Topic 5 Part 2 *[360 marks]*

The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} ae^{-x}, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

1a. State the mode of X . *[1 mark]*

1b. Determine the value of a . *[3 marks]*

1c. Find $\mathrm{E}(X)$. *[4 marks]*

2. The probability density function of the random variable X is defined as *[5 marks]*

$$f(x) = \begin{cases} \sin x, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Find

$\mathrm{E}(X)$.

3. Two events A and B are such that *[6 marks]*

$\mathrm{P}(A \cup B) = 0.7$ and

$\mathrm{P}(A|B') = 0.6$.

Find

$\mathrm{P}(B)$.

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is

$\frac{2}{3}$.

4a. Show that the probability that Alfred wins exactly 4 of the games is *[3 marks]*

$\frac{80}{243}$.

4b. (i) Explain why the total number of possible outcomes for the results of the 6 games is 64. *[4 marks]*

(ii) By expanding

$(1 + x)^6$ and choosing a suitable value for x, prove

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$

(iii) State the meaning of this equality in the context of the 6 games played.

4c. The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still *[9 marks]*

$\frac{2}{3}$.

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2 \left(\frac{2}{3}\right)^s \left(\frac{1}{3}\right)^t$ where the values of r, s and t are to be found.

(ii) Using your answer to (c) (i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that

$\binom{12}{6} = \binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2$.

4d. Alfred and Beatrice play n games. Let A denote the number of games Alfred wins. The expected value of A can be written as *[6 marks]*

$\mathrm{E}(A) = \sum_{r=0}^{n} r \binom{n}{r} \frac{a^r}{b^n}$.

(i) Find the values of a and b.

(ii) By differentiating the expansion of

$(1+x)^n$, prove that the expected number of games Alfred wins is

$\frac{2n}{3}$.

5. The marks obtained by a group of students in a class test are shown below. *[4 marks]*

Marks	Frequency
5	6
6	k
7	3
8	1
9	2
10	1

Given the mean of the marks is 6.5, find the value of k.

Emily walks to school every day. The length of time this takes can be modelled by a normal distribution with a mean of 11 minutes and a standard deviation of 3 minutes. She is late if her journey takes more than 15 minutes.

6a. Find the probability she is late next Monday. *[2 marks]*

6b. Find the probability she is late at least once during the next week (Monday to Friday). *[3 marks]*

A ferry carries cars across a river. There is a fixed time of T minutes between crossings. The arrival of cars at the crossing can be assumed to follow a Poisson distribution with a mean of one car every four minutes. Let X denote the number of cars that arrive in T minutes.

7a. Find T, to the nearest minute, if *[3 marks]*

$\mathrm{P}(X \leqslant 3) = 0.6$.

7b. It is now decided that the time between crossings, T, will be 10 minutes. The ferry can carry a maximum of three cars on each trip. *[4 marks]*

One day all the cars waiting at 13:00 get on the ferry. Find the probability that all the cars that arrive in the next 20 minutes will get on either the 13:10 or the 13:20 ferry.

Tim and Caz buy a box of 16 chocolates of which 10 are milk and 6 are dark. Caz randomly takes a chocolate and eats it. Then Tim randomly takes a chocolate and eats it.

8a. Draw a tree diagram representing the possible outcomes, clearly labelling each branch with the correct probability. *[3 marks]*

8b. Find the probability that Tim and Caz eat the same type of chocolate. *[2 marks]*

It is believed that the lifespans of Manx cats are normally distributed with a mean of 13.5 years and a variance of 9.5 years^2.

9a. Calculate the range of lifespans of Manx cats whose lifespans are within one standard deviation of the mean. *[2 marks]*

9b. Estimate the number of Manx cats in a population of 10 000 that will have a lifespan of less than 10 years. Give your answer to the nearest whole number. *[3 marks]*

The length, X metres, of a species of fish has the probability density function

$$f(x) = \begin{cases} ax^2, & \text{for } 0 \leqslant x \leqslant 0.5 \\ 0.5a(1-x), & \text{for } 0.5 \leqslant x \leqslant 1 \\ 0, & \text{otherwise .} \end{cases}$$

10a. Show that $a = 9.6$. *[3 marks]*

10b. Sketch the graph of the distribution. *[2 marks]*

10c. Find *[2 marks]*

$\mathrm{P}(X < 0.6)$.

