^3y\frac{\mathrm{d}y}{\mathrm{d}x} = -\pi\sin(\pi y)\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1A1***

At $(-1,\ 1)$, $3 - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}$ ***A1***

[6 marks]

METHOD 2

$3x^2y^2 + 2x^3y\frac{\mathrm{d}y}{\mathrm{d}x} = -\pi\sin(\pi y)\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2y^2}{-\pi\sin(\pi y) - 2x^3y}$ ***A1***

At $(-1,\ 1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(-1)^2(1)^2}{-\pi\sin(\pi) - 2(-1)^3(1)} = \frac{3}{2}$ ***M1A1***

[6 marks]

Examiners report

A large number of candidates obtained full marks on this question. Some candidates missed π and/or $\frac{\mathrm{d}y}{\mathrm{d}x}$ when differentiating the trigonometric function. Some candidates attempted to rearrange before differentiating, and some made algebraic errors in rearranging.

35.

[6 marks]

Markscheme

Let

$u = \ln y \Rightarrow \mathrm{d}u = \frac{1}{y}\mathrm{d}y$ ***A1(A1)***

$\int \frac{\tan(\ln y)}{y}\mathrm{d}y = \int \tan u\,\mathrm{d}u$ ***A1***

$= \int \frac{\sin u}{\cos u}\mathrm{d}u = -\ln|\cos u| + c$ ***A1***

EITHER

$\int \frac{\tan(\ln y)}{y}\mathrm{d}y = -\ln|\cos(\ln y)| + c$ ***A1A1***

OR

$\int \frac{\tan(\ln y)}{y}\mathrm{d}y = \ln|\sec(\ln y)| + c$ ***A1A1***

[6 marks]

Examiners report

Many candidates obtained the first three marks, but then attempted various methods unsuccessfully. Quite a few candidates attempted integration by parts rather than substitution. The candidates who successfully integrated the expression often failed to put the absolute value sign in the final answer.

36.

[13 marks]

Markscheme

(a) (i)

$f'_k(x) = 3k^2x^2 - 2kx + 1$ ***A1***

$f''_k(x) = 6k^2x - 2k$ ***A1***

(ii) Setting

$f''(x) = 0$ ***M1***

$\Rightarrow 6k^2x - 2k = 0 \Rightarrow x = \frac{1}{3k}$ ***A1***

$f\left(\frac{1}{3k}\right) = k^2\left(\frac{1}{3k}\right)^3 - k\left(\frac{1}{3k}\right)^2 + \left(\frac{1}{3k}\right)$ ***M1***

$= \frac{7}{27k}$ ***A1***

Hence,

P_k is $\left(\frac{1}{3k}, \frac{7}{27k}\right)$

[6 marks]

(b) Equation of the straight line is

$y = \frac{7}{9}x$ ***A1***

As this equation is independent of k, all P_k lie on this straight line ***R1***

[2 marks]

(c) Gradient of tangent at P_k:

$f'(P_k) = f'\left(\frac{1}{3k}\right) = 3k^2\left(\frac{1}{3k}\right)^2 - 2k\left(\frac{1}{3k}\right) + 1 = \frac{2}{3}$ ***M1A1***

As the gradient is independent of k, the tangents are parallel. ***R1***

$\frac{7}{27k} = \frac{2}{3} \times \frac{1}{3k} + c \Rightarrow c = \frac{1}{27k}$ ***(A1)***

The equation is

$y = \frac{2}{3}x + \frac{1}{27k}$ ***A1***

[5 marks]

Total [13 marks]

Examiners report

Many candidates scored the full 6 marks for part (a). The main mistake evidenced was to treat k as a variable, and hence use the product rule to differentiate. Of the many candidates who attempted parts (b) and (c), few scored the R1 marks in either part, but did manage to get the equations of the straight lines.

37.

[6 marks]

Markscheme

(a) Attempting implicit differentiation ***M1***

A1
$2x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

EITHER

Substituting

e.g.
$x = -1,\ y = k$

M1
$-2 + k - \frac{\mathrm{d}y}{\mathrm{d}x} + 2k\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

Attempting to make

$\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject ***M1***

OR

Attempting to make

$\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject *e.g.*

M1
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(2x+y)}{x+2y}$

Substituting

M1
$x = -1,\ y = k$ into $\frac{\mathrm{d}y}{\mathrm{d}x}$

THEN

A1 ***N1***
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-k}{2k-1}$

(b) Solving

A1
$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for k gives $k = 2$

[6 marks]

Examiners report

Part (a) was generally well answered, almost all candidates realising that implicit differentiation was involved. A few failed to differentiate the right hand side of the relationship. A surprising number of candidates made an error in part (b), even when they had scored full marks on the first part.

38. *[6 marks]*

Markscheme

Using integration by parts ***(M1)***

(A1)
$u = x, \frac{\mathrm{d}u}{\mathrm{d}x} = 1, \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x$ and $v = -\frac{1}{2}\cos 2x$
A1
$\left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \left(-\frac{1}{2}\cos 2x\right)\mathrm{d}x$
A1
$= \left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} + \left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}}$
Note: Award the ***A1A1*** above if the limits are not included.

A1
$\left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} = -\frac{\pi}{24}$
A1
$\left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{8}$
AG N0
$\int_0^{\frac{\pi}{6}} x\sin 2x\mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$
Note: Allow ***FT*** on the last two ***A1*** marks if the expressions are the negative of the correct ones.

[6 marks]

Examiners report

This question was reasonably well done, with few candidates making the inappropriate choice of u and

$\frac{\mathrm{d}v}{\mathrm{d}x}$. The main source of a loss of marks was in finding v by integration. A few candidates used the double angle formula for sine, with poor results.

39. *[6 marks]*

Markscheme

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x-1)) = \frac{1}{1+(x-1)^2}$ (or equivalent) ***A1***
(R1)
$m_N = -2$ and so $m_T = \frac{1}{2}$
Attempting to solve

$\frac{1}{1+(x-1)^2} = \frac{1}{2}$ (or equivalent) for x ***M1***
A1
$x = 2$ (as $x > 0$)
Substituting

$x = 2$ and

$y = \frac{\pi}{4}$ to find c ***M1***
A1 N1
$c = 4 + \frac{\pi}{4}$
[6 marks]

Examiners report

There was a disappointing response to this question from a fair number of candidates. The differentiation was generally correctly performed, but it was then often equated to

rather than the correct numerical value. A few candidates either didn't simplify arctan(1) to
$-2x+c$

, or stated it to be 45 or
$\frac{\pi}{4}$

.
$\frac{\pi}{2}$

40. *[18 marks]*

Markscheme

(a)

$\text{AQ} = \sqrt{x^2+4}$ (km) ***(A1)***

$\text{QY} = (2-x)$ (km) ***(A1)***

$T = 5\sqrt{5}\text{AQ} + 5\text{QY}$ ***(M1)***

$= 5\sqrt{5}\sqrt{(x^2+4)} + 5(2-x)$ (mins) ***A1***

[4 marks]

(b) Attempting to use the chain rule on $5\sqrt{5}\sqrt{(x^2+4)}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(5\sqrt{5}\sqrt{(x^2+4)}\right) = 5\sqrt{5} \times \frac{1}{2}(x^2+4)^{-\frac{1}{2}} \times 2x$ ***A1***

$\left(= \frac{5\sqrt{5}x}{\sqrt{x^2+4}}\right)$

$\frac{\mathrm{d}}{\mathrm{d}x}(5(2-x)) = -5$ ***A1***

$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$ ***AG N0***

[3 marks]

(c) (i)

$\sqrt{5}x = \sqrt{x^2+4}$ ***A1***

Squaring both sides and rearranging to obtain $5x^2 = x^2+4$ ***M1***

$x = 1$ ***A1 N1***

Note: Do not award the final ***A1*** for stating a negative solution in final answer.

