Topic 8 Part 1 *[382 marks]*

A group with the binary operation of multiplication modulo 15 is shown in the following Cayley table.

$\times_{15}$	**1**	**2**	**4**	**7**	**8**	**11**	**13**	**14**
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	a	b	c
13	13	11	7	1	14	d	e	f
14	14	13	11	8	7	g	h	i

1a. Find the values represented by each of the letters in the table. *[3 marks]*

1b. Find the order of each of the elements of the group. *[3 marks]*

1c. Write down the three sets that form subgroups of order 2. *[2 marks]*

1d. Find the three sets that form subgroups of order 4. *[4 marks]*

Define $f : \mathbb{R}\backslash\{0.5\} \to \mathbb{R}$ by $f(x) = \frac{4x+1}{2x-1}$.

2a. Prove that f is an injection. *[4 marks]*

2b. Prove that f is not a surjection. *[4 marks]*

. . . .

. . . .

. . . .

Consider the set A consisting of all the permutations of the integers $1, 2, 3, 4, 5$.

3a. Two members of A are given by $p = (1\ 2\ 5)$ and $q = (1\ 3)(2\ 5)$. *[4 marks]*

Find the single permutation which is equivalent to $q \circ p$.

. . . .

. . . .

. . . .

3b. State a permutation belonging to A of order *[3 marks]*

(i) 4;

(ii) 6.

. . . .

. . . .

. . . .

3c. Let $P = \{$all permutations in A where exactly two integers change position$\}$, *[4 marks]*

and $Q = \{$all permutations in A where the integer 1 changes position$\}$.

(i) List all the elements in $P \cap Q$.

(ii) Find $n(P \cap Q')$.

. . . .

. . . .

. . . .

The group $\{G, *\}$ has identity e_G and the group $\{H, \circ\}$ has identity e_H. A homomorphism f is such that $f : G \to H$. It is given that $f(e_G) = e_H$.

4a. Prove that for all $a \in G,\ f(a^{-1}) = (f(a))^{-1}$. *[4 marks]*

. . . .

. . . .

. . . .

4b. Let $\{H, \circ\}$ be the cyclic group of order seven, and let p be a generator. *[4 marks]*

Let $x \in G$ such that $f(x) = p^2$.

Find $f(x^{-1})$.

. . . .

. . . .

. . . .

4c. Given that $f(x * y) = p$, find $f(y)$. *[4 marks]*

. . . .

. . . .

. . . .

5a. State Lagrange's theorem. *[2 marks]*

. . . .

. . . .

. . . .

$\{G,\ *\}$ is a group with identity element e. Let $a,\ b \in G$.

5b. Verify that the inverse of $a * b^{-1}$ is equal to $b * a^{-1}$. *[3 marks]*

. . . .

. . . .

. . . .

5c. Let $\{H, *\}$ be a subgroup of $\{G, *\}$. Let R be a relation defined on G by *[8 marks]*

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

Prove that R is an equivalence relation, indicating clearly whenever you are using one of the four properties required of a group.

. . . .

. . . .

. . . .

5d. Let $\{H, *\}$ be a subgroup of $\{G, *\}$ .Let R be a relation defined on G by *[3 marks]*

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

Show that $aRb \Leftrightarrow a \in Hb$, where Hb is the right coset of H containing b.

. . . .

. . . .

. . . .

5e. Let $\{H, *\}$ be a subgroup of $\{G, *\}$ .Let R be a relation defined on G by *[3 marks]*

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

It is given that the number of elements in any right coset of H is equal to the order of H.

Explain how this fact together with parts (c) and (d) prove Lagrange’s theorem.

. . . .

. . . .

. . . .

Consider the set $S_3 = \{p,\ q,\ r,\ s,\ t,\ u\}$ of permutations of the elements of the set $\{1,\ 2,\ 3\}$, defined by

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ r = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},\ s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ u = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Let $\circ$ denote composition of permutations, so $a \circ b$ means b followed by a. You may assume that $(S_3,\ \circ)$ forms a group.

6a. Complete the following Cayley table *[4 marks]*

$\circ$	p	q	r	s	t	u
p						
q			t			s
r		u		t	s	q
s		t	u			r
t		s	q	r		
u		r	s	q		

[5 marks]

. . . .

. . . .

. . . .

6b. (i) State the inverse of each element. *[6 marks]*

(ii) Determine the order of each element.

. . . .

. . . .

. . . .

6c. Write down the subgroups containing [2 marks]

(i) r,

(ii) u.

The binary operation $*$ is defined for $x,\ y \in S = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ by

$$x * y = (x^3 y - xy) \bmod 7.$$

7a. Find the element e such that $e * y = y$, for all $y \in S$. [2 marks]

7b. (i) Find the least solution of $x * x = e$. [5 marks]

(ii) Deduce that $(S,\ *)$ is not a group.

7c. Determine whether or not e is an identity element. *[3 marks]*

The relation R is defined on $\mathbb{Z}$ by xRy if and only if $x^2y \equiv y \bmod 6$.

8a. Show that the product of three consecutive integers is divisible by 6. *[2 marks]*

8b. Hence prove that R is reflexive. *[3 marks]*

8c. Find the set of all y for which $5Ry$. *[3 marks]*

8d. Find the set of all y for which $3Ry$. *[2 marks]*

8e. Using your answers for (c) and (d) show that R is not symmetric. *[2 marks]*

Let X and Y be sets. The functions $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f$ is the identity function on X.

9a. Prove that: *[6 marks]*

(i) f is an injection,

(ii) g is a surjection.