A small car hire company has two cars. Each car can be hired for one whole day at a time. The rental charge is US$60 per car per day. The number of requests to hire a car for one whole day may be modelled by a Poisson distribution with mean 1.2.

11a. Find the probability that on a particular weekend, three requests are received on Saturday and none are received on Sunday. *[2 marks]*

11b. Over a weekend of two days, it is given that a total of three requests are received. *[5 marks]*

Find the expected total rental income for the weekend.

12a. (i) Express the sum of the first *n* positive odd integers using sigma notation. *[4 marks]*

(ii) Show that the sum stated above is

n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers.

12b. A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points. *[7 marks]*

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is

$\frac{n(n-3)}{2}$ if there are n points, where

$n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible.

12c. The random variable *[8 marks]*

$X \sim B(n,\, p)$ has mean 4 and variance 3.

(i) Determine *n* and *p*.

(ii) Find the probability that in a single experiment the outcome is 1 or 3.

The random variable *X* has probability distribution Po(8).

13a. (i) Find *[5 marks]*

$\mathrm{P}(X = 6)$.

(ii) Find

$\mathrm{P}(X = 6 | 5 \leqslant X \leqslant 8)$.

13b. $\bar{X}$ denotes the sample mean of *[3 marks]*

$n > 1$ independent observations from

X.

(i) Write down

$\mathrm{E}(\bar{X})$ and

$\mathrm{Var}(\bar{X})$.

(ii) Hence, give a reason why

$\bar{X}$ is not a Poisson distribution.

13c. A random sample of 40 observations is taken from the distribution for X. *[6 marks]*

(i) Find $\mathrm{P}(7.1 < \bar{X} < 8.5)$.

(ii) Given that $\mathrm{P}\left(\left|\bar{X} - 8\right| \leqslant k\right) = 0.95$, find the value of k.

14. The probability density function of a random variable X is defined as: *[13 marks]*

$$f(x) = \begin{cases} ax\cos x, \ 0 \le x \le \frac{\pi}{2}, \text{where } a \in \mathbb{R} \\ 0, \ \text{elsewhere} \end{cases}$$

(a) Show that $a = \frac{2}{\pi - 2}$.

(b) Find $\mathrm{P}\left(X < \frac{\pi}{4}\right)$.

(c) Find:

(i) the mode of X;

(ii) the median of X.

(d) Find $\mathrm{P}\left(X < \frac{\pi}{8} | X < \frac{\pi}{4}\right)$.

15. The discrete random variable X has probability distribution: *[6 marks]*

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	a

(a) Find the value of a.

(b) Find $\mathrm{E}(X)$.

(c) Find $\mathrm{Var}(X)$.

16. The duration of direct flights from London to Singapore in a particular year followed a normal distribution with mean μ and standard deviation σ. *[6 marks]*

92% of flights took under 13 hours, while only 12% of flights took under 12 hours 35 minutes.

Find μ and σ to the nearest minute.

At the start of each week, Eric and Marina pick a night at random on which they will watch a movie. If they choose a Saturday night, the probability that they watch a French movie is $\frac{7}{9}$ and if they choose any other night the probability that they watch a French movie is $\frac{4}{9}$.

17a. Find the probability that they watch a French movie. *[3 marks]*

17b. Given that last week they watched a French movie, find the probability that it was on a Saturday night. *[2 marks]*

18a. The number of cats visiting Helena's garden each week follows a Poisson distribution with mean $\lambda = 0.6$. *[9 marks]*

Find the probability that

(i) in a particular week no cats will visit Helena's garden;

(ii) in a particular week at least three cats will visit Helena's garden;

(iii) over a four-week period no more than five cats in total will visit Helena's garden;

(iv) over a twelve-week period there will be exactly four weeks in which at least one cat will visit Helena's garden.

18b. A continuous random variable X has probability distribution function f given by *[9 marks]*

$f(x) = k \ln x$ $\quad 1 \leqslant x \leqslant 3$

$f(x) = 0$ otherwise

(i) Find the value of k to six decimal places.

(ii) Find the value of $\mathrm{E}(X)$.

(iii) State the mode of X.

(iv) Find the median of X.

