Topic 5 Part 3 *[262 marks]*

A mathematics test is given to a class of 20 students. One student scores 0, but all the other students score 10.

1a. Find the mean score for the class. *[2 marks]*

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1b. Write down the median score. *[1 mark]*

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1c. Write down the number of students who scored *[2 marks]*

(i) above the mean score;

(ii) below the median score.

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2. A football team, Melchester Rovers are playing a tournament of five matches. *[6 marks]*

The probabilities that they win, draw or lose a match are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively.

These probabilities remain constant; the result of a match is independent of the results of other matches. At the end of the tournament their coach Roy loses his job if they lose three **consecutive** matches, otherwise he does not lose his job. Find the probability that Roy loses his job.

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A and B are two events such that $\mathrm{P}(A) = 0.25,\ \mathrm{P}(B) = 0.6$ and $\mathrm{P}(A \cup B) = 0.7$.

3a. Find $\mathrm{P}(A \cap B)$. *[2 marks]*

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3b. Determine whether events A and B are independent. *[2 marks]*

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The finishing times in a marathon race follow a normal distribution with mean 210 minutes and standard deviation 22 minutes.

4a. Find the probability that a runner finishes the race in under three hours. *[2 marks]*

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4b. The fastest 90% of the finishers receive a certificate. *[2 marks]*

Find the time, below which a competitor has to complete the race, in order to gain a certificate.

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A mosaic is going to be created by randomly selecting 1000 small tiles, each of which is either black or white. The probability that a tile is white is 0.1. Let the random variable W be the number of white tiles.

5a. State the distribution of W, including the values of any parameters. *[2 marks]*

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5b. Write down the mean of W. *[1 mark]*

5c. Find $\mathrm{P}(W > 89)$. *[2 marks]*

The random variable X follows a Poisson distribution with mean $m \neq 0$.

6a. Given that $2\mathrm{P}(X = 4) = \mathrm{P}(X = 5)$, show that $m = 10$. *[3 marks]*

6b. Given that $X \leq 11$, find the probability that $X = 6$. *[4 marks]*

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The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 0,\ x < 0 \\ \frac{\sin x}{4},\ 0 \leq x \leq \pi \\ a(x-\pi),\ \pi < x \leq 2\pi \\ 0,\ 2\pi < x \end{cases}.$$

7a. Sketch the graph $y = f(x)$. *[2 marks]*

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7b. Find $\mathrm{P}(X \leq \pi)$. *[2 marks]*

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7c. Show that $a = \frac{1}{\pi^2}$. *[3 marks]*

7d. Write down the median of X. *[1 mark]*

7e. Calculate the mean of X. *[3 marks]*

7f. Calculate the variance of X. *[3 marks]*

7g. Find $\mathrm{P}\left(\frac{\pi}{2} \leq X \leq \frac{3\pi}{2}\right)$. *[2 marks]*

7h. Given that $\frac{\pi}{2} \leq X \leq \frac{3\pi}{2}$ find the probability that $\pi \leq X \leq 2\pi$. *[4 marks]*

Emma acquires a new cell phone for her birthday and receives texts from her friends. It is assumed that the daily number of texts Emma receives follows a Poisson distribution with mean $m = 5$.

8a. (i) Find the probability that on a certain day Emma receives more than 7 texts. *[4 marks]*

(ii) Determine the expected number of days in a week on which Emma receives more than 7 texts.

8b. Find the probability that Emma receives fewer than 30 texts during a week. *[3 marks]*

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Natasha lives in Chicago and has relatives in Nashville and St. Louis.

Each time she visits her relatives, she either flies or drives.

When travelling to Nashville, the probability that she drives is $\frac{4}{5}$, and when travelling to St. Louis, the probability that she flies is $\frac{1}{3}$.