(ii)

$T = 5\sqrt{5}\sqrt{1+4} + 5(2-1)$ ***M1***

$= 30$ (mins) ***A1 N1***

Note: Allow ***FT*** on incorrect *x* value.

(iii) **METHOD 1**

Attempting to use the quotient rule ***M1***

$u = x$, $v = \sqrt{x^2+4}$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = x(x^2+4)^{-1/2}$ ***(A1)***

$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 5\sqrt{5}\left[\frac{\sqrt{x^2+4} - \frac{1}{2}(x^2+4)^{-1/2} \times 2x^2}{(x^2+4)}\right]$ ***A1***

Attempt to simplify ***(M1)***

$= \frac{5\sqrt{5}}{(x^2+4)^{3/2}}[x^2+4-x^2]$ or equivalent ***A1***

$= \frac{20\sqrt{5}}{(x^2+4)^{3/2}}$ ***AG***

When $x = 1$, $\frac{20\sqrt{5}}{(1^2+4)^{3/2}} > 0$ and hence $T = 30$ is a minimum ***R1 N0***

Note: Allow ***FT*** on incorrect *x* value, $0 \leqslant x \leqslant 2$.

METHOD 2

Attempting to use the product rule ***M1***

$u = x$, $v = \sqrt{x^2+4}$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = x(x^2+4)^{-1/2}$ ***(A1)***

$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 5\sqrt{5}(x^2+4)^{-1/2} - \frac{5\sqrt{5}x}{2}(x^2+4)^{-3/2} \times 2x$ ***A1***

$\left(= \frac{5\sqrt{5}}{(x^2+4)^{1/2}} - \frac{5\sqrt{5}x^2}{(x^2+4)^{3/2}}\right)$

Attempt to simplify ***(M1)***

$= \frac{5\sqrt{5}(x^2+4) - 5\sqrt{5}x^2}{(x^2+4)^{3/2}}$ $\left(= \frac{5\sqrt{5}(x^2+4-x^2)}{(x^2+4)^{3/2}}\right)$ ***A1***

$= \frac{20\sqrt{5}}{(x^2+4)^{3/2}}$ ***AG***

When $x = 1$, $\frac{20\sqrt{5}}{(1^2+4)^{3/2}} > 0$ and hence $T = 30$ is a minimum ***R1 N0***

Note: Allow ***FT*** on incorrect *x* value,

.

$0 \leqslant x \leqslant 2$

[11 marks]

Total [18 marks]

Examiners report

Most candidates scored well on this question. The question tested their competence at algebraic manipulation and differentiation. A few candidates failed to extract from the context the correct relationship between velocity, distance and time.

41. *[5 marks]*

Markscheme

(a) Either solving

for x, stating
$\mathrm{e}^{-x} - x + 1 = 0$
, stating P(x, 0) or using an appropriate sketch graph. ***M1***
$\mathrm{e}^{-x} - x + 1 = 0$
$x = 1.28$ ***A1*** ***N1***

Note: Accept P(1.28, 0) .

(b) Area

M1A1
$= \int_0^{1.278...} (\mathrm{e}^{-x} - x + 1)\mathrm{d}x$
$= 1.18$ ***A1*** ***N1***

Note: Award ***M1A0A1*** if the dx is absent.

[5 marks]

Examiners report

This was generally well done. In part (a), most candidates were able to find $x = 1.28$ successfully. A significant number of candidates were awarded an accuracy penalty for expressing answers to an incorrect number of significant figures.

Part (b) was generally well done. A number of candidates unfortunately omitted the dx in the integral while some candidates omitted to write down the definite integral and instead offered detailed instructions on how they obtained the answer using their GDC.

42. *[7 marks]*

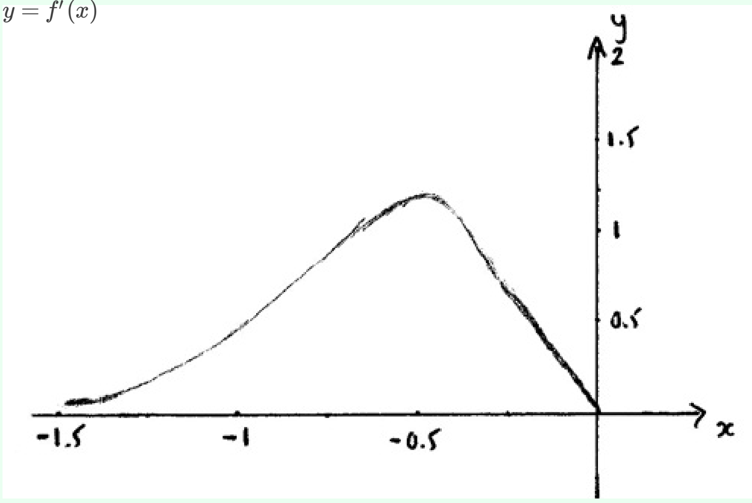
Markscheme

METHOD 1

EITHER

Using the graph of

(M1)
$y = f'(x)$



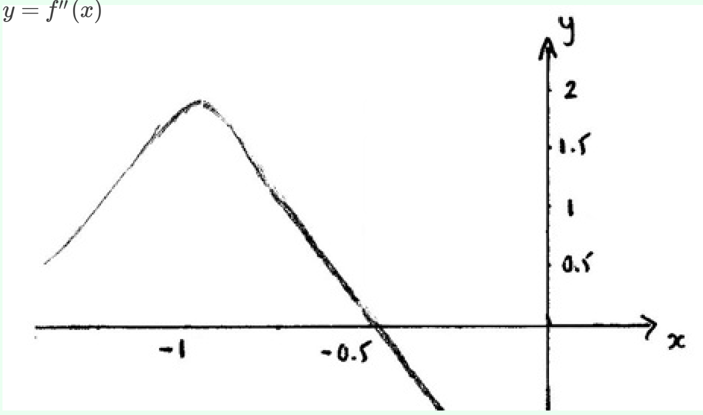
A1

The maximum of

occurs at $x = -0.5$. *A1*
$f'(x)$
OR

Using the graph of

. *(M1)*
$y = f''(x)$



A1

The zero of

occurs at $x = -0.5$. *A1*
$f''(x)$
THEN

Note: Do not award this *A1* for stating $x = \pm 0.5$ as the final answer for x .

A2
$f(-0.5) = 0.607 (= \mathrm{e}^{-0.5})$
Note: Do not award this *A1* for also stating (0.5, 0.607) as a coordinate.

EITHER

Correctly labelled graph of

for
$f'(x)$

$f'(x)$
denoting the maximum
$x < 0$
R1
$f'(x)$
(*e.g.*

and
$f'(-0.6) = 1.17$
stated) ***A1*** ***N2***
$f'(-0.4) = 1.16$
OR

Correctly labelled graph of

for
$f''(x)$
denoting the maximum
$x < 0$
R1
$f'(x)$
(*e.g.*

and
$f''(-0.6) = 0.857$
stated) ***A1*** ***N2***
$f''(-0.4) = -1.05$
OR

.
$f'(0.5) \approx 1.21$
just to the left of
$f'(x) < 1.21$

$x = -\frac{1}{2}$
and

just to the right of
$f'(x) < 1.21$
R1
$x = -\frac{1}{2}$
(*e.g.*

and
$f'(-0.6) = 1.17$
stated) ***A1*** ***N2***
$f'(-0.4) = 1.16$
OR

just to the left of
$f''(x) > 0$
and
$x = -\frac{1}{2}$
just to the right of
$f''(x) < 0$
R1
$x = -\frac{1}{2}$
(*e.g.*

and
$f''(-0.6) = 0.857$
stated) ***A1*** ***N2***
$f''(-0.4) = -1.05$
[7 marks]