A traffic radar records the speed, v kilometres per hour (km h^{-1}), of cars on a section of a road.

The following table shows a summary of the results for a random sample of 1000 cars whose speeds were recorded on a given day.

Speed	**Number of cars**
$50 \le v < 60$	5
$60 \le v < 70$	13
$70 \le v < 80$	52
$80 \le v < 90$	68
$90 \le v < 100$	98
$100 \le v < 110$	105
$110 \le v < 120$	289
$120 \le v < 130$	142
$130 \le v < 140$	197
$140 \le v < 150$	31

19a. Using the data in the table, *[4 marks]*

(i) show that an estimate of the mean speed of the sample is 113.21 km h^{-1};

(ii) find an estimate of the variance of the speed of the cars on this section of the road.

19b. Find the 95% confidence interval, I, for the mean speed. *[2 marks]*

19c. Let J be the 90% confidence interval for the mean speed. *[2 marks]*

Without calculating J, explain why $J \subset I$.

20. The probability density function of the continuous random variable X is given by *[4 marks]*

$$f(x) = \begin{cases} k2^{\frac{1}{x}}, & 1 \leqslant x \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant. Find the expected value of X .

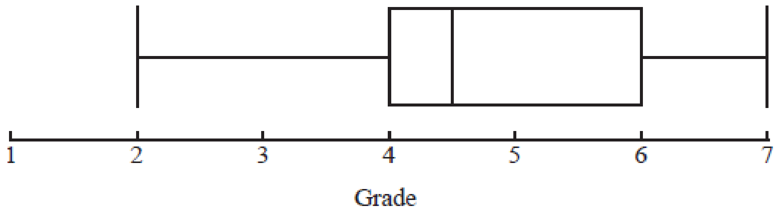
A team of 6 players is to be selected from 10 volleyball players, of whom 8 are boys and 2 are girls.

21a. In how many ways can the team be selected? *[2 marks]*

21b. In how many of these selections is exactly one girl in the team? *[3 marks]*

21c. If the selection of the team is made at random, find the probability that exactly one girl is in the team. *[2 marks]*

The box and whisker plot below illustrates the IB grades obtained by 100 students.



IB grades can only take integer values.

22a. How many students obtained a grade of more than 4? *[1 mark]*

22b. State, with reasons, the maximum possible number and minimum possible number of students who obtained a 4 in the exam. *[4 marks]*

23. A fisherman notices that in any hour of fishing, he is equally likely to catch exactly two fish, as he is to catch less than two fish. Assuming the number of fish caught can be modelled by a Poisson distribution, calculate the expected value of the number of fish caught when he spends four hours fishing. *[5 marks]*

A ski resort finds that the mean number of accidents on any given weekday (Monday to Friday) is 2.2 . The number of accidents can be modelled by a Poisson distribution.

24a. Find the probability that in a certain week (Monday to Friday only) *[6 marks]*

(i) there are fewer than 12 accidents;

(ii) there are more than 8 accidents, given that there are fewer than 12 accidents.

24b. Due to the increased usage, it is found that the probability of more than 3 accidents in a day at the weekend (Saturday and Sunday) is 0.24. *[10 marks]*

Assuming a Poisson model,

(i) calculate the mean number of accidents per day at the weekend (Saturday and Sunday);

(ii) calculate the probability that, in the four weekends in February, there will be more than 5 accidents during at least two of the weekends.

24c. It is found that 20 % of skiers having accidents are at least 25 years of age and 40 % are under 18 years of age. Assuming that the ages of skiers having accidents are normally distributed, find the mean age of skiers having accidents. *[6 marks]*

On a particular day, the probability that it rains is $\frac{2}{5}$. The probability that the “Tigers” soccer team wins on a day when it rains is $\frac{2}{7}$ and the probability that they win on a day when it does not rain is $\frac{4}{7}$.