Given that the probability that she drives when visiting her relatives is $\frac{13}{18}$, find the probability that for a particular trip,

9a. she travels to Nashville; *[3 marks]*

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9b. she is on her way to Nashville, given that she is flying. *[3 marks]*

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10a. Farmer Suzie grows turnips and the weights of her turnips are normally distributed with a mean of $122g$ and standard deviation of $14.7g$. *[6 marks]*

(i) Calculate the percentage of Suzie's turnips that weigh between $110g$ and $130g$.

(ii) Suzie has 100 turnips to take to market. Find the expected number weighing more than $130g$.

(iii) Find the probability that at least 30 of the $100g$ turnips weigh more than $130g$.

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10b. Farmer Ray also grows turnips and the weights of his turnips are normally distributed with a mean of $144g$. Ray only takes to market turnips that weigh more than $130g$. Over a period of time, Ray finds he has to reject 1 in 15 turnips due to their being underweight. *[6 marks]*

(i) Find the standard deviation of the weights of Ray's turnips.

(ii) Ray has 200 turnips to take to market. Find the expected number weighing more than $150g$.

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Engine oil is sold in cans of two capacities, large and small. The amount, in millilitres, in each can, is normally distributed according to Large $\sim \mathrm{N}(5000,\ 40)$ and Small $\sim \mathrm{N}(1000,\ 25)$.

11a. A large can is selected at random. Find the probability that the can contains at least 4995 millilitres of oil. *[2 marks]*

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11b. A large can and a small can are selected at random. Find the probability that the large can contains at least 30 milliliters more than five times the amount contained in the small can. *[6 marks]*

11c. A large can and five small cans are selected at random. Find the probability that the large can contains at least 30 milliliters less than the total amount contained in the small cans. *[5 marks]*

12a. Determine the probability generating function for $X \sim \mathrm{B}(1,\ p)$. *[4 marks]*

12b. Explain why the probability generating function for $\mathrm{B}(n,\ p)$ is a polynomial of degree n. *[2 marks]*

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12c. Two independent random variables X_1 and X_2 are such that $X_1 \sim \mathrm{B}(1,\ p_1)$ and $X_2 \sim \mathrm{B}(1,\ p_2)$. Prove that if $X_1 + X_2$ has a binomial distribution then $p_1 = p_2$. *[5 marks]*

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13. Four numbers are such that their mean is 13, their median is 14 and their mode is 15. Find the four numbers. *[4 marks]*

14. A student sits a national test and is told that the marks follow a normal distribution with mean 100. The student receives a mark of 124 and is told that he is at the 68^{th} percentile. *[5 marks]*

Calculate the variance of the distribution.

15a. Find the term in x^5 in the expansion of $(3x + A)(2x + B)^6$. *[4 marks]*

15b. Mina and Norbert each have a fair cubical die with faces labelled 1, 2, 3, 4, 5 and 6; they throw it to decide if they are going to eat a cookie. *[4 marks]*

Mina throws her die just once and she eats a cookie if she throws a four, a five or a six.

Norbert throws his die six times and each time eats a cookie if he throws a five or a six.

Calculate the probability that five cookies are eaten.

The number of birds seen on a power line on any day can be modelled by a Poisson distribution with mean 5.84.

16a. Find the probability that during a certain seven-day week, more than 40 birds have been seen on the power line. *[2 marks]*

16b. On Monday there were more than 10 birds seen on the power line. Show that the probability of there being more than 40 birds seen on the power line from that Monday to the following Sunday, inclusive, can be expressed as: *[5 marks]*

$\frac{P(X>40)+\sum_{r=11}^{40} P(X=r)P(Y>40-r)}{P(X>10)}$ where

$X \sim \mathrm{Po}(5.84)$ and

$Y \sim \mathrm{Po}(35.04)$.

17. A random variable *[21 marks]*

X has probability density function

$$f(x) = \begin{cases} ax+b, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}, a, b \in \mathbb{R}$$

(a) Show that

$5a + 2b = 2$.

Let

$\mathrm{E}(X) = \mu$.

(b) (i) Show that

$a = 12\mu - 30$.

(ii) Find a similar expression for b in terms of

μ.

