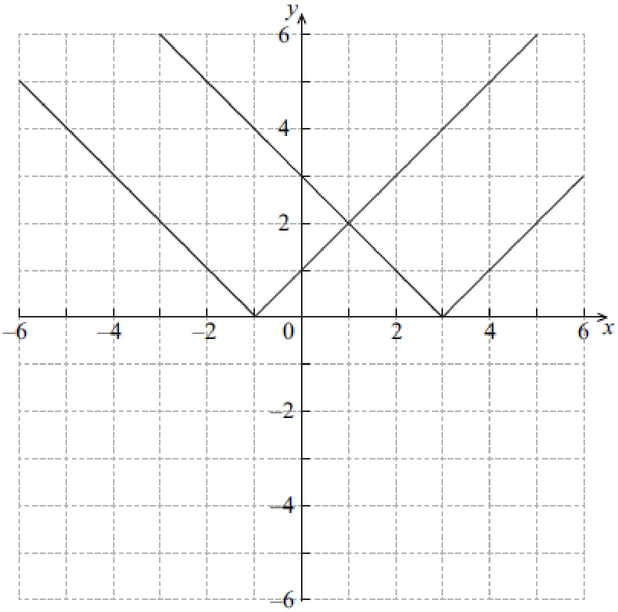
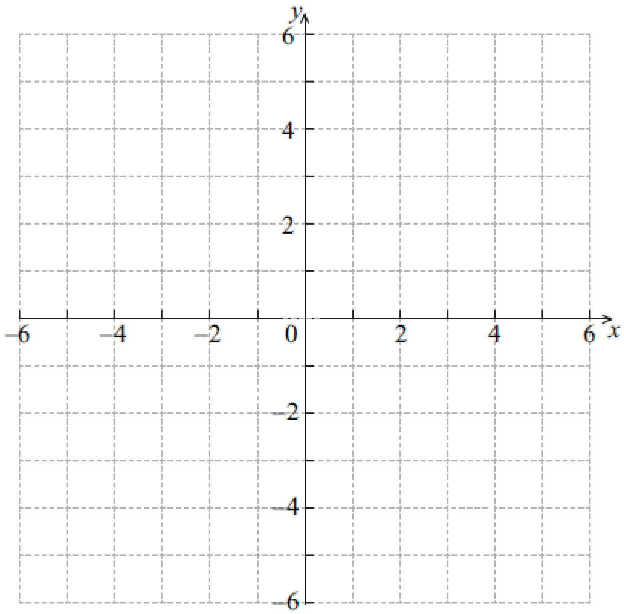
Topic 2 Part 3 *[438 marks]*

The graphs of
$y = |x + 1|$ and
$y = |x - 3|$ are shown below.



Let $f(x) =$
$|x + 1| - |x - 3|$.

1a. Draw the graph of $y = f(x)$ on the blank grid below. *[4 marks]*



1b. Hence state the value of *[4 marks]*
(i)
$f'(-3)$;
(ii)
$f'(2.7)$;
(iii)
$\int_{-3}^{-2} f(x)dx$.

2. Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of k . *[5 marks]*

3. Let $f(x) = \ln x$. The graph of f is transformed into the graph of the function g by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, followed by a reflection in the x-axis. Find an expression for $g(x)$, giving your answer as a single logarithm. *[5 marks]*

The function $f(x) = 3\sin x + 4\cos x$ is defined for $0 < x < 2\pi$.

4a. Write down the coordinates of the minimum point on the graph of f . *[1 mark]*

4b. The points $\mathrm{P}(p,\ 3)$ and $\mathrm{Q}(q,\ 3)$, $q > p$, lie on the graph of $y = f(x)$. *[2 marks]*

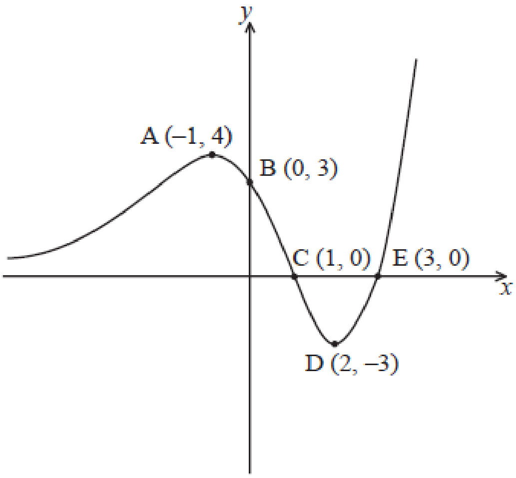
Find p and q .

4c. Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3. *[4 marks]*

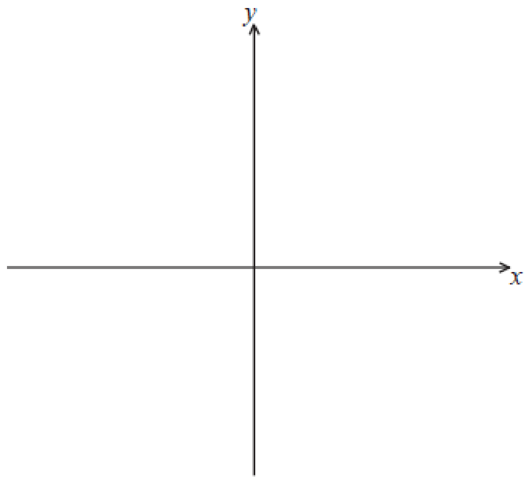
4d. Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. *[7 marks]*