25a. Draw a tree diagram to represent these events and their outcomes. *[1 mark]*

25b. What is the probability that the “Tigers” soccer team wins? *[2 marks]*

25c. Given that the “Tigers” soccer team won, what is the probability that it rained on that day? *[2 marks]*

Consider the following functions:

$$f(x) = \frac{2x^2 + 3}{75},\ x \geqslant 0$$

$$g(x) = \frac{|3x - 4|}{10},\ x \in \mathbb{R}\ .$$

26a. State the range of *f* and of *g* . *[2 marks]*

26b. Find an expression for the composite function $f \circ g(x)$ in the form $\frac{ax^2+bx+c}{3750}$, where $a,\ b$ and $c \in \mathbb{Z}$. *[4 marks]*

26c. (i) Find an expression for the inverse function $f^{-1}(x)$. *[4 marks]*

(ii) State the domain and range of f^{-1} .

26d. The domains of *f* and *g* are now restricted to $\{0,\ 1,\ 2,\ 3,\ 4\}$. *[6 marks]*

By considering the values of *f* and *g* on this new domain, determine which of *f* and *g* could be used to find a probability distribution for a discrete random variable *X* , stating your reasons clearly.

26e. Using this probability distribution, calculate the mean of *X* . *[2 marks]*

The random variable *X* has the distribution $\mathrm{B}(30,\ p)$. Given that $\mathrm{E}(X) = 10$, find

27a. the value of *p* ; *[1 mark]*

27b. $\mathrm{P}(X = 10)$; *[2 marks]*

27c. $\mathrm{P}(X \geqslant 15)$. *[2 marks]*

28. A set of 15 observations has mean 11.5 and variance 9.3. One observation of 22.1 is considered unreliable and is removed. Find the mean and variance of the remaining 14 observations. *[6 marks]*

Kathy plays a computer game in which she has to find the path through a maze within a certain time. The first time she attempts the game, the probability of success is known to be 0.75. In subsequent attempts, if Kathy is successful, the difficulty increases and the probability of success is half the probability of success on the previous attempt. However, if she is unsuccessful, the probability of success remains the same. Kathy plays the game three times consecutively.

29a. Find the probability that she is successful in all three games. *[2 marks]*

29b. Assuming that she is successful in the first game, find the probability that she is successful in exactly two games. *[6 marks]*

The number of visitors that arrive at a museum every minute can be modelled by a Poisson distribution with mean 2.2.

30a. If the museum is open 6 hours daily, find the expected number of visitors in 1 day. *[2 marks]*

30b. Find the probability that the number of visitors arriving during an hour exceeds 100. *[3 marks]*

30c. Find the probability that the number of visitors in each of the 6 hours the museum is open exceeds 100. *[2 marks]*

30d. The ages of the visitors to the museum can be modelled by a normal distribution with mean μ and variance σ^2. The records show that 29 % of the visitors are under 35 years of age and 23 % are at least 55 years of age. *[6 marks]*

Find the values of μ and σ.

30e. The ages of the visitors to the museum can be modelled by a normal distribution with mean μ and variance σ^2. The records show that 29 % of the visitors are under 35 years of age and 23 % are at least 55 years of age. *[5 marks]*

One day, 100 visitors under 35 years of age come to the museum. Estimate the number of visitors under 50 years of age that were at the museum on that day.

Bill also has a box with 10 biscuits in it. 4 biscuits are chocolate and 6 are plain. Bill takes a biscuit from his box at random, looks at it and replaces it in the box. He repeats this process until he has looked at 5 biscuits in total. Let B be the number of chocolate biscuits that Bill takes and looks at.

31a. State the distribution of B. *[1 mark]*

31b. Find $\mathrm{P}(B = 3)$. *[2 marks]*

31c. Find $\mathrm{P}(B = 5)$. *[2 marks]*

32. Over a one month period, Ava and Sven play a total of n games of tennis. *[6 marks]*

The probability that Ava wins any game is 0.4. The result of each game played is independent of any other game played.

Let X denote the number of games won by Ava over a one month period.

(a) Find an expression for $\mathrm{P}(X = 2)$ in terms of n.

(b) If the probability that Ava wins two games is 0.121 correct to three decimal places, find the value of n.

33a. The distance travelled by students to attend Gauss College is modelled by a normal distribution with mean 6 km and standard deviation 1.5 km. *[7 marks]*

(i) Find the probability that the distance travelled to Gauss College by a randomly selected student is between 4.8 km and 7.5 km.

(ii) 15 % of students travel less than d km to attend Gauss College. Find the value of d.