Let the median of the distribution be 2.3.

(c) (i) Find the value of

μ.

(ii) Find the value of the standard deviation of X.

18. Events *[6 marks]*

A and

B are such that

$\mathrm{P}(A) = \frac{2}{5}$, $\mathrm{P}(B) = \frac{11}{20}$ and

$\mathrm{P}(A|B) = \frac{2}{11}$.

(a) Find

$\mathrm{P}(A \cap B)$.

(b) Find

$\mathrm{P}(A \cup B)$.

(c) State with a reason whether or not events

A and

B are independent.

19a. Mobile phone batteries are produced by two machines. Machine A produces 60% of the daily output and machine B produces 40%. It is found by testing that on average 2% of batteries produced by machine A are faulty and 1% of batteries produced by machine B are faulty. *[6 marks]*

(i) Draw a tree diagram clearly showing the respective probabilities.

(ii) A battery is selected at random. Find the probability that it is faulty.

(iii) A battery is selected at random and found to be faulty. Find the probability that it was produced by machine A.

19b. In a pack of seven transistors, three are found to be defective. Three transistors are selected from the pack at random without replacement. The discrete random variable X represents the number of defective transistors selected. *[6 marks]*

(i) Find

$\mathrm{P}(X = 2)$.

(ii) **Copy** and complete the following table:

x	0	1	2	3
$\mathrm{P}(X = x)$				

(iii) Determine

$\mathrm{E}(X)$.

20. The weights, in kg, of one-year-old bear cubs are modelled by a normal distribution with mean *[5 marks]*

μ and standard deviation

σ.

(a) Given that the upper quartile weight is 21.3 kg and the lower quartile weight is 17.1 kg, calculate the value of

μ and the value of

σ.

A random sample of 100 of these bear cubs is selected.

(b) Find the expected number of bear cubs weighing more than 22 kg.

21. Six customers wait in a queue in a supermarket. A customer can choose to pay with cash or a credit card. Assume that whether or not a customer pays with a credit card is independent of any other customers' methods of payment. *[7 marks]*

It is known that 60% of customers choose to pay with a credit card.

(a) Find the probability that:

(i) the first three customers pay with a credit card and the next three pay with cash;

(ii) exactly three of the six customers pay with a credit card.

There are n customers waiting in another queue in the same supermarket. The probability that at least one customer pays with cash is greater than 0.995.

(b) Find the minimum value of n.

22. The random variable *[4 marks]*

X has a Poisson distribution with mean

μ.

Given that

$\mathrm{P}(X = 2) + \mathrm{P}(X = 3) = \mathrm{P}(X = 5)$,

(a) find the value of

μ;

(b) find the probability that X lies within one standard deviation of the mean.

Events A and B are such that $\mathrm{P}(A) = 0.2$ and $\mathrm{P}(B) = 0.5$.

23a. Determine the value of $\mathrm{P}(A \cup B)$ when *[4 marks]*

(i) A and B are mutually exclusive;

(ii) A and B are independent.

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23b. Determine the range of possible values of $\mathrm{P}\left(A|B\right)$. *[3 marks]*

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A continuous random variable T has probability density function f defined by

$$f(t) = \begin{cases} |2-t|, & 1 \le t \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

24a. Sketch the graph of $y = f(t)$. *[2 marks]*

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24b. Find the interquartile range of T. *[4 marks]*

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The wingspans of a certain species of bird can be modelled by a normal distribution with mean 60.2 cm and standard deviation 2.4 cm.

According to this model, 99% of wingspans are greater than x cm.

25a. Find the value of x. *[2 marks]*

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25b. In a field experiment, a research team studies a large sample of these birds. The wingspans of each bird are measured correct to the nearest 0.1 cm. *[3 marks]*

Find the probability that a randomly selected bird has a wingspan measured as 60.2 cm.

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26. Consider the data set $\{2,\ x,\ y,\ 10,\ 17\}$, $x,\ y \in \mathbb{Z}^{+}$ and $x < y$. *[6 marks]*

The mean of the data set is 8 and its variance is 27.6.