9b. Given that $X = \mathbb{R}^+ \cup \{0\}$ and $Y = \mathbb{R}$, choose a suitable pair of functions f and g to show that g is not necessarily a bijection. *[3 marks]*

Consider the sets

$$G = \left\{ \frac{n}{6^i} | n \in \mathbb{Z},\ i \in \mathbb{N} \right\},\ H = \left\{ \frac{m}{3^j} | m \in \mathbb{Z},\ j \in \mathbb{N} \right\}.$$

10a. Show that $(G,\ +)$ forms a group where $+$ denotes addition on $\mathbb{Q}$. Associativity may be assumed. *[5 marks]*

10b. Assuming that $(H, +)$ forms a group, show that it is a proper subgroup of $(G, +)$. *[4 marks]*

. . . .

. . . .

. . . .

10c. The mapping $\phi : G \to G$ is given by $\phi(g) = g + g$, for $g \in G$. *[7 marks]*

Prove that ϕ is an isomorphism.

. . . .

. . . .

. . . .

The binary operation

$*$ is defined on

$\mathbb{N}$ by

$a * b = 1 + ab$.

Determine whether or not

$*$

11a. is closed; *[2 marks]*

11b. is commutative; *[2 marks]*

11c. is associative; *[3 marks]*

11d. has an identity element. *[3 marks]*

Consider the set $S = \{1, 3, 5, 7, 9, 11, 13\}$ under the binary operation multiplication modulo 14 denoted by $\times_{14}$.

12a. Copy and complete the following Cayley table for this binary operation. *[4 marks]*

$\times_{14}$	**1**	**3**	**5**	**7**	**9**	**11**	**13**
1	1	3	5	7	9	11	13
3	3				13	5	11
5	5				3	13	9
7	7						
9	9	13	3				
11	11	5	13				
13	13	11	9				

12b. Give one reason why *[1 mark]*

$\{S, \times_{14}\}$ is not a group.

12c. Show that a new set G can be formed by removing one of the elements of S such that *[5 marks]*

$\{G, \times_{14}\}$ is a group.

12d. Determine the order of each element of *[4 marks]*

$\{G, \times_{14}\}$.

12e. Find the proper subgroups of *[2 marks]*

$\{G, \times_{14}\}$.

The function

$f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 2x + 1 & \text{for } x \leqslant 2 \\ x^2 - 2x + 5 & \text{for } x > 2. \end{cases}$$

13a. (i) Sketch the graph of f. *[5 marks]*

(ii) By referring to your graph, show that f is a bijection.

13b. Find *[8 marks]*

$f^{-1}(x)$.

The relation R is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ by aRb if and only if

$a(a+1) \equiv b(b+1)(\bmod 5)$.

14a. Show that R is an equivalence relation. *[6 marks]*

14b. Show that the equivalence defining R can be written in the form *[3 marks]*

$$(a-b)(a+b+1) \equiv 0(\bmod 5).$$

14c. Hence, or otherwise, determine the equivalence classes. *[4 marks]*

15. The binary operation *[12 marks]*
Δ is defined on the set
$S = \{1, 2, 3, 4, 5\}$ by the following Cayley table.

Δ	1	2	3	4	5
1	1	1	2	3	4
2	1	2	1	2	3
3	2	1	3	1	2
4	3	2	1	4	1
5	4	3	2	1	5

(a) State whether S is closed under the operation Δ and justify your answer.

(b) State whether Δ is commutative and justify your answer.

(c) State whether there is an identity element and justify your answer.

(d) Determine whether Δ is associative and justify your answer.

(e) Find the solutions of the equation
$a\Delta b = 4\Delta b$, for
$a \neq 4$.

Consider the set S defined by
$S = \{s \in \mathbb{Q} : 2s \in \mathbb{Z}\}$.
You may assume that
$+$ (addition) and
$\times$ (multiplication) are associative binary operations
on
$\mathbb{Q}$.

16a. (i) Write down the six smallest non-negative elements of
S. *[9 marks]*

(ii) Show that
$\{S, +\}$ is a group.

(iii) Give a reason why
$\{S, \times\}$ is not a group. Justify your answer.

16b. The relation R is defined on S by $s_1 R s_2$ if $3s_1 + 5s_2 \in \mathbb{Z}$. *[10 marks]*

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence classes.

Sets X and Y are defined by $X =]0,\ 1[;\ Y = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$.

17a. (i) Sketch the set $X \times Y$ in the Cartesian plane. *[5 marks]*

(ii) Sketch the set $Y \times X$ in the Cartesian plane.

(iii) State $(X \times Y) \cap (Y \times X)$.

17b. Consider the function $f : X \times Y \to \mathbb{R}$ defined by $f(x,\ y) = x + y$ and the function $g : X \times Y \to \mathbb{R}$ defined by $g(x,\ y) = xy$. *[10 marks]*

(i) Find the range of the function f.

(ii) Find the range of the function g.

(iii) Show that f is an injection.

(iv) Find $f^{-1}(\pi)$, expressing your answer in exact form.

(v) Find all solutions to $g(x,\ y) = \frac{1}{2}$.

Let $f : G \to H$ be a homomorphism of finite groups.

18a. Prove that $f(e_G) = e_H$, where e_G is the identity element in G and e_H is the identity element in H. *[2 marks]*

18b. (i) Prove that the kernel of $f,\ K = \mathrm{Ker}(f)$, is closed under the group operation. *[6 marks]*

(ii) Deduce that K is a subgroup of G.

18c. (i) Prove that

$gkg^{-1} \in K$ for all

$g \in G,\ k \in K$.

(ii) Deduce that each left coset of K in G