METHOD 2

A1
$f'(x) = -4x\mathrm{e}^{-2x^2}$
A1
$f''(x) = -4\mathrm{e}^{-2x^2} + 16x^2\mathrm{e}^{-2x^2} \quad \left(= (16x^2 - 4)\mathrm{e}^{-2x^2}\right)$
Attempting to solve

(M1)
$f''(x) = 0$
A1
$x = -\frac{1}{2}$
Note: Do not award this ***A1*** for stating

as the final answer for *x* .
$x = \pm\frac{1}{2}$

A1
$f\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{\mathrm{e}}} \; (= 0.607)$
Note: Do not award this ***A1*** for also stating

as a coordinate.
$\left(\frac{1}{2}, \frac{1}{\sqrt{e}}\right)$

EITHER

Correctly labelled graph of

for
$f'(x)$
denoting the maximum
$x < 0$
R1
$f'(x)$
(*e.g.*

and
$f'(-0.6) = 1.17$
stated) ***A1*** ***N2***
$f'(-0.4) = 1.16$
OR

Correctly labelled graph of

for
$f''(x)$
denoting the maximum
$x < 0$
R1
$f'(x)$
(*e.g.*

and
$f''(-0.6) = 0.857$
stated) ***A1*** ***N2***
$f''(-0.4) = -1.05$
OR

.
$f'(0.5) \approx 1.21$
just to the left of
$f'(x) < 1.21$

$x = -\frac{1}{2}$
and

just to the right of
$f'(x) < 1.21$
R1
$x = -\frac{1}{2}$
(*e.g.*

and
$f'(-0.6) = 1.17$
stated) ***A1*** ***N2***
$f'(-0.4) = 1.16$
OR

just to the left of
$f''(x) > 0$
and
$x = -\frac{1}{2}$
just to the right of
$f''(x) < 0$
R1
$x = -\frac{1}{2}$
(*e.g.*

and
$f''(-0.6) = 0.857$
stated) ***A1*** ***N2***
$f''(-0.4) = -1.05$
[7 marks]

Examiners report

Most candidates adopted an algebraic approach rather than a graphical approach. Most candidates found $f'(x)$ correctly, however when attempting to find $f''(x)$, a surprisingly large number either made algebraic errors using the product rule or seemingly used an incorrect form of the product rule. A large number ignored the domain restriction and either expressed $x = \pm\frac{1}{2}$ as the x-coordinates of the point of inflection or identified $x = \frac{1}{2}$ rather than $x = -\frac{1}{2}$. Most candidates were unsuccessful in their attempts to justify the existence of the point of inflection.

43. *[19 marks]*

Markscheme

(a) (i) **EITHER**

Attempting to separate the variables ***(M1)***

$\frac{\mathrm{d}v}{-v(1+v^2)} = \frac{\mathrm{d}t}{50}$ ***(A1)***

OR

Inverting to obtain $\frac{\mathrm{d}t}{\mathrm{d}v}$ ***(M1)***

$\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{-50}{v(1+v^2)}$ ***(A1)***

THEN

$t = -50\int_{10}^{5} \frac{1}{v(1+v^2)}\mathrm{d}v \quad \left(= 50\int_{5}^{10} \frac{1}{v(1+v^2)}\mathrm{d}v\right)$ ***A1 N3***

(ii)

$t = 0.732\ (\text{sec}) \quad \left(= 25\ln\frac{104}{101}(\text{sec})\right)$ ***A2 N2***

[5 marks]

(b) (i)

$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

Must see division by v

$(v > 0)$ ***A1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-(1+v^2)}{50}$ ***AG N0***

(ii) Either attempting to separate variables or inverting to obtain $\frac{\mathrm{d}x}{\mathrm{d}v}$ ***(M1)***

$\int\frac{\mathrm{d}v}{1+v^2} = -\frac{1}{50}\int\mathrm{d}x$ (or equivalent) ***A1***

Attempting to integrate both sides ***M1***

$\arctan v = -\frac{x}{50} + C$ ***A1A1***

Note: Award ***A1*** for a correct LHS and ***A1*** for a correct RHS that must include C.

When $x = 0$, $v = 10$ and so $C = \arctan 10$ ***M1***

$x = 50(\arctan 10 - \arctan v)$ ***A1 N1***

(iii) Attempting to make $\arctan v$ the subject. ***M1***

$\arctan v = \arctan 10 - \frac{x}{50}$ ***A1***

$v = \tan\left(\arctan 10 - \frac{x}{50}\right)$ ***M1A1***

Using $\tan(A-B)$ formula to obtain the desired form. ***M1***

$v = \frac{10 - \tan\frac{x}{50}}{1 + 10\tan\frac{x}{50}}$ ***AG N0***

[14 marks]

Total [19 marks]

Examiners report

No comment.

44.

[5 marks]

Markscheme

Recognition of integration by parts ***M1***

A1A1

$\int x^2 \ln x \mathrm{d}x = \left[\frac{x^3}{3}\ln x\right] - \int \frac{x^3}{3} \times \frac{1}{x}\mathrm{d}x$

$= \left[\frac{x^3}{3}\ln x\right] - \int \frac{x^2}{3}\mathrm{d}x$

A1

$= \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]$

A1

$\Rightarrow \int_1^{\mathrm{e}} x^2 \ln x \mathrm{d}x = \left(\frac{\mathrm{e}^3}{3} - \frac{\mathrm{e}^3}{9}\right) - \left(0 - \frac{1}{9}\right) \quad \left(= \frac{2\mathrm{e}^3+1}{9}\right)$

[5 marks]

Examiners report

Most candidates recognised that a method of integration by parts was appropriate for this question. However, although a good number of correct answers were seen, a number of candidates made algebraic errors in the process. A number of students were also unable to correctly substitute the limits.

45.

[7 marks]

Markscheme

A1A1A1

$5y^2 + 10xy\frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0$

Note: Award ***A1A1*** for correct differentiation of $5xy^2$.

A1 for correct differentiation of $-2x^2$ and 18.

At the point (1, 2),

$20 + 20\frac{\mathrm{d}y}{\mathrm{d}x} - 4 = 0$

(A1)

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{5}$

Gradient of normal $= \frac{5}{4}$ ***A1***

Equation of normal

M1

$y - 2 = \frac{5}{4}(x - 1)$

$y = \frac{5}{4}x - \frac{5}{4} + \frac{8}{4}$

A1

$y = \frac{5}{4}x + \frac{3}{4} \quad (4y = 5x + 3)$

[7 marks]

Examiners report

It was pleasing to see that a significant number of candidates understood that implicit differentiation was required and that they were able to make a reasonable attempt at this. A small number of candidates tried to make the equation explicit. This method will work, but most candidates who attempted this made either arithmetic or algebraic errors, which stopped them from gaining the correct answer.

46. *[8 marks]*

Markscheme

(a) Area of hexagon

$= 6 \times \frac{1}{2} \times x \times x \times \sin 60°$ ***M1***

$= \frac{3\sqrt{3}x^2}{2}$ ***AG***

(b) Let the height of the box be h

Volume

$= \frac{3\sqrt{3}hx^2}{2} = 90$ ***M1***

Hence

$h = \frac{60}{\sqrt{3}x^2}$ ***A1***

Surface area,

$A = 3\sqrt{3}x^2 + 6hx$ ***M1***

$= 3\sqrt{3}x^2 + \frac{360}{\sqrt{3}}x^{-1}$ ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x} = 6\sqrt{3}x - \frac{360}{\sqrt{3}}x^{-2}$ ***A1***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} = 0\right)$ ***M1***

$6\sqrt{3}x^3 = \frac{360}{\sqrt{3}}$

$x^3 = 20$

$x = \sqrt[3]{20}$ ***AG***

$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 6\sqrt{3} + \frac{720x^{-3}}{\sqrt{3}}$

which is positive when

$x = \sqrt[3]{20}$

, and hence gives a minimum value. ***R1***

[8 marks]

Examiners report

There were a number of wholly correct answers seen and the best candidates tackled the question well. However, many candidates did not seem to understand what was expected in such a problem. It was disappointing that a significant number of candidates were unable to find the area of the hexagon.

