Topic 2 Part 2 *[375 marks]*

1a. Express *[2 marks]*

$4x^2 - 4x + 5$ in the form

$a(x - h)^2 + k$ where a, h,

$k \in \mathbb{Q}$.

1b. The graph of *[3 marks]*

$y = x^2$ is transformed onto the graph of

$y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear.

The function f is defined by

$f(x) = \frac{1}{4x^2 - 4x + 5}$.

1c. Sketch the graph of *[2 marks]*

$y = f(x)$.

1d. Find the range of f. *[2 marks]*

1e. By using a suitable substitution show that *[3 marks]*

$\int f(x)\mathrm{d}x = \frac{1}{4}\int \frac{1}{u^2 + 1}\mathrm{d}u$.

1f. Prove that *[7 marks]*

$\int_1^{3.5} \frac{1}{4x^2 - 4x + 5}\mathrm{d}x = \frac{\pi}{16}$.

2. A polynomial *[7 marks]*

$p(x)$ with real coefficients is of degree five. The equation

$p(x) = 0$ has a complex root 2 + i. The graph of

$y = p(x)$ has the x-axis as a tangent at (2, 0) and intersects the coordinate axes at (–1, 0) and (0, 4).

Find

$p(x)$ in factorised form with real coefficients.

3a. Prove that the equation *[4 marks]*

$3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of

$k \in \mathbb{R}$.

3b. Find the value of k for which the two roots of the equation are closest together. *[3 marks]*

A particle, A, is moving along a straight line. The velocity,

v_A ms^{-1}, of A t seconds after its motion begins is given by

$$v_A = t^3 - 5t^2 + 6t.$$

4a. Sketch the graph of $v_A = t^3 - 5t^2 + 6t$ for $t \geqslant 0$, with v_A on the vertical axis and t on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the t-axis. *[3 marks]*

4b. Write down the times for which the velocity of the particle is increasing. *[2 marks]*

4c. Write down the times for which the magnitude of the velocity of the particle is increasing. *[3 marks]*

4d. At $t = 0$ the particle is at point O on the line. *[3 marks]*

Find an expression for the particle's displacement, x_A m, from O at time t.

4e. A second particle, B, moving along the same line, has position x_B m, velocity v_B ms^{-1} and acceleration, a_B ms^{-2}, where $a_B = -2v_B$ for $t \geqslant 0$. At $t = 0,\ x_B = 20$ and $v_B = -20$. *[4 marks]*

Find an expression for v_B in terms of t.

4f. Find the value of t when the two particles meet. *[6 marks]*

Given the complex numbers

$z_1 = 1 + 3\mathrm{i}$ and

$z_2 = -1 - \mathrm{i}$.

5a. Write down the exact values of $|z_1|$ and $\arg(z_2)$. *[2 marks]*

5b. Find the minimum value of $|z_1 + \alpha z_2|$, where $\alpha \in \mathbb{R}$. *[5 marks]*

The function f is given by

$f(x) = \frac{3^x + 1}{3^x - 3^{-x}}$, for $x > 0$.

6a. Show that $f(x) > 1$ for all $x > 0$. *[3 marks]*

6b. Solve the equation $f(x) = 4$. *[4 marks]*

The function f is defined by

$f(x) = \frac{2x-1}{x+2}$, with domain $D = \{x : -1 \leqslant x \leqslant 8\}$.

7a. Express $f(x)$ in the form $A + \frac{B}{x+2}$, where A and $B \in \mathbb{Z}$. *[2 marks]*

7b. Hence show that $f'(x) > 0$ on D. *[2 marks]*

7c. State the range of f. *[2 marks]*

7d. (i) Find an expression for $f^{-1}(x)$. *[8 marks]*

(ii) Sketch the graph of $y = f(x)$, showing the points of intersection with both axes.

(iii) On the same diagram, sketch the graph of $y = f'(x)$.

7e. (i) On a different diagram, sketch the graph of [7 marks]

$y = f(|x|)$ where

$x \in D$.

(ii) Find all solutions of the equation

$f(|x|) = -\frac{1}{4}$.

8a. (i) Express each of the complex numbers [9 marks]

$z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and

$z_3 = -2\mathrm{i}$ in modulus-argument form.