5. The same remainder is found when $2x^3 + kx^2 + 6x + 32$ and $x^4 - 6x^2 - k^2x + 9$ are divided by $x + 1$. Find the possible values of k . *[6 marks]*

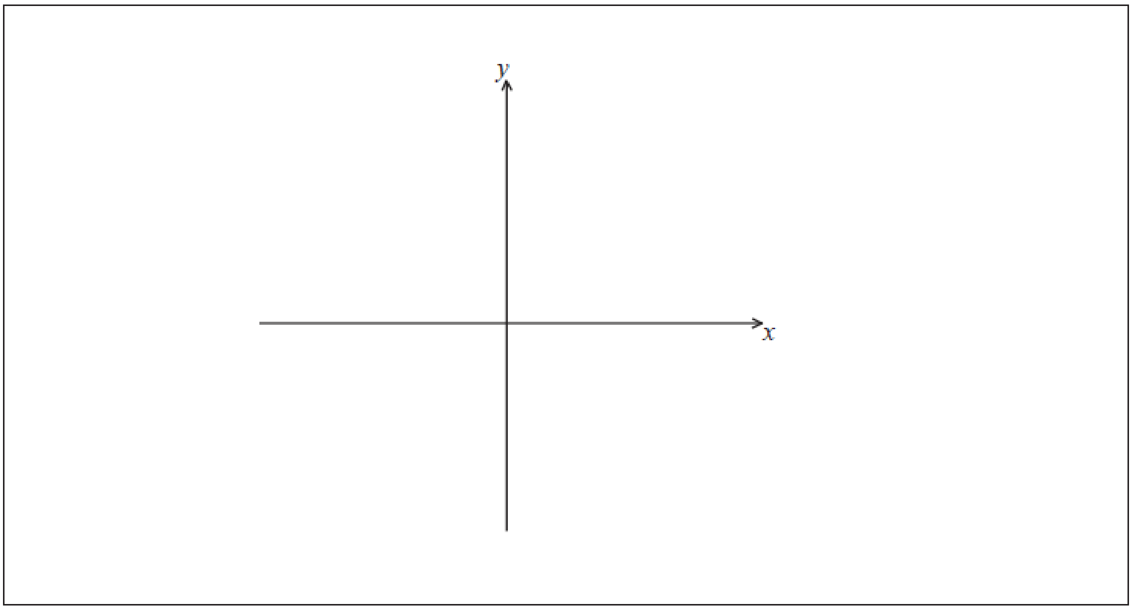
The graph of
$y = f(x)$ is shown below, where A is a local maximum point and D is a local minimum point.



6a. On the axes below, sketch the graph of $y = \frac{1}{f(x)}$, clearly showing the coordinates of the images of the points A, B and D, labelling them A', B', and D' respectively, and the equations of any vertical asymptotes. *[3 marks]*



6b. On the axes below, sketch the graph of the derivative $y = f'(x)$, clearly showing the coordinates of the images of the points A and D, labelling them A'' and D'' respectively. *[3 marks]*



The function f is defined on the domain $\left[0, \frac{3\pi}{2}\right]$ by $f(x) = e^{-x}\cos x$.

7a. State the two zeros of f . *[1 mark]*

7b. Sketch the graph of f . *[1 mark]*

7c. The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B . Show that the ratio of the area of A to the area of B is *[7 marks]*

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

Consider the following functions:

$$f(x) = \frac{2x^2+3}{75},\ x \geqslant 0$$

$$g(x) = \frac{|3x-4|}{10},\ x \in \mathbb{R}\ .$$

8a. State the range of f and of g . *[2 marks]*

8b. Find an expression for the composite function $f \circ g(x)$ in the form $\frac{ax^2+bx+c}{3750}$, where a, b and $c \in \mathbb{Z}$. *[4 marks]*

8c. (i) Find an expression for the inverse function $f^{-1}(x)$.
(ii) State the domain and range of f^{-1} . *[4 marks]*

8d. The domains of f and g are now restricted to $\{0, 1, 2, 3, 4\}$. *[6 marks]*
By considering the values of f and g on this new domain, determine which of f and g could be used to find a probability distribution for a discrete random variable X , stating your reasons clearly.

8e. Using this probability distribution, calculate the mean of X . *[2 marks]*

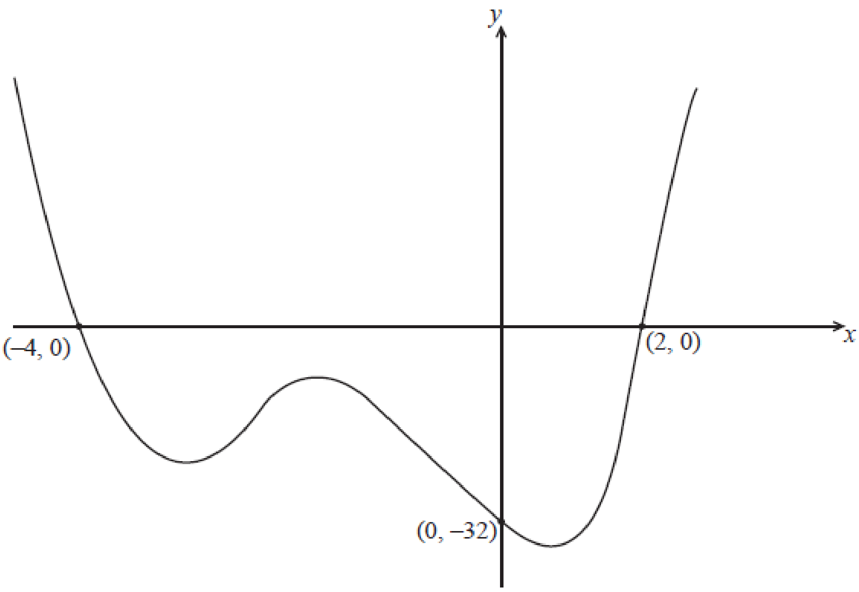
9a. Given that *[2 marks]*
$(x+\mathrm{i}y)^2 = -5+12\mathrm{i},\ x,\ y \in \mathbb{R}$. Show that
(i)
$x^2 - y^2 = -5$;
(ii)
$xy = 6$.