47. *[7 marks]*

Markscheme

$y = \ln\left(\frac{1}{3}(1 + \mathrm{e}^{-2x})\right)$

EITHER

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{2}{3}\mathrm{e}^{-2x}}{\frac{1}{3}(1+\mathrm{e}^{-2x})}$ ***M1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\mathrm{e}^{-2x}}{1+\mathrm{e}^{-2x}}$ ***A1***

$\mathrm{e}^y = \frac{1}{3}(1 + \mathrm{e}^{-2x})$ ***M1***

Now

$\mathrm{e}^{-2x} = 3\mathrm{e}^y - 1$ ***A1***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2(3\mathrm{e}^y-1)}{1+3\mathrm{e}^y-1}$ ***A1***

$= -\frac{2}{3\mathrm{e}^y}(3\mathrm{e}^y - 1)$ ***A1***

$= -\frac{2}{3}(3 - \mathrm{e}^{-y})$ ***AG***

$= \frac{2}{3}(\mathrm{e}^{-y} - 3)$

OR

$\mathrm{e}^y = \frac{1}{3}(1 + \mathrm{e}^{-2x})$ ***M1A1***

$\mathrm{e}^y \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}\mathrm{e}^{-2x}$ ***M1A1***

Now

$\mathrm{e}^{-2x} = 3\mathrm{e}^y - 1$ ***(A1)***

$\Rightarrow \mathrm{e}^y \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}(3\mathrm{e}^y - 1)$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}\mathrm{e}^{-y}(3\mathrm{e}^y - 1)$ ***(A1)***

$= \frac{2}{3}(-3 + \mathrm{e}^{-y})$ ***(A1)***

$= \frac{2}{3}(\mathrm{e}^{-y} - 3)$ **AG**

Note: Only two of the three ***(A1)*** marks may be implied.

[7 marks]

Examiners report

Solutions were generally disappointing with many candidates being awarded the first 2 or 3 marks, but then going no further.

48.

[21 marks]

Markscheme

(a) For $x\sqrt{9-x^2}$, $-3 \leqslant x \leqslant 3$ and for $2\arcsin\left(\frac{x}{3}\right)$, $-3 \leqslant x \leqslant 3$

A1

A1

$\Rightarrow D \text{ is } -3 \leqslant x \leqslant 3$

[2 marks]

(b)

$V = \pi\int_0^{2.8}\left(x\sqrt{9-x^2} = 2\arcsin\frac{x}{3}\right)^2 \mathrm{d}x$ ***M1A1***

= 181 ***A1***

[3 marks]

(c)

$\frac{\mathrm{d}y}{\mathrm{d}x} = (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{\frac{2}{3}}{\sqrt{1-\frac{x^2}{9}}}$ ***M1A1***

$= (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{2}{(9-x^2)^{\frac{1}{2}}}$ ***A1***

$= \frac{9-x^2-x^2+2}{(9-x^2)^{\frac{1}{2}}}$ ***A1***

$= \frac{11-2x^2}{\sqrt{9-x^2}}$ ***A1***

[5 marks]

(d)

$\int_{-p}^{p}\frac{11-2x^2}{\sqrt{9-x^2}}\mathrm{d}x = \left[x\sqrt{9-x^2} + 2\arcsin\frac{x}{3}\right]_{-p}^{p}$ ***M1***

$= p\sqrt{9-p^2} + 2\arcsin\frac{p}{3} + p\sqrt{9-p^2} + 2\arcsin\frac{p}{3}$ ***A1***

$= 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$ ***AG***

[2 marks]

(e)

$11 - 2p^2 = 0$ ***M1***

$p = 2.35\left(\sqrt{\frac{11}{2}}\right)$ ***A1***

Note: Award ***A0*** for $p = \pm 2.35$.

[2 marks]

(f) (i)

$f''(x) = \frac{(9-x^2)^{\frac{1}{2}}(-4x) + x(11-2x^2)(9-x^2)^{-\frac{1}{2}}}{9-x^2}$ ***M1A1***

$= \frac{-4x(9-x^2)+x(11-2x^2)}{(9-x^2)^{\frac{3}{2}}}$ ***A1***

$= \frac{-36x+4x^3+11x-2x^3}{(9-x^2)^{\frac{3}{2}}}$ ***A1***

$= \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$ ***AG***

(ii) **EITHER**

When $0 < x < 3$, $f''(x) < 0$. When $-3 < x < 0$, $f''(x) > 0$. ***A1***

OR

$f''(0) = 0$ ***A1***

$f''(0) = 0$

THEN

Hence

$f''(x)$ changes sign through $x = 0$, giving a point of inflexion. ***R1***

EITHER

$x = \pm\sqrt{\frac{25}{2}}$ is outside the domain of *f*. ***R1***

OR

$x = \pm\sqrt{\frac{25}{2}}$ is not a root of $f''(x) = 0$. ***R1***

[7 marks]

Total [21 marks]

Examiners report

It was disappointing to note that some candidates did not know the domain for arcsin. Most candidates knew what to do in (b) but sometimes the wrong answer was obtained due to the calculator being in the wrong mode. In (c), the differentiation was often disappointing with $\arcsin\left(\frac{x}{5}\right)$ causing problems. In (f)(i), some candidates who failed to do (c) guessed the correct form of $f'(x)$ (presumably from (d)) and then went on to find $f''(x)$ correctly. In (f)(ii), the justification of a point of inflexion at $x = 0$ was sometimes incorrect – for example, some candidates showed simply that $f'(x)$ is positive on either side of the origin which is not a valid reason.

49a. *[3 marks]*

Markscheme

$f(-x) = 2\cos(-x) + (-x)\sin(-x)$ ***M1***

$= 2\cos x + x\sin x \quad (= f(x))$ ***A1***

therefore *f* is even ***A1***

[3 marks]

Examiners report

[N/A]

49b. *[2 marks]*

Markscheme

$f'(x) = -2\sin x + \sin x + x\cos x \quad (= -\sin x + x\cos x)$ ***A1***

$f''(x) = -\cos x + \cos x - x\sin x \quad (= -x\sin x)$ ***A1***

so

$f''(0) = 0$ ***AG***

[2 marks]

Examiners report

[N/A]

49c. *[2 marks]*

Markscheme

John's statement is incorrect because

either; there is a stationary point at (0, 2) and since *f* is an even function and therefore symmetrical about the *y*-axis it must be a maximum or a minimum

or;

$f''(x)$ is even and therefore has the same sign either side of (0, 2) ***R2***

[2 marks]

Examiners report

[N/A]

50. *[5 marks]*

Markscheme

(a)

$\int \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \int \frac{(1-\cos\theta)'}{1-\cos\theta}\mathrm{d}\theta = \ln(1-\cos\theta)+C$ ***(M1)A1A1***

Note: Award ***A1*** for $\ln(1-\cos\theta)$ and ***A1*** for *C*.

(b)

$\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \frac{1}{2} \Rightarrow [\ln(1-\cos\theta)]_{\frac{\pi}{2}}^{a} = \frac{1}{2}$ ***M1***

$1-\cos a = \mathrm{e}^{\frac{1}{2}} \Rightarrow a = \arccos(1-\sqrt{\mathrm{e}})$ ***A1***

$a = 2.28$ ***N2***

[5 marks]

Examiners report

Generally well answered, although many students did not include the constant of integration.