(ii) Hence show that the points in the complex plane representing

z_1,

z_2 and

z_3 form the vertices of an equilateral triangle.

(iii) Show that

$\mathrm{z}_1^{3n} + z_2^{3n} = 2z_3^{3n}$ where

$n \in \mathbb{N}$.

8b. (i) State the solutions of the equation [9 marks]

$z^7 = 1$ for

$z \in \mathbb{C}$, giving them in modulus-argument form.

(ii) If w is the solution to

$z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of

π.

(iii) Show that

$z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial

$z^7 - 1$. State the two other quadratic factors with real coefficients.

The function f is defined as
$f(x) = -3 + \frac{1}{x-2}$, $x \neq 2$.

9a. (i) Sketch the graph of [4 marks]
$y = f(x)$, clearly indicating any asymptotes and axes intercepts.
(ii) Write down the equations of any asymptotes and the coordinates of any axes intercepts.

9b. Find the inverse function [4 marks]
f^{-1}, stating its domain.

Particle A moves such that its velocity v ms^{-1}, at time t seconds, is given by $v(t) = \frac{t}{12+t^4}$, $t \geqslant 0$.

10a. Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. *[2 marks]*

10b. Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4} \mathrm{d}t$. *[4 marks]*

10c. Find the exact distance travelled by particle A between $t = 0$ and $t = 6$ seconds. *[3 marks]*

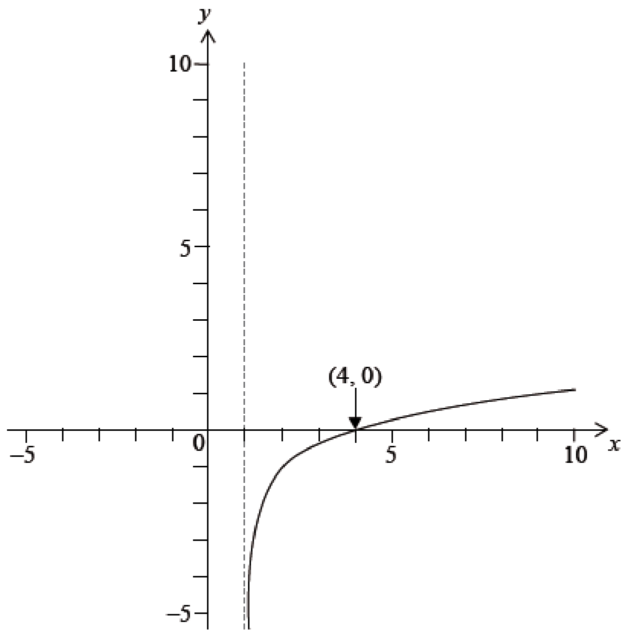
Give your answer in the form $k\arctan(b)$, k, $b \in \mathbb{R}$.

Particle B moves such that its velocity v ms^{-1} is related to its displacement s m, by the equation $v(s) = \arcsin(\sqrt{s})$.

10d. Find the acceleration of particle B when $s = 0.1$ m. *[3 marks]*

11. The cubic polynomial $3x^3 + px^2 + qx - 2$ has a factor $(x+2)$ and leaves a remainder 4 when divided by $(x+1)$. Find the value of p and the value of q. *[5 marks]*

The diagram below shows a sketch of the graph of $y = f(x)$.



12a. Sketch the graph of $y = f^{-1}(x)$ on the same axes. *[2 marks]*

12b. State the range of f^{-1}. *[1 mark]*

12c. Given that $f(x) = \ln(ax + b),\ x > 1$, find the value of a and the value of b. *[4 marks]*

13. Solve the following equations: *[8 marks]*

(a) $\log_2(x-2) = \log_4(x^2 - 6x + 12)$;

(b) $x^{\ln x} = e^{(\ln x)^3}$.

The function f is given by $f(x) = xe^{-x}\ (x \geqslant 0)$.

14a. (i) Find an expression for $f'(x)$. *[3 marks]*

(ii) Hence determine the coordinates of the point A, where $f'(x) = 0$.

14b. Find an expression for $f''(x)$ and hence show the point A is a maximum. *[3 marks]*

14c. Find the coordinates of B, the point of inflexion. *[2 marks]*

14d. The graph of the function g is obtained from the graph of f by stretching it in the x-direction by a scale factor 2. *[5 marks]*

(i) Write down an expression for $g(x)$.