9b. Hence find the two square roots of *[5 marks]*
$-5+12\mathrm{i}$.

9c. For any complex number z , show that *[3 marks]*
$(z^*)^2 = (z^2)^*$.

9d. Hence write down the two square roots of *[2 marks]*
$-5-12\mathrm{i}$.

The graph of a polynomial function f of degree 4 is shown below.



9e. Explain why, of the four roots of the equation *[2 marks]*
$f(x) = 0$, two are real and two are complex.

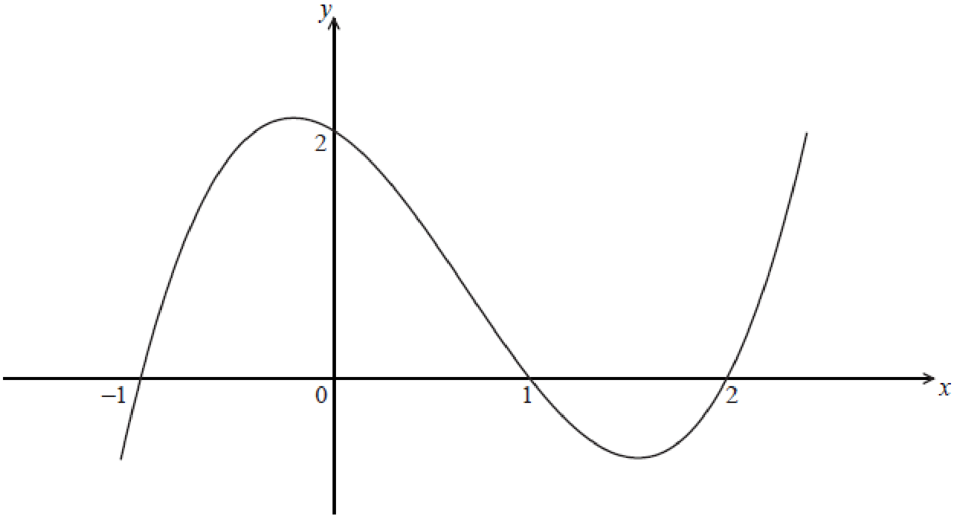
9f. The curve passes through the point *[5 marks]*
$(-1, -18)$. Find
$f(x)$ in the form
$f(x) = (x-a)(x-b)(x^2+cx+d)$, where $a,\ b,\ c,\ d \in \mathbb{Z}$.

9g. Find the two complex roots of the equation *[2 marks]*
$f(x) = 0$ in Cartesian form.

9h. Draw the four roots on the complex plane (the Argand diagram). *[2 marks]*

9i. Express each of the four roots of the equation in the form $r\mathrm{e}^{\mathrm{i}\theta}$. *[6 marks]*

Let $f(x) = x^3 + ax^2 + bx + c$, where a , b , $c \in \mathbb{Z}$. The diagram shows the graph of $y = f(x)$.



10a. Using the information shown in the diagram, find the values of a , b and c . *[4 marks]*

10b. If $g(x) = 3f(x - 2)$, *[3 marks]*
(i) state the coordinates of the points where the graph of g intercepts the x-axis.
(ii) Find the y-intercept of the graph of g .

Consider a function f , defined by $f(x) = \frac{x}{2-x}$ for $0 \leqslant x \leqslant 1$.

11a. Find an expression for $(f \circ f)(x)$. *[3 marks]*

11b. Let $F_n(x) = \frac{x}{2^n - (2^n - 1)x}$, where $0 \leqslant x \leqslant 1$. *[8 marks]*
Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$$

.

11c. Show that $F_{-n}(x)$ is an expression for the inverse of F_n . *[6 marks]*

11d. (i) State $F_n(0)$ and $F_n(1)$. *[6 marks]*

(ii) Show that $F_n(x) < x$, given 0 < *x* < 1, $n \in \mathbb{Z}^+$.

(iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1} , the *x*-axis and the line *x* = 1. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n .

12. Show that the quadratic equation $x^2 - (5 - k)x - (k + 2) = 0$ has two distinct real roots for all real values of k . *[4 marks]*

Let c be a positive, real constant. Let G be the set

$\{x \in \mathbb{R} | -c < x < c\}$. The binary operation

$*$ is defined on the set G by

$x * y = \frac{x+y}{1+\frac{xy}{c^2}}$.

13a. *[2 marks]*

Simplify

$\frac{c}{2} * \frac{3c}{4}$.

13b. State the identity element for G under *[1 mark]*

$*$.

13c. For *[1 mark]*

$x \in G$ find an expression for

x^{-1} (the inverse of *x* under

$*$).

13d. Show that the binary operation *[2 marks]*

$*$ is commutative on G .

13e. Show that the binary operation *[4 marks]*

$*$ is associative on G .

13f. (i) If *[2 marks]*

$x,\, y \in G$ explain why

$(c-x)(c-y) > 0$.