51. *[7 marks]*

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= \sec^2\left(\arcsin\frac{y}{r}\right) \times \left(\arcsin\frac{y}{r}\right)' \times \frac{r}{1000}$$

$$= \frac{1}{\cos^2\left(\arcsin\frac{y}{r}\right)} \times \frac{\frac{1}{r}}{\sqrt{1-\left(\frac{y}{r}\right)^2}} \times \frac{r}{1000}$$

$$= \frac{\frac{1}{\sqrt{r^2-y^2}}}{\frac{r^2-y^2}{r^2}}\frac{r}{1000}$$

$$= \frac{r^3}{10^3\sqrt{(r^2-y^2)^3}}$$

$$= \left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$$

$$\frac{\mathrm{d}m}{\mathrm{d}t}$$

Examiners report

Few students were able to complete this question successfully, although many did obtain partial marks. Many students failed to recognise the difference between differentiating with respect to t or with respect to y. Very few were able to give a satisfactory geometrical meaning in part (b).

52. [19 marks]

Markscheme

(a) **METHOD 1**

using GDC

$a = 1$, $b = 5$, $c = 3$ ***A1A2A1***

METHOD 2

$x = x + 2\cos x \Rightarrow \cos x = 0$

$\Rightarrow x = \frac{\pi}{2}$, $\frac{3\pi}{2}$ ***M1***

$a = 1$, $c = 3$ ***A1***

$1 - 2\sin x = 0$ ***M1***

or $\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$, $\frac{5\pi}{6}$

$b = 5$ ***A1***

Note: Final ***M1A1*** is independent of previous work.

[4 marks]

(b)

$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3}$ (or 0.886) ***(M1)***

$f(2\pi) = 2\pi + 2$ (or 8.28) ***(M1)***

the range is $\left[\frac{5\pi}{6} - \sqrt{3},\ 2\pi + 2\right]$ (or [0.886, 8.28]) ***A1***

[3 marks]

(c)

$f'(x) = 1 - 2\sin x$ ***(M1)***

$f'\left(\frac{3\pi}{6}\right) = 3$ ***A1***

gradient of normal $= -\frac{1}{3}$ ***(M1)***

equation of the normal is

$y - \frac{3\pi}{2} = -\frac{1}{3}\left(x - \frac{3\pi}{2}\right)$ ***(M1)***

$y = -\frac{1}{3}x + 2\pi$ (or equivalent decimal values) ***A1*** ***N4***

[5 marks]

(d) (i)

$V = \pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(x^2 - (x + 2\cos x)^2\right)\mathrm{d}x$ (or equivalent) ***A1A1***

Note: Award ***A1*** for limits and ***A1*** for π and integrand.

(ii)

$V = \pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(x^2 - (x + 2\cos x)^2\right)\mathrm{d}x$

$= -\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(4x\cos x + 4\cos^2 x\right)\mathrm{d}x$

using integration by parts ***M1***

and the identity

, *M1*
$4\cos^2 x = 2\cos 2x + 2$
A1A1
$V = -\pi[(4x\sin x + 4\cos x) + (\sin 2x + 2x)]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$
Note: Award *A1* for $4x\sin x + 4\cos x$ and *A1* for $\sin 2x + 2x$.

A1
$= -\pi\left[\left(6\pi\sin\frac{3\pi}{2} + 4\cos\frac{3\pi}{2} + \sin 3\pi + 3\pi\right) - \left(2\pi sin\frac{\pi}{2} + 4\cos\frac{\pi}{2} + \sin\pi + \pi\right)\right]$
$= -\pi(-6\pi + 3\pi - \pi)$
AG *N0*
$= 6\pi^2$
Note: Do not accept numerical answers.

[7 marks]

Total [19 marks]

Examiners report

Generally there were many good attempts to this, more difficult, question. A number of students found b to be equal to 1, rather than 5. In the final part few students could successfully work through the entire integral successfully.

53.

[6 marks]

Markscheme

(a)

$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}}$ $\left(= \frac{1-2x}{\sqrt{1-x^2}}\right)$ *M1A1A1*

Note: Award *A1* for first term,
M1A1 for second term (*M1* for attempting chain rule).

(b)

$f'(x) = 0$ *(M1)*
$x = 0.5$, $y = 2.26$ (accept $\frac{\pi}{6} + \sqrt{3}$) or $(0.500, 2.26)$ *A1A1* *N3*

[6 marks]

Examiners report

Most candidates scored well on this question, showing competence at non-trivial differentiation. The follow through rules allowed candidates to recover from minor errors in part (a). Some candidates demonstrated their resourcefulness in using their GDC to answer part (b) even when they had been unable to gain full marks on part (a).

54. *[8 marks]*

Markscheme

(a)

$\frac{\mathrm{d}y}{\mathrm{d}x} = 24x^2 + 2bx + c$ ***(A1)***

$24x^2 + 2bx + c = 0$ ***(M1)***

$\Delta = (2b)^2 - 96(c)$ ***(A1)***

$4b^2 - 96c > 0$ ***A1***

$b^2 > 24c$ ***AG***

(b)

$1 + \frac{1}{4}b + \frac{1}{2}c + d = -12$

$6 + b + c = 0$

$-27 + \frac{9}{4}b - \frac{3}{2}c + d = 20$

$54 - 3b + c = 0$ ***A1A1A1***

Note: Award ***A1*** for each correct equation, up to 3, not necessarily simplified.

$b = 12$, $c = -18$, $d = -7$ ***A1***

[8 marks]

Examiners report

Many candidates throughout almost the whole mark range were able to score well on this question. It was pleasing that most candidates were aware of the discriminant condition for distinct real roots of a quadratic. Some who dropped marks on part (b) either didn't write down a sufficient number of linear equations to determine the three unknowns or made arithmetic errors in their manual solution – few GDC solutions were seen.

55. *[8 marks]*

Markscheme

$\int \sqrt{4-x^2}\,\mathrm{d}x$

$x = 2\sin\theta$

A1

$\mathrm{d}x = 2\cos\theta\mathrm{d}\theta$

M1A1

$= \int \sqrt{4-4\sin^2\theta} \times 2\cos\theta\mathrm{d}\theta$

$= \int 2\cos\theta \times 2\cos\theta\mathrm{d}\theta$

$= 4\int \cos^2\theta\mathrm{d}\theta$

now

$\int \cos^2\theta\mathrm{d}\theta$

M1A1

$= \int \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right)\mathrm{d}\theta$

A1

$= \left(\frac{\sin 2\theta}{4} + \frac{1}{2}\theta\right)$

so original integral

$= 2\sin 2\theta + 2\theta$

$= 2\sin\theta\cos\theta + 2\theta$

$= \left(2 \times \frac{x}{2} \times \frac{\sqrt{4-x^2}}{2}\right) + 2\arcsin\left(\frac{x}{2}\right)$

A1A1

$= \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\left(\frac{x}{2}\right) + C$

Note: Do not penalise omission of C.

$\left(A = \frac{1}{2},\ B = 2\right)$

[8 marks]

Examiners report

For many candidates this was an all or nothing question. Examiners were surprised at the number of candidates who were unable to change the variable in the integral using the given substitution. Another stumbling block, for some candidates, was a lack of care with the application of the trigonometric version of Pythagoras' Theorem to reduce the integrand to a multiple of $\cos^2\theta$. However, candidates who obtained the latter were generally successful in completing the question.

56.