(ii) State the coordinates of the maximum C of g.

(iii) Determine the x-coordinates of D and E, the two points where $f(x) = g(x)$.

14e. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, showing clearly the points A, B, C, D and E. *[4 marks]*

14f. Find an exact value for the area of the region bounded by the curve $y = g(x)$, the x-axis and the line $x = 1$. *[3 marks]*

Consider $f(x) = \ln x - e^{\cos x},\ 0 < x \leqslant 10$.

15a. Sketch the graph of $y = f(x)$, stating the coordinates of any maximum and minimum points and points of intersection with the x-axis. *[5 marks]*

15b. Solve the inequality $\ln x \leqslant e^{\cos x},\ 0 < x \leqslant 10$. *[2 marks]*

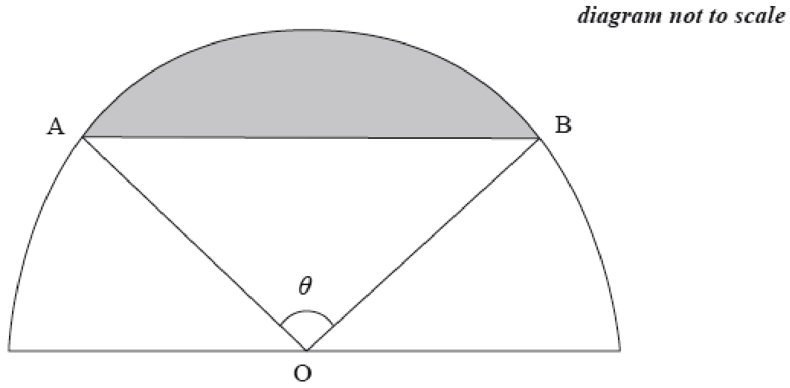
The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are such that $\boldsymbol{a} = (3\cos\theta + 6)\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{b} = (\cos\theta - 2)\boldsymbol{i} + (1 + \sin\theta)\boldsymbol{j}$.

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular,

16a. show that $3\sin^2\theta - 7\sin\theta + 2 = 0$; *[3 marks]*

16b. find the smallest possible positive value of θ. *[3 marks]*

The diagram below shows a semi-circle of diameter 20 cm, centre O and two points A and B such that $\mathrm{A\hat{O}B} = \theta$, where θ is in radians.



17a. Show that the shaded area can be expressed as $50\theta - 50\sin\theta$. *[2 marks]*

17b. Find the value of θ for which the shaded area is equal to half that of the unshaded area, giving your answer correct to four significant figures. *[3 marks]*

The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8.

18a. Find the first term and the common difference. *[4 marks]*

18b. Find the smallest value of n such that the sum of the first n terms is greater than 600. *[3 marks]*

19a. Sketch the curve $y = \frac{\cos x}{\sqrt{x^2+1}},\ -4 \leqslant x \leqslant 4$ showing clearly the coordinates of the x-intercepts, any maximum points and any minimum points. *[4 marks]*

19b. Write down the gradient of the curve at $x = 1$. *[1 mark]*

19c. Find the equation of the normal to the curve at $x = 1$. *[3 marks]*

A particle moves in a straight line with velocity v metres per second. At any time t seconds, $0 \leqslant t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} + v^2 + 1 = 0$.
It is also given that $v = 1$ when $t = 0$.

20a. Find an expression for v in terms of t . *[7 marks]*

20b. Sketch the graph of v against t , clearly showing the coordinates of any intercepts, and the equations of any asymptotes. *[3 marks]*

20c. (i) Write down the time T at which the velocity is zero. *[3 marks]*
(ii) Find the distance travelled in the interval $[0, T]$.