(ii) Hence show that

$x + y < c + \frac{xy}{c}$.

13g. Show that G is closed under *[2 marks]*

$*$.

13h. Explain why *[2 marks]*

$\{G, *\}$ is an Abelian group.

A particle, A, is moving along a straight line. The velocity,

v_A ms^{-1}, of A t seconds after its motion begins is given by

$$v_A = t^3 - 5t^2 + 6t.$$

14a. Sketch the graph of *[3 marks]*

$v_A = t^3 - 5t^2 + 6t$ for

$t \geqslant 0$, with

v_A on the vertical axis and t on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the t-axis.

14b. Write down the times for which the velocity of the particle is increasing. *[2 marks]*

14c. Write down the times for which the magnitude of the velocity of the particle is increasing. *[3 marks]*

14d. At $t = 0$ the particle is at point O on the line. *[3 marks]*

Find an expression for the particle's displacement,

x_A m, from O at time t.

14e. A second particle, B, moving along the same line, has position [4 marks]

x_B m, velocity

v_B ms^{-1} and acceleration,

a_B ms^{-2}, where

$a_B = -2v_B$ for

$t \geqslant 0$. At

$t = 0,\ x_B = 20$ and

$v_B = -20$.

Find an expression for

v_B in terms of *t*.

14f. Find the value of *t* when the two particles meet. [6 marks]

The function *f* has inverse

f^{-1} and derivative

$f'(x)$ for all

$x \in \mathbb{R}$. For all functions with these properties you are given the result that for

$a \in \mathbb{R}$ with

$b = f(a)$ and

$f'(a) \neq 0$

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

15a. Verify that this is true for [6 marks]

$f(x) = x^3 + 1$ at $x = 2$.

15b. Given that [3 marks]

$g(x) = xe^{x^2}$, show that

$g'(x) > 0$ for all values of *x*.

15c. Using the result given at the start of the question, find the value of the gradient function of [4 marks]

$y = g^{-1}(x)$ at $x = 2$.

15d. (i) With *f* and *g* as defined in parts (a) and (b), solve [6 marks]

$g \circ f(x) = 2$.

(ii) Let

$h(x) = (g \circ f)^{-1}(x)$. Find

$h'(2)$.

16a. Find [4 marks]

$\int x\sec^2 x\mathrm{d}x$.

16b. Determine the value of m if [2 marks]

$\int_0^m x\sec^2 x\mathrm{d}x = 0.5$, where $m > 0$.

The arithmetic sequence

$\{u_n : n \in \mathbb{Z}^+\}$ has first term

$u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence

$\{v_n : n \in \mathbb{Z}^+\}$ has first term

$v_1 = 3$ and common ratio $r = 1.2$.

17a. Find an expression for [2 marks]

$u_n - v_n$ in terms of n.

17b. Determine the set of values of n for which [3 marks]

$u_n > v_n$.

17c. Determine the greatest value of [1 mark]

$u_n - v_n$. Give your answer correct to four significant figures.

18a. (i) Express the sum of the first n positive odd integers using sigma notation. [4 marks]

(ii) Show that the sum stated above is

n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers.

18b. A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points. [7 marks]

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is

$\frac{n(n-3)}{2}$ if there are n points, where

$n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible.

18c. The random variable *[8 marks]*

$X \sim B(n,\, p)$ has mean 4 and variance 3.

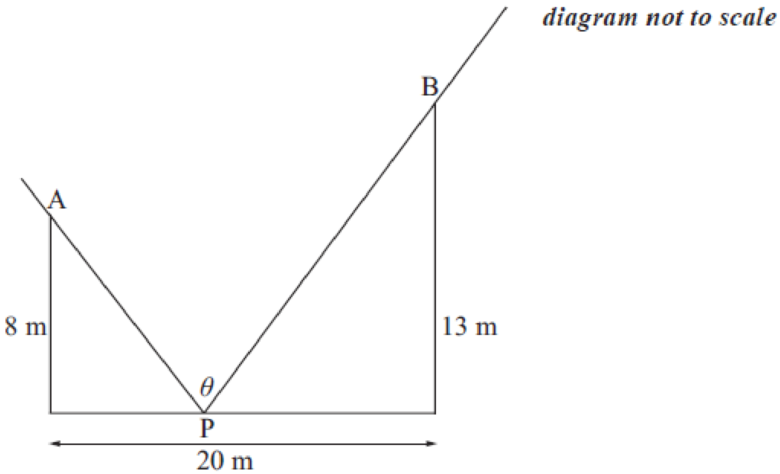
(i) Determine n and p.

(ii) Find the probability that in a single experiment the outcome is 1 or 3.

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle

θ where

$\theta = \mathrm{A\hat{P}B}$, as shown in the diagram.



19a. Find an expression for *[2 marks]*

θ in terms of x, where x is the distance of P from the base of the wall of height 8 m.

19b. (i) Calculate the value of *[2 marks]*

θ when $x = 0$.

(ii) Calculate the value of

θ when $x = 20$.

19c. Sketch the graph of *[2 marks]*

θ, for

$0 \leqslant x \leqslant 20$.