[7 marks]

Markscheme

(a) let

$\mathrm{H\hat{P}Q} = \theta$

$\tan\theta = \frac{h}{40}$

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{40}\frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1(A1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{4\sec^2\theta}$

$= \frac{16}{4\times25}$ ***A1***

$\left(\sec\theta = \frac{5}{4} \text{ or } \theta = 0.6435\right)$

$= 0.16$ radians per second ***AG***

(b)

$x^2 = h^2 + 1600$, where PH $= x$

$2x\frac{\mathrm{d}x}{\mathrm{d}t} = 2h\frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1A1***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{h}{x} \times 10$

$= \frac{10h}{\sqrt{h^2+1600}}$ ***(A1)***

$h = 80$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 6$ ms^{-1} ***A1***

Note: Accept solutions that begin $x = 40\sec\theta$ or use $\dot{h} = 10t$

[7 marks]

Examiners report

For those candidates who realized this was an applied calculus problem involving related rates of change, the main source of error was in differentiating inverse tan in part (a). Some found part (b) easier than part (a), involving a changing length rather than an angle. A number of alternative approaches were reported by examiners.

57. [18 marks]

Markscheme

(a) PQ $= 50$ and non-intersecting **R1**

[1 mark]

(b) a construction QT (where T is on the radius MP), parallel to MN, so that $\text{QTM} = 90°$ (angle between tangent and radius $= 90°$) **M1**

lengths $x - 10$, 50 and angle θ marked on a diagram, or equivalent **R1**

Note: Other construction lines are possible.

[2 marks]

(c) (i) MN $= \sqrt{50^2 - (x-10)^2}$ **A1**

(ii) maximum for MN occurs when $x = 10$ **A1**

[2 marks]

(d) (i) $\alpha = 2\pi - 2\theta$ **M1**

$= 2\pi - 2\arccos\left(\frac{x-10}{50}\right)$ **A1**

(ii) $\beta = 2\pi - \alpha$ ($= 2\theta$) **A1**

$= 2\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)$ **A1**

[4 marks]

(e) (i) $b(x) = x\alpha + 10\beta + 2\sqrt{50^2 - (x-10)^2}$ **A1A1A1**

$= x\left(2\pi - 2\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)\right) + 20\left(\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)\right) + 2\sqrt{50^2 - (x-10)^2}$ **M1A1**

(ii) maximum value of perimeter $= 276$ **A2**

(iii) perimeter of 200 cm $\Rightarrow b(x) = 200$ **(M1)**

when $x = 21.2$ **A1**

[9 marks]

Total [18 marks]

Examiners report

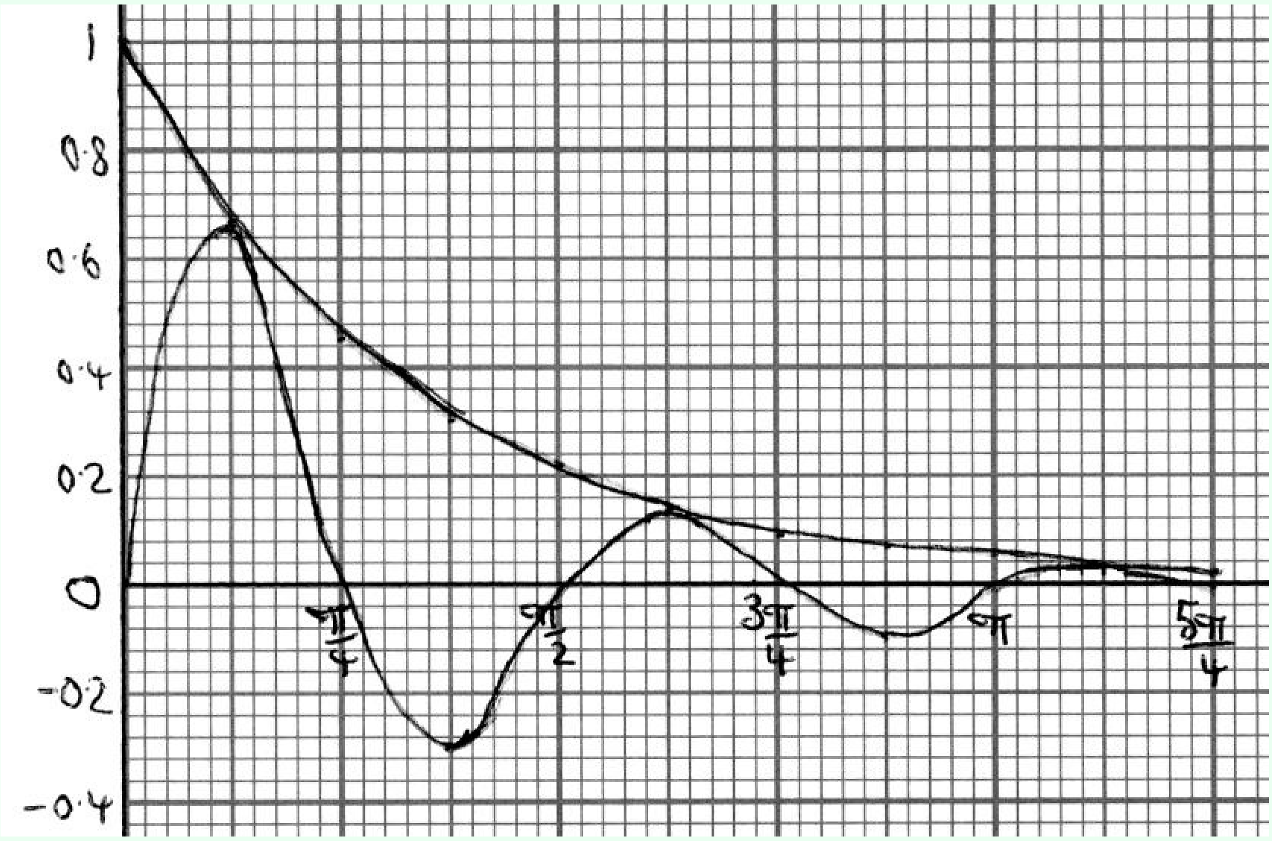
This is not an inherently difficult question, but candidates either made heavy weather of it or avoided it almost entirely. The key to answering the question is in obtaining the displayed answer to part (b), for which a construction line parallel to MN through Q is required. Diagrams seen by examiners on some scripts tend to suggest that the perpendicularity property of a tangent to a circle and the associated radius is not as firmly known as they had expected. Some candidates mixed radians and degrees in their expressions.

58.

[22 marks]

Markscheme

(a)



A3

Note: Award ***A1*** for each correct shape,

A1 for correct relative position.

[3 marks]

(b)

$\mathrm{e}^{-x}\sin(4x) = 0$ ***(M1)***

$\sin(4x) = 0$ ***A1***

$4x = 0$, π, 2π, 3π, 4π, 5π ***A1***

$x = 0$, $\frac{\pi}{4}$, $\frac{2\pi}{4}$, $\frac{3\pi}{4}$, $\frac{4\pi}{4}$, $\frac{5\pi}{4}$ ***AG***

[3 marks]

(c)

$\mathrm{e}^{-x} = \mathrm{e}^{-x}\sin(4x) = 0$ or reference to graph

$\sin 4x = 1$, ***M1***

$\sin 4x = 1$, $\frac{\pi}{2}$, $\frac{5\pi}{2}$, $\frac{9\pi}{2}$, ***A1***

$x = \frac{\pi}{8}$, $\frac{5\pi}{8}$, $\frac{9\pi}{8}$ ***A1*** ***N3***

[3 marks]

(d) (i)

$y = \mathrm{e}^{-x}\sin 4x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}\sin 4x + 4\mathrm{e}^{-x}\cos 4x$ ***M1A1***

$y = \mathrm{e}^{-x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}$ ***A1***

verifying equality of gradients at one point ***R1***

verifying at the other two ***R1***

(ii) since $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ at these points they cannot be local maxima ***R1***

[6 marks]

(e) (i) maximum when $y' = 4\mathrm{e}^{-x}\cos 4x - \mathrm{e}^{-x}\sin 4x = 0$ ***M1***

$x = \frac{\arctan(4)}{4}$, $\frac{\arctan(4)+\pi}{4}$, $\frac{\arctan(4)+2\pi}{4}$, ...

maxima occur at

$x = \frac{\arctan(4)}{4}$, $\frac{\arctan(4)+2\pi}{4}$, $\frac{\arctan(4)+4\pi}{4}$ ***A1***

so

$y_1 = \mathrm{e}^{-\frac{1}{4}(\arctan(4))}\sin(\arctan(4))$ $(= 0.696)$ ***A1***

$y_2 = \mathrm{e}^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4) + 2\pi)$ ***A1***

$\left(= \mathrm{e}^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4)) = 0.145\right)$

$y_3 = \mathrm{e}^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4) + 4\pi)$ ***A1*** ***N3***

$\left(= \mathrm{e}^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4)) = 0.0301\right)$

(ii) for finding and comparing $\frac{y_3}{y_2}$ and $\frac{y_2}{y_1}$ ***M1***

$r = \mathrm{e}^{-\frac{\pi}{2}}$ ***A1***

Note: Exact values must be used to gain the ***M1*** and the ***A1***.