20d. Find an expression for s , the displacement, in terms of t , given that $s = 0$ when $t = 0$. *[5 marks]*

20e. Hence, or otherwise, show that $s = \frac{1}{2}\ln\frac{2}{1+v^2}$. *[4 marks]*

The quadratic function

$f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

21a. Find the value of p and the value of q . *[4 marks]*

21b. The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph. *[2 marks]*

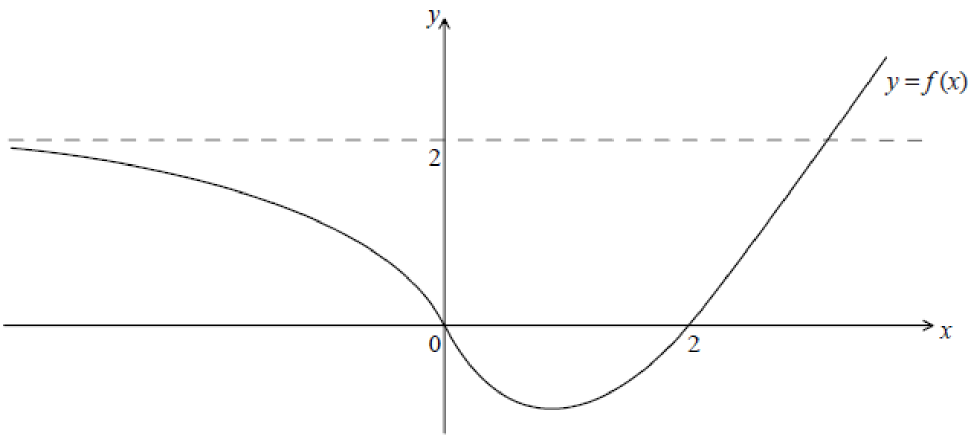
The random variable X has probability density function f where

$$f(x) = \begin{cases} kx(x+1)(2-x), & 0 \leqslant x \leqslant 2 \\ 0, & \text{otherwise .} \end{cases}$$

22a. Sketch the graph of the function. You are not required to find the coordinates of the maximum. *[1 mark]*

22b. Find the value of k . *[5 marks]*

The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 2$.



23a. Sketch the graph of *[3 marks]*

$y = \frac{1}{f(x)}$.

23b. Sketch the graph of *[3 marks]*

$y = x\ f(x)$.

24. A function is defined by *[6 marks]*

$h(x) = 2e^x - \frac{1}{e^x},\ x \in \mathbb{R}$. Find an expression for

$h^{-1}(x)$.

25a. Factorize

$z^3 + 1$ into a linear and quadratic factor. *[2 marks]*

25b. Let *[9 marks]*

$\gamma = \frac{1+\mathrm{i}\sqrt{3}}{2}$.

(i) Show that

γ is one of the cube roots of -1.

(ii) Show that

$\gamma^2 = \gamma - 1$.

(iii) Hence find the value of

$(1-\gamma)^6$.

26a. (i) Sketch the graphs of *[9 marks]*

$y = \sin x$ and

$y = \sin 2x$, on the same set of axes, for

$0 \leqslant x \leqslant \frac{\pi}{2}$.

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain

$0 \leqslant x \leqslant \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

26b. Find the value of *[8 marks]*

$\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x$ using the substitution

$x = 4\sin^2\theta$.

26c. The increasing function *f* satisfies *[8 marks]*

$f(0) = 0$ and

$f(a) = b$, where

$a > 0$ and

$b > 0$.

(i) By reference to a sketch, show that

$\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of

$\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

27. Sketch the graph of *[7 marks]*

$f(x) = x + \frac{8x}{x^2-9}$. Clearly mark the coordinates of the two maximum points and the two minimum points. Clearly mark and state the equations of the vertical asymptotes and the oblique asymptote.

Consider the functions

$f(x) = x^3 + 1$ and

$g(x) = \frac{1}{x^3+1}$. The graphs of

$y = f(x)$ and

$y = g(x)$ meet at the point (0, 1) and one other point, P.

28a. Find the coordinates of P. *[1 mark]*

28b. Calculate the size of the acute angle between the tangents to the two graphs at the point P. *[4 marks]*

29. Find the set of values of x for which *[4 marks]*

$|x-1| > |2x-1|$.

30. Consider the function *[8 marks]*

$f : x \to \sqrt{\frac{\pi}{4} - \arccos x}$.

(a) Find the largest possible domain of f.