19d. Show that *[6 marks]*

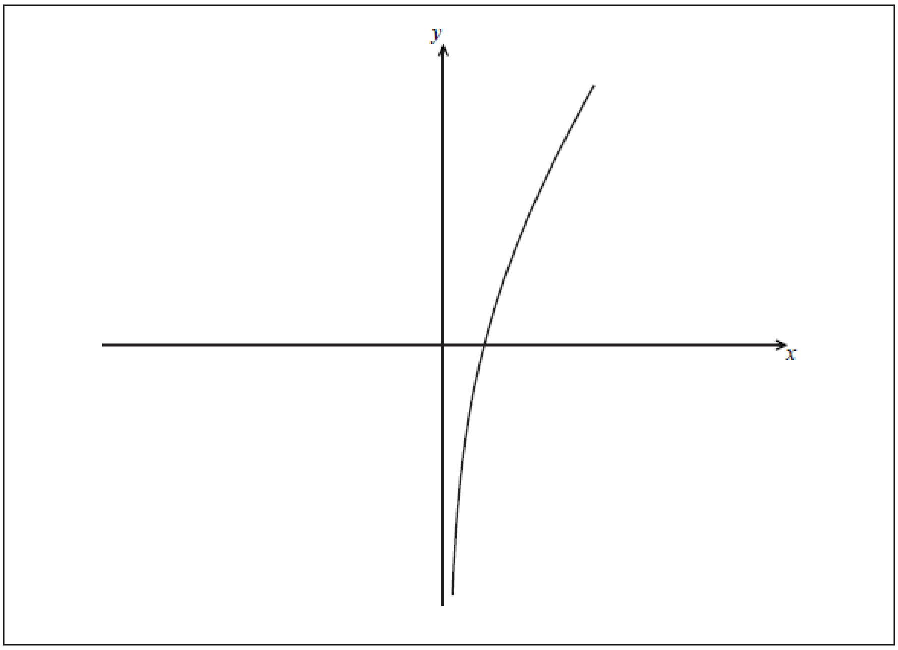
$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{5(744 - 64x - x^2)}{(x^2 + 64)(x^2 - 40x + 569)}$.

19e. Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. *[3 marks]*

19f. The point P moves across the street with speed $0.5\ \mathrm{ms}^{-1}$. Determine the rate of change of θ with respect to time when P is at the midpoint of the street. *[4 marks]*

20. The graph below shows $y = f(x)$, where $f(x) = x + \ln x$. *[6 marks]*

(a) On the graph below, sketch the curve $y = f^{-1}(x)$.



(b) Find the coordinates of the point of intersection of the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$.

21. Given that $Ax^3 + Bx^2 + x + 6$ is exactly divisible by $(x+1)(x-2)$, find the value of A and the value of B . *[5 marks]*

22. The functions f and g are defined as: *[8 marks]*

$$f(x) = \mathrm{e}^{x^2},\ x \geqslant 0$$

$$g(x) = \frac{1}{x+3},\ x \neq -3.$$

(a) Find

$h(x)$ where $h(x) = g \circ f(x)$.

(b) State the domain of

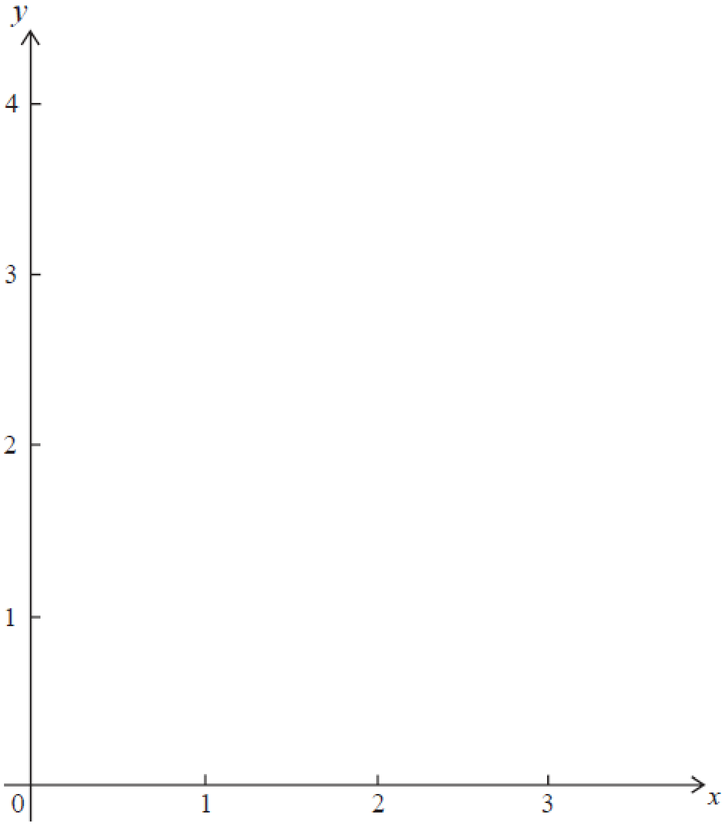
$h^{-1}(x)$.

(c) Find

$h^{-1}(x)$.

23. (a) Sketch the curve [6 marks]

$f(x) = |1 + 3\sin(2x)|$, for $0 \leqslant x \leqslant \pi$. Write down on the graph the values of the x and y intercepts.



(b) By adding **one** suitable line to your sketch, find the number of solutions to the equation

$\pi f(x) = 4(\pi - x)$.

24. The polynomial [6 marks]

$P(x) = x^3 + ax^2 + bx + 2$ is divisible by $(x + 1)$ and by $(x - 2)$.

Find the value of a and of b, where

$a, b \in \mathbb{R}$.

25. Let [6 marks]

$f(x) = \frac{4}{x+2}$, $x \neq -2$ and $g(x) = x - 1$.