[7 marks]

Total [22 marks]

Examiners report

Although the final question on the paper it had parts accessible even to the weakest candidates. The vast majority of candidates earned marks on part (a), although some graphs were rather scruffy. Many candidates also tackled parts (b), (c) and (d). In part (b), however, as the answer was given, it should have been clear that some working was required rather than reference to a graph, which often had no scale indicated. In part d(i), although the functions were usually differentiated correctly, it was often the case that only one point was checked for the equality of the gradients. In part e(i) many candidates who got this far were able to determine the -coordinates of the local maxima numerically using a GDC, and that was given credit. Only the exact values, however, could be used in part e(ii).

59a. *[3 marks]*

Markscheme

(i)

A1
$f'(x) = \mathrm{e}^x - \mathrm{e}x^{\mathrm{e}-1}$

(ii) by inspection the two roots are 1, e ***A1A1***

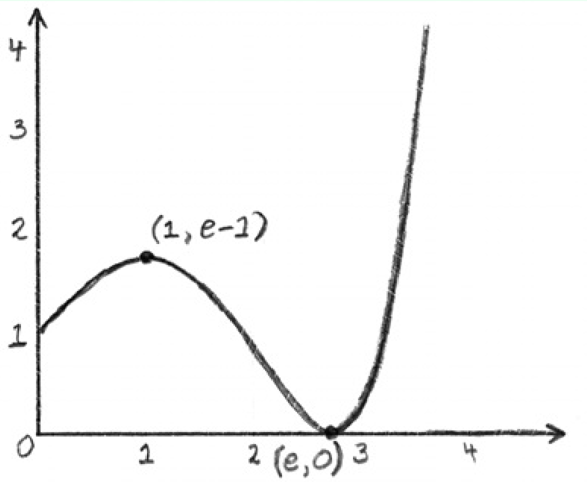
[3 marks]

Examiners report

[N/A]

59b. *[3 marks]*

Markscheme



A3

Note: Award ***A1*** for maximum, ***A1*** for minimum and ***A1*** for general shape.

[3 marks]

Examiners report

[N/A]

59c.

[1 mark]

Markscheme

from the graph:

$\mathrm{e}^x > x^\mathrm{e}$ for all $x > 0$ except $x = \mathrm{e}$ ***R1***

putting $x = \pi$, conclude that

$\mathrm{e}^\pi > \pi^\mathrm{e}$ ***AG***

[1 mark]

Examiners report

[N/A]

60a.

[7 marks]

Markscheme

let

$x = 2\sin\theta$ ***M1***

$\mathrm{d}x = 2\cos\theta\mathrm{d}\theta$ ***A1***

$I = \int_0^{\frac{\pi}{4}} 2\cos\theta \times 2\cos\theta\mathrm{d}\theta \left(= 4\int_0^{\frac{\pi}{4}} \cos^2\theta\mathrm{d}\theta\right)$ ***A1A1***

Note: Award ***A1*** for limits and ***A1*** for expression.

$= 2\int_0^{\frac{\pi}{4}} (1 + \cos 2\theta)\mathrm{d}\theta$ ***A1***

$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$ ***A1***

$= 1 + \frac{\pi}{2}$ ***A1***

[7 marks]

Examiners report

[N/A]

60b.

[5 marks]

Markscheme

$I = [x\arcsin x]_0^{0.5} - \int_0^{0.5} x \times \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$ ***M1A1A1***

$= [x\arcsin x]_0^{0.5} + \left[\sqrt{1-x^2}\right]_0^{0.5}$ ***A1***

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ ***A1***

[5 marks]

Examiners report

[N/A]

60c. *[7 marks]*

Markscheme

A1(A1)

$\mathrm{d}t = \sec^2\theta \mathrm{d}\theta\ ,\ \left[0, \frac{\pi}{4}\right] \to [0,\ 1]$

M1(A1)

$I = \int_0^1 \frac{\frac{\mathrm{d}t}{(1+t^2)}}{\frac{3}{(1+t^2)} + \frac{t^2}{(1+t^2)}}$

A1

$= \int_0^1 \frac{\mathrm{d}t}{3+t^2}$

A1

$= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$

A1

$= \frac{\pi}{6\sqrt{3}}$

[7 marks]

Examiners report

[N/A]

61a. *[3 marks]*

Markscheme

A1

$f'(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x$

A1

$f''(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x + \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$

A1

$= 2\mathrm{e}^x \cos x$

AG

$= 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$

[3 marks]

Examiners report

[N/A]

61b. *[4 marks]*

Markscheme

A1

$f'''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right)$

A1

$f^{(4)}(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right) - 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$

A1

$= 4\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right)$

A1

$= 4\mathrm{e}^x \sin(x + \pi)$

[4 marks]

Examiners report

[N/A]

61c. [8 marks]

Markscheme

the conjecture is that

$f^{(2n)}(x) = 2^n \mathrm{e}^x \sin\left(x + \frac{n\pi}{2}\right)$ ***A1***

for $n = 1$, this formula gives

$f''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ which is correct ***A1***

let the result be true for $n = k$,

$\left(i.e.\ f^{(2k)}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right)\right)$ ***M1***

consider

$f^{(2k+1)}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right)$ ***M1***

$f^{(2(k+1))}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right) - 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} \mathrm{e}^x \sin\left(x + \frac{(k+1)\pi}{2}\right)$ ***A1***

therefore true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 1$

the result is proved by induction. ***R1***

Note: Award the final ***R1*** only if the two ***M*** marks have been awarded.

[8 marks]

Examiners report

[N/A]

62a. [6 marks]

Markscheme

f continuous

$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ ***M1***

$4a + 2b = 8$ ***A1***

$f'(x) = \begin{cases} 2, & x < 2 \\ 2ax + b, & 2 < x < 3 \end{cases}$ ***A1***

f' continuous $\Rightarrow \lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x)$ ***A1***

$4a + b = 2$

solve simultaneously ***M1***

to obtain $a = -1$ and $b = 6$ ***A1***

[6 marks]

Examiners report

[N/A]

62b.

[3 marks]

Markscheme

for $x \leqslant 2,\ f'(x) = 2 > 0$ ***A1***

for $2 < x < 3,\ f'(x) = -2x + 6 > 0$ ***A1***

since $f'(x) > 0$ for all values in the domain of f, f is increasing ***R1***

therefore one-to-one ***AG***

[3 marks]

Examiners report

[N/A]

62c.

[5 marks]

Markscheme

$x = 2y - 1 \Rightarrow y = \frac{x+1}{2}$ ***M1***

$x = -y^2 + 6y - 5 \Rightarrow y^2 - 6y + x + 5 = 0$ ***M1***

$y = 3 \pm \sqrt{4-x}$

therefore

$$f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & x \leqslant 3 \\ 3 - \sqrt{4-x}, & 3 < x < 4 \end{cases}$$ ***A1A1A1***

Note: Award ***A1*** for the first line and ***A1A1*** for the second line.