(b) Determine an expression for the inverse function,

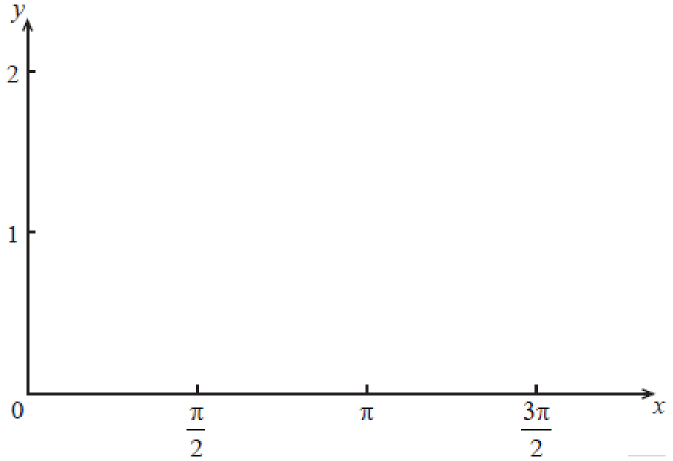
f^{-1}, and write down its domain.

Given that

$f(x) = 1 + \sin x,\ 0 \leqslant x \leqslant \frac{3\pi}{2}$,

31a. sketch the graph of *[1 mark]*

f;



31b. show that *[1 mark]*

$(f(x))^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$;

31c. find the volume of the solid formed when the graph of f is rotated through *[4 marks]*

2π radians about the x-axis.

Consider the equation

$yx^2 + (y-1)x + (y-1) = 0$.

32a. Find the set of values of y for which this equation has real roots. *[4 marks]*

32b. Hence determine the range of the function *[3 marks]*

$f : x \to \frac{x+1}{x^2+x+1}$.

32c. Explain why *f* has no inverse. *[1 mark]*

Consider the graph of

$y = x + \sin(x - 3),\ -\pi \leqslant x \leqslant \pi$.

33a. Sketch the graph, clearly labelling the *x* and *y* intercepts with their values. *[3 marks]*

33b. Find the area of the region bounded by the graph and the *x* and *y* axes. *[2 marks]*

Given that

$f(x) = \frac{1}{1+e^{-x}}$,

34a. find *[6 marks]*

$f^{-1}(x)$, stating its domain;

34b. find the value of *x* such that *[1 mark]*

$f(x) = f^{-1}(x)$.

Consider the functions given below.

$$f(x) = 2x + 3$$

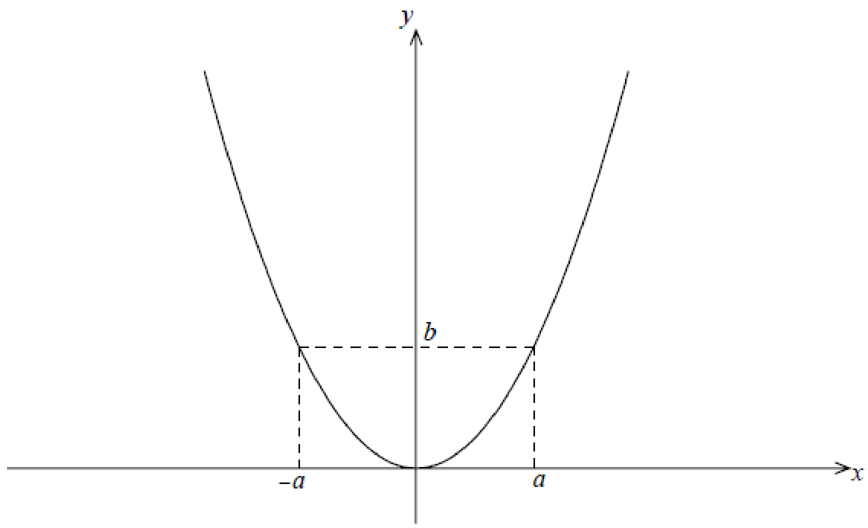
$$g(x) = \frac{1}{x}, x \neq 0$$

35a. (i) Find *[2 marks]*
$(g \circ f)(x)$ and write down the domain of the function.

(ii) Find
$(f \circ g)(x)$ and write down the domain of the function.

35b. Find the coordinates of the point where the graph of *[4 marks]*
$y = f(x)$ and the graph of
$y = \left(g^{-1} \circ f \circ g\right)(x)$ intersect.