Find the value of x and the value of y.

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The number of complaints per day received by customer service at a department store follows a Poisson distribution with a mean of 0.6.

27a. On a randomly chosen day, find the probability that *[3 marks]*

(i) there are no complaints;

(ii) there are at least three complaints.

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27b. In a randomly chosen five-day week, find the probability that there are no complaints. *[2 marks]*

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27c. On a randomly chosen day, find the most likely number of complaints received. *[3 marks]*

Justify your answer.

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27d. The department store introduces a new policy to improve customer service. The number of complaints received per day now follows a Poisson distribution with mean λ. *[2 marks]*

On a randomly chosen day, the probability that there are no complaints is now 0.8.

Find the value of λ.

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Ava and Barry play a game with a bag containing one green marble and two red marbles. Each player in turn randomly selects a marble from the bag, notes its colour and replaces it. Ava wins the game if she selects a green marble. Barry wins the game if he selects a red marble. Ava starts the game.

28a. Find the probability that Ava wins on her first turn. *[1 mark]*

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28b. Find the probability that Barry wins on his first turn. *[2 marks]*

28c. Find the probability that Ava wins in one of her first three turns. *[4 marks]*

28d. Find the probability that Ava eventually wins. *[4 marks]*

A random variable X has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ \frac{1}{4} & 1 \leq x < 3 \\ 0 & x \geq 3 \end{cases}$$

29a. Sketch the graph of $y = f(x)$. *[1 mark]*

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29b. Find the cumulative distribution function for X. *[5 marks]*

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29c. Find the interquartile range for X. *[3 marks]*

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If X is a random variable that follows a Poisson distribution with mean $\lambda > 0$ then the probability generating function of X is $G(t) = e^{\lambda(t-1)}$.

30a. (i) Prove that $\mathrm{E}(X) = \lambda$. *[6 marks]*

(ii) Prove that $\mathrm{Var}(X) = \lambda$.

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30b. Y is a random variable, independent of X, that also follows a Poisson distribution with mean λ. *[3 marks]*

If $S = 2X - Y$ find

(i) $\mathrm{E}(S)$;

(ii) $\mathrm{Var}(S)$.

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30c. Let $T = \frac{Y}{2} + \frac{Y}{2}$. *[3 marks]*

(i) Show that T is an unbiased estimator for λ.

(ii) Show that T is a more efficient unbiased estimator of λ than S.

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30d. Could either S or T model a Poisson distribution? Justify your answer. *[1 mark]*

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30e. By consideration of the probability generating function, $G_{X+Y}(t)$, of $X+Y$, prove that $X+Y$ follows a Poisson distribution with mean 2λ. *[3 marks]*

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30f. Find *[2 marks]*

(i) $G_{X+Y}(1)$;

(ii) $G_{X+Y}(-1)$.

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30g. Hence find the probability that $X+Y$ is an even number. *[3 marks]*

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Two species of plant, A and B, are identical in appearance though it is known that the mean length of leaves from a plant of species A is 5.2 cm, whereas the mean length of leaves from a plant of species B is 4.6 cm. Both lengths can be modelled by normal distributions with standard deviation 1.2 cm.

In order to test whether a particular plant is from species A or species B, 16 leaves are collected at random from the plant. The length, x, of each leaf is measured and the mean length evaluated. A one-tailed test of the sample mean, $\bar{X}$, is then performed at the 5% level, with the hypotheses: $H_0 : \mu = 5.2$ and $H_1 : \mu < 5.2$.

31a. Find the critical region for this test. *[3 marks]*

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31b. It is now known that in the area in which the plant was found 90% of all the plants are of species A and 10% are of species B. *[2 marks]*

Find the probability that $\bar{X}$ will fall within the critical region of the test.

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31c. If, having done the test, the sample mean is found to lie within the critical region, find the probability that the leaves came from a plant of species A. *[3 marks]*

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