33b. At Euler College, the distance travelled by students to attend their school is modelled by a normal distribution with mean μ km and standard deviation σ km. *[6 marks]*

If 10 % of students travel more than 8 km and 5 % of students travel less than 2 km, find the value of μ and of σ.

33c. The number of telephone calls, T, received by Euler College each minute can be modelled by a Poisson distribution with a mean of 3.5. *[8 marks]*

(i) Find the probability that at least three telephone calls are received by Euler College in **each** of two successive one-minute intervals.

(ii) Find the probability that Euler College receives 15 telephone calls during a randomly selected five-minute interval.

A bag contains three balls numbered 1, 2 and 3 respectively. Bill selects one of these balls at random and he notes the number on the selected ball. He then tosses that number of fair coins.

34a. Calculate the probability that no head is obtained. *[3 marks]*

34b. Given that no head is obtained, find the probability that he tossed two coins. *[3 marks]*

The continuous variable X has probability density function

$$f(x) = \begin{cases} 12x^2(1-x), & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

35a. Determine *[3 marks]*

$\mathrm{E}(X)$.

35b. Determine the mode of X . *[3 marks]*

The weights, in kg, of male birds of a certain species are modelled by a normal distribution with mean

μ and standard deviation

σ .

36a. Given that 70 % of the birds weigh more than 2.1 kg and 25 % of the birds weigh more than 2.5 kg, calculate the value of μ and the value of *[4 marks]*

σ .

36b. A random sample of ten of these birds is obtained. Let X denote the number of birds in the sample weighing more than 2.5 kg. *[5 marks]*

(i) Calculate

$\mathrm{E}(X)$.

(ii) Calculate the probability that exactly five of these birds weigh more than 2.5 kg.

(iii) Determine the most likely value of X .

36c. The number of eggs, Y , laid by female birds of this species during the nesting season is modelled by a Poisson distribution with mean [8 marks]

λ . You are given that

$\mathrm{P}(Y \geqslant 2) = 0.80085$, correct to 5 decimal places.

(i) Determine the value of

λ .

(ii) Calculate the probability that two randomly chosen birds lay a total of

two eggs between them.

(iii) Given that the two birds lay a total of two eggs between them, calculate the probability that they each lay one egg.

When Andrew throws a dart at a target, the probability that he hits it is

$\frac{1}{3}$; when Bill throws a dart at the target, the probability that he hits the it is

$\frac{1}{4}$. Successive throws are independent. One evening, they throw darts at the target alternately, starting with Andrew, and stopping as soon as one of their darts hits the target. Let X denote the total number of darts thrown.

37a. Write down the value of [2 marks]

$\mathrm{P}(X = 1)$ and show that

$\mathrm{P}(X = 2) = \frac{1}{6}$.

37b. Show that the probability generating function for X is given by [6 marks]

$$G(t) = \frac{2t + t^2}{6 - 3t^2}.$$

37c. Hence determine [4 marks]

$\mathrm{E}(X)$.

The discrete random variable X has the following probability distribution, where

$0 < \theta < \frac{1}{3}$.

x	1	2	3
$\mathrm{P}(X = x)$	θ	2θ	$1 - 3\theta$

38a. Determine [4 marks]

$\mathrm{E}(X)$ and show that

$\mathrm{Var}(X) = 6\theta - 16\theta^2$.

38b. In order to estimate [10 marks]

θ, a random sample of n observations is obtained from the distribution of X .

(i) Given that

$\bar{X}$ denotes the mean of this sample, show that

$$\hat{\theta}_1 = \frac{3 - \bar{X}}{4}$$

is an unbiased estimator for

θ and write down an expression for the variance of

$\hat{\theta}_1$ in terms of n and

θ.

(ii) Let Y denote the number of observations that are equal to 1 in the sample. Show that Y has the binomial distribution

$\mathrm{B}(n,\, \theta)$ and deduce that

$\hat{\theta}_2 = \frac{Y}{n}$ is another unbiased estimator for

θ. Obtain an expression for the variance of

$\hat{\theta}_2$.

(iii) Show that

$\mathrm{Var}(\hat{\theta}_1) < \mathrm{Var}(\hat{\theta}_2)$ and state, with a reason, which is the more efficient estimator,

$\hat{\theta}_1$ or

$\hat{\theta}_2$.