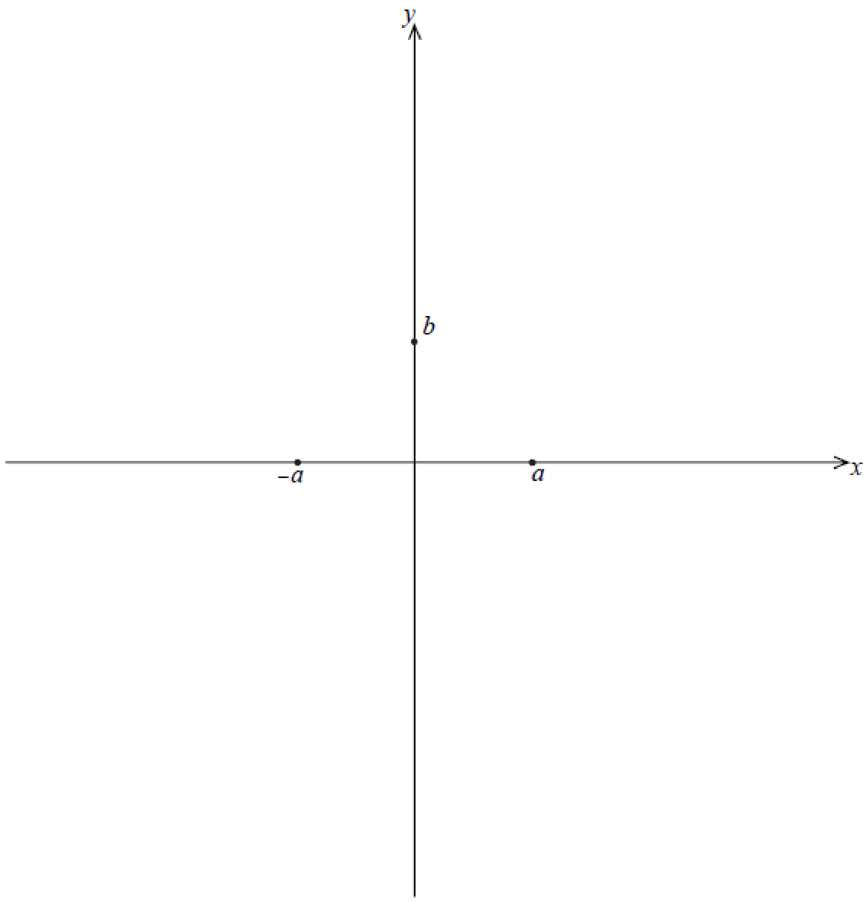
The diagram below shows the graph of the function
$y = f(x)$, defined for all
$x \in \mathbb{R}$,
where
$b > a > 0$.



Consider the function
$g(x) = \frac{1}{f(x-a)-b}$.

36a. Find the largest possible domain of the function
g . *[2 marks]*

36b. On the axes below, sketch the graph of
$y = g(x)$. On the graph, indicate any asymptotes and local maxima or minima, and write down their equations and coordinates. *[6 marks]*



Consider the function
$f(x) = \frac{\ln x}{x}$,
$0 < x < \mathrm{e}^2$.

37a. (i) Solve the equation
$f'(x) = 0$. [5 marks]

(ii) Hence show the graph of
f has a local maximum.

(iii) Write down the range of the function
f .

37b. Show that there is a point of inflexion on the graph and determine its coordinates. [5 marks]

37c. Sketch the graph of
$y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum. [3 marks]

37d. Now consider the functions
$g(x) = \frac{\ln|x|}{x}$ and
$h(x) = \frac{\ln|x|}{|x|}$, where
$0 < x < \mathrm{e}^2$. [6 marks]

(i) Sketch the graph of
$y = g(x)$.

(ii) Write down the range of
g .

(iii) Find the values of
x such that
$h(x) > g(x)$.

The function
$f(x) = 4x^3 + 2ax - 7a$,
$a \in \mathbb{R}$, leaves a remainder of
-10 when divided by
$(x - a)$.

38a. Find the value of
a . [3 marks]

38b. Show that for this value of
a there is a unique real solution to the equation
$f(x) = 0$. [2 marks]

39a. Write down the quadratic expression
$2x^2 + x - 3$ as the product of two linear factors. [1 mark]

39b. Hence, or otherwise, find the coefficient of
x in the expansion of
$\left(2x^2 + x - 3\right)^8$. [4 marks]