If

$h = g \circ f$, find

(a) $h(x)$;

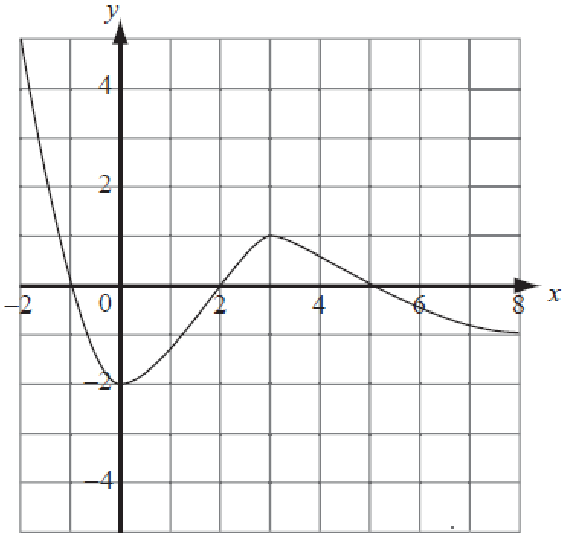
(b)

$h^{-1}(x)$, where

h^{-1} is the inverse of h.

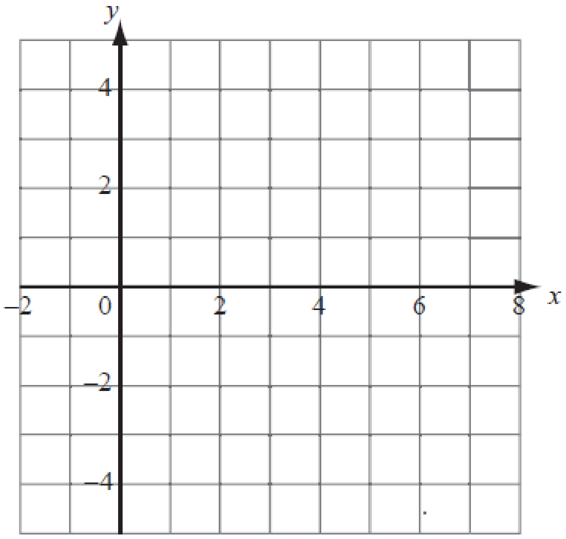
26. The graph of [5 marks]

$y = f(x)$ for $-2 \leqslant x \leqslant 8$ is shown.



On the set of axes provided, sketch the graph of

$y = \frac{1}{f(x)}$, clearly showing any asymptotes and indicating the coordinates of any local maxima or minima.



27. Find the set of values of x for which [6 marks]

$|0.1x^2 - 2x + 3| < \log_{10} x$.

28. When [6 marks]

$f(x) = x^4 + 3x^3 + px^2 - 2x + q$ is divided by $(x - 2)$ the remainder is 15, and $(x + 3)$ is a factor of $f(x)$.

Find the values of p and q .

29. Write [5 marks]

$\ln(x^2 - 1) - 2\ln(x + 1) + \ln(x^2 + x)$ as a single logarithm, in its simplest form.

30. (a) Sketch the curve *[7 marks]*

$y = |\ln x| - |\cos x| - 0.1$,

$0 < x < 4$ showing clearly the coordinates of the points of intersection with the *x*-axis and the coordinates of any local maxima and minima.

(b) Find the values of *x* for which

$|\ln x| > |\cos x| + 0.1,$

$0 < x < 4$.

31. Let *[5 marks]*

$g(x) = \log_5 |2\log_3 x|$. Find the product of the zeros of g .

32. The diagram below shows a solid with volume *V* , obtained from a cube with edge *[8 marks]*

$a > 1$ when a smaller cube with edge

$\frac{1}{a}$ is removed.

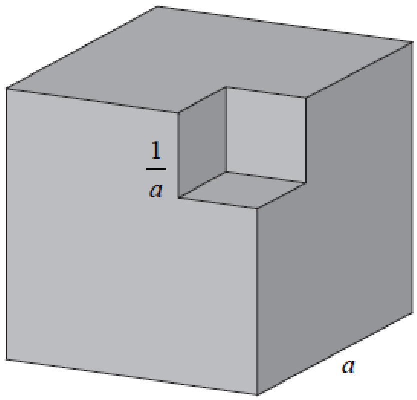


diagram not to scale

Let

$x = a - \frac{1}{a}$

(a) Find *V* in terms of *x* .

(b) Hence or otherwise, show that the only value of *a* for which *V* = 4*x* is

$a = \frac{1+\sqrt{5}}{2}$.

33. When the function *[5 marks]*

$q(x) = x^3 + kx^2 - 7x + 3$ is divided by (*x* + 1) the remainder is seven times the remainder that is found when the function is divided by (*x* + 2) .

Find the value of *k* .

34. A function is defined as [16 marks]

$f(x) = k\sqrt{x}$, with

$k > 0$ and

$x \geqslant 0$.

(a) Sketch the graph of

$y = f(x)$.

(b) Show that f is a one-to-one function.

(c) Find the inverse function,

$f^{-1}(x)$ and state its domain.

(d) If the graphs of

$y = f(x)$ and

$y = f^{-1}(x)$ intersect at the point (4, 4) find the value of k .

(e) Consider the graphs of

$y = f(x)$ and

$y = f^{-1}(x)$ using the value of k found in part (d).

(i) Find the area enclosed by the two graphs.

(ii) The line $x = c$ cuts the graphs of

$y = f(x)$ and

$y = f^{-1}(x)$ at the points P and Q respectively. Given that the tangent to

$y = f(x)$ at point P is parallel to the tangent to

$y = f^{-1}(x)$ at point Q find the value of c .