[5 marks]

Examiners report

[N/A]

63a.

[4 marks]

Markscheme

(i) displacement

$= \int_0^3 v\,\mathrm{d}t$ ***(M1)***

$= 0.703\ (\mathrm{m})$ ***A1***

(ii) total distance

$= \int_0^3 |v|\,\mathrm{d}t$ ***(M1)***

$= 2.05\ (\mathrm{m})$ ***A1***

[4 marks]

Examiners report

[N/A]

63b. *[2 marks]*

Markscheme

solving the equation

(M1)

$\int_0^t |\cos(u^2)| \, \mathrm{d}u = 1$

A1

$t = 1.39$ (s)

[2 marks]

Examiners report

[N/A]

64. *[7 marks]*

Markscheme

let x, y (m) denote respectively the distance of the bottom of the ladder from the wall and the distance of the top of the ladder from the ground

then,

M1A1

$x^2 + y^2 = 100$

M1A1

$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$

when

$x = 4,\ y = \sqrt{84}$

and

A1

$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.5$

substituting,

A1

$2 \times 4 \times 0.5 + 2\sqrt{84}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$

A1

$\frac{\mathrm{d}y}{\mathrm{d}t} = -0.218\ \mathrm{ms}^{-1}$

(speed of descent is

$0.218\ \mathrm{ms}^{-1}$

)

[7 marks]

Examiners report

[N/A]

65a. *[3 marks]*

Markscheme

M1A1A1

$f'(x) = \frac{1}{x+1}\sin(\pi x) + \pi \ln(x+1)\cos(\pi x)$

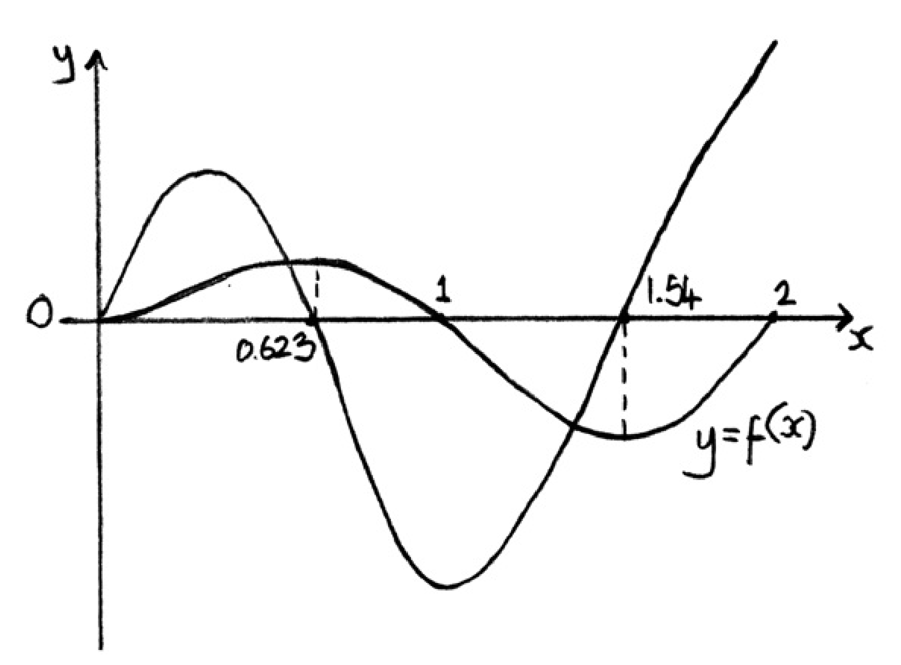
[3 marks]

Examiners report

[N/A]

65b. *[4 marks]*

Markscheme



A4

Note: Award ***A1A1*** for graphs, ***A1A1*** for intercepts.

[4 marks]

Examiners report

[N/A]

65c. *[2 marks]*

Markscheme

0.310, 1.12 ***A1A1***

[2 marks]

Examiners report

[N/A]

65d. *[3 marks]*

Markscheme

A1

$f'(0.75) = -0.839092$

so equation of normal is

M1

$y - 0.39570812 = \frac{1}{0.839092}(x - 0.75)$

A1

$y = 1.19x - 0.498$

[3 marks]

Examiners report

[N/A]

65e. *[6 marks]*

Markscheme

A(0, 0)

B$(\underbrace{0.548\ldots}_{c}, \underbrace{0.432\ldots}_{d})$ ***A1***

C$(\underbrace{1.44\ldots}_{e}, \underbrace{-0.881\ldots}_{f})$ ***A1***

Note: Accept coordinates for B and C rounded to 3 significant figures.

area

$\Delta\text{ABC} = \frac{1}{2}\left|(c\boldsymbol{i} + d\boldsymbol{j}) \times (e\boldsymbol{i} + f\boldsymbol{j})\right|$ ***M1A1***

$= \frac{1}{2}(de - cf)$ ***A1***

$= 0.554$ ***A1***

[6 marks]

Examiners report

[N/A]

66a. *[4 marks]*

Markscheme

$f'(x) = \frac{\cos x}{1+\sin x}$ ***A1***

$f''(x) = \frac{-\sin x(1+\sin x) - \cos x\cos x}{(1+\sin x)^2}$ ***M1A1***

$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$ ***A1***

$= -\frac{1}{1+\sin x}$ ***AG***

[4 marks]

Examiners report

[N/A]

66b. *[7 marks]*

Markscheme

(i)

$f'''(x) = \frac{\cos x}{(1+\sin x)^2}$ ***A1***

$f^{(4)}(x) = \frac{-\sin x(1+\sin x)^2 - 2(1+\sin x)\cos^2 x}{(1+\sin x)^4}$ ***M1A1***

$f(0) = 0,\ f'(0) = 1,\ f''(0) = -1$ ***M1***

$f'''(0) = 1,\ f^{(4)}(0) = -2$ ***A1***

$f(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$ ***A1***

(ii) the series contains even and odd powers of *x* ***R1***

[7 marks]

Examiners report

[N/A]

66c. *[3 marks]*

Markscheme

$\lim\limits_{x\to 0} \frac{\ln(1+\sin x) - x}{x^2} = \lim\limits_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots - x}{x^2}$ ***M1***

$= \lim\limits_{x\to 0} \frac{-\frac{1}{2} + \frac{x}{6} + \ldots}{1}$ ***(A1)***

$= -\frac{1}{2}$ ***A1***

Note: Use of l'Hopital's Rule is also acceptable.

[3 marks]

Examiners report

[N/A]

67a. *[7 marks]*

Markscheme

we note that

$f(0) = 0,\ f(x) = 3x$ for $x > 0$ and

$f(x) = x$ for $x < 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x = 0$ ***M1A1***

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 3x = 0$ ***A1***

since $f(0) = 0$, the function is continuous when $x = 0$ ***AG***

$\lim\limits_{x\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x\to 0^-} \frac{h}{h} = 1$ ***M1A1***

$\lim\limits_{x\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x\to 0^+} \frac{3h}{h} = 3$ ***A1***

these limits are unequal ***R1***

so *f* is not differentiable when $x = 0$ ***AG***

[7 marks]

Examiners report

[N/A]

67b. *[3 marks]*

Markscheme

M1

$\int_{-a}^{a} f(x)\mathrm{d}x = \int_{-a}^{0} x\mathrm{d}x + \int_{0}^{a} 3x\mathrm{d}x$

A1

$= \left[\frac{x^2}{2}\right]_{-a}^{0} + \left[\frac{3x^2}{2}\right]_{0}^{a}$

A1

$= a^2$

[3 marks]

Examiners report

[N/A]