35. When [5 marks]

$3x^5 - ax + b$ is divided by $x - 1$ and $x + 1$ the remainders are equal. Given that a ,

$b \in \mathbb{R}$, find

(a) the value of a ;

(b) the set of values of b .

36. Consider the function f , where [6 marks]

$f(x) = \arcsin(\ln x)$.

(a) Find the domain of f .

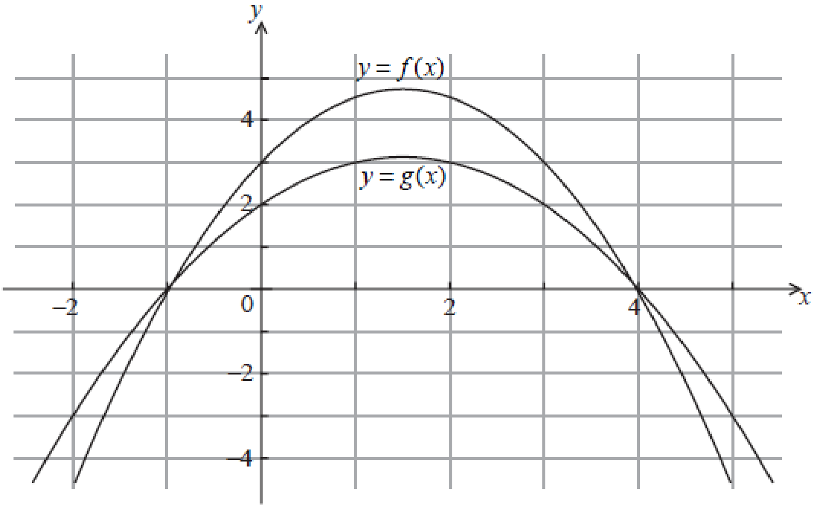
(b) Find

$f^{-1}(x)$.

37. Shown below are the graphs of [4 marks]

$y = f(x)$ and

$y = g(x)$.

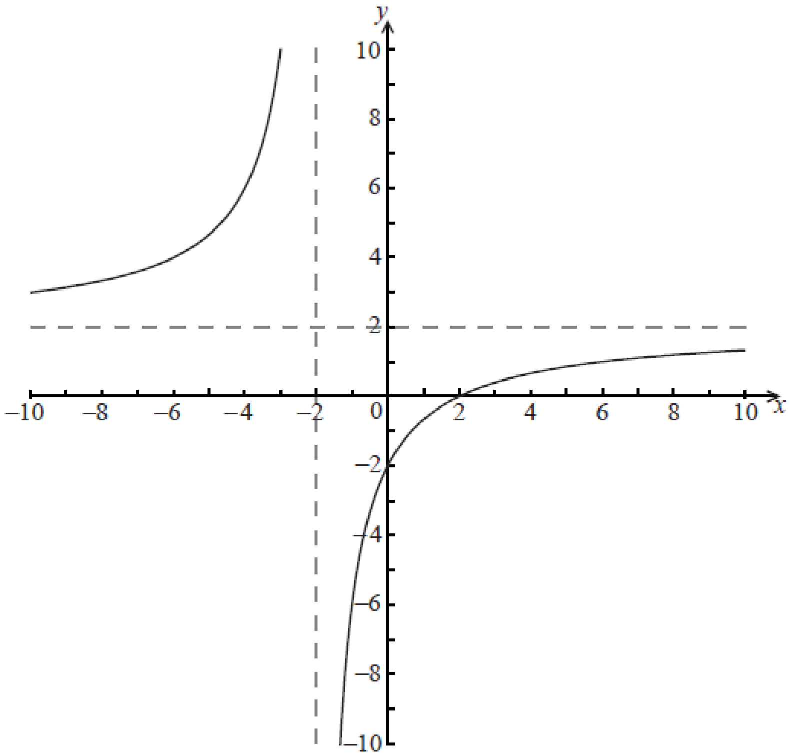


If

$(f \circ g)(x) = 3$, find all possible values of x.

38. The graph of [8 marks]

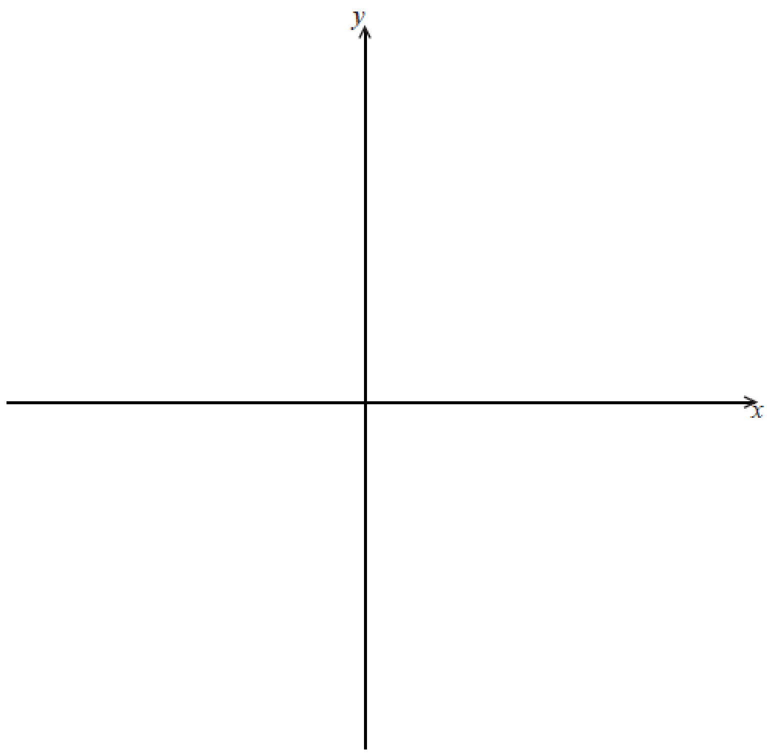
$y = \frac{a+x}{b+cx}$ is drawn below.



(a) Find the value of a, the value of b and the value of c.

(b) Using the values of a, b and c found in part (a), sketch the graph of

$y = \left|\frac{b+cx}{a+x}\right|$ on the axes below, showing clearly all intercepts and asymptotes.



39. Let *[6 marks]*

$f(x) = \frac{4-x^2}{4-\sqrt{x}}$.

(a) State the largest possible domain for *f*.

(b) Solve the inequality

$f(x) \geqslant 1$.

40. (a) Express the quadratic *[6 marks]*

$3x^2 - 6x + 5$ in the form

$a(x+b)^2 + c$, where a, b, c

$\in \mathbb{Z}$.

(b) Describe a sequence of transformations that transforms the graph of

$y = x^2$ to the graph of

$y = 3x^2 - 6x + 5$.

41. A function *f* is defined by *[6 marks]*

$f(x) = \frac{2x-3}{x-1}$, $x \neq 1$.

(a) Find an expression for

$f^{-1}(x)$.

(b) Solve the equation

$|f^{-1}(x)| = 1 + f^{-1}(x)$.

42. (a) Find the solution of the equation *[6 marks]*

$$\ln 2^{4x-1} = \ln 8^{x+5} + \log_2 16^{1-2x},$$

expressing your answer in terms of

$\ln 2$.

(b) Using this value of *x*, find the value of *a* for which

$\log_a x = 2$, giving your answer to three decimal places.

43. (a) Simplify the difference of binomial coefficients *[6 marks]*

$$\binom{n}{3} - \binom{2n}{2}, \text{ where } n \geqslant 3.$$

(b) Hence, solve the inequality

$$\binom{n}{3} - \binom{2n}{2} > 32n, \text{ where } n \geqslant 3.$$

44. The function f is defined by [14 marks]

$$f(x) = (x^3 + 6x^2 + 3x - 10)^{\frac{1}{2}}\text{ , for } x \in D,$$

where

$D \subseteq \mathbb{R}$ is the greatest possible domain of f.

(a) Find the roots of

$f(x) = 0.$

(b) Hence specify the set D.

(c) Find the coordinates of the local maximum on the graph

$y = f(x).$

(d) Solve the equation

$f(x) = 3.$

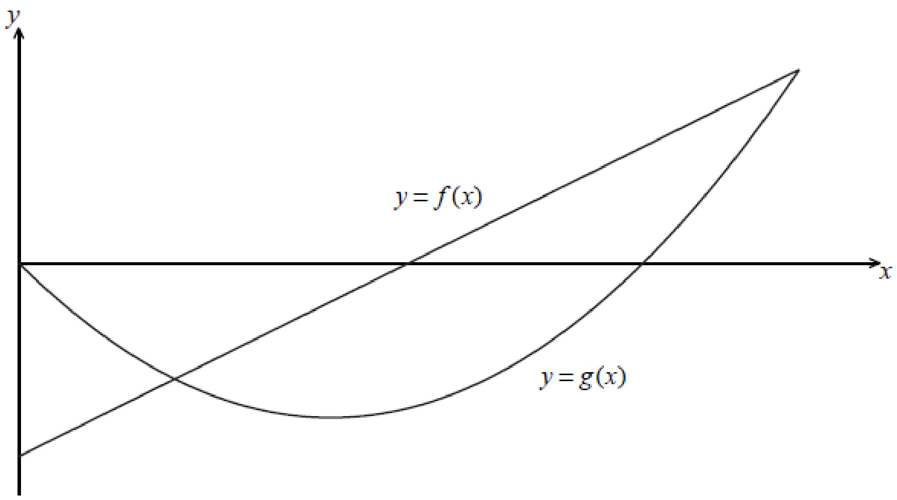
(e) Sketch the graph of

$|y| = f(x),\text{ for } x \in D.$

(f) Find the area of the region completely enclosed by the graph of

$|y| = f(x)$

45. The diagram shows the graphs of a linear function f and a quadratic function g. [5 marks]



On the same axes sketch the graph of

$\frac{f}{g}$. Indicate clearly where the x-intercept and the asymptotes occur.

46. Find the values of [5 marks]
k such that the equation
$x^3 + x^2 - x + 2 = k$ has three distinct real solutions.

Consider the function g, where $g(x) = \frac{3x}{5+x^2}$.

47. (a) Given that the domain of g is $x \geqslant a$, find the least value of a such that g has an inverse function. *[8 marks]*

(b) On the same set of axes, sketch

(i) the graph of g for this value of a;

(ii) the corresponding inverse, g^{-1}.

(c) Find an expression for $g^{-1}(x)$.

48. Let $f(x) = \frac{1-x}{1+x}$ and $g(x) = \sqrt{x+1}$, $x > -1$. *[7 marks]*

Find the set of values of x for which $f'(x) \leqslant f(x) \leqslant g(x)$.

49a. The graph of $y = \ln(x)$ is transformed into the graph of $y = \ln(2x+1)$. *[2 marks]*

Describe two transformations that are required to do this.

49b. Solve $\ln(2x+1) > 3\cos(x)$, $x \in [0, 10]$. *[4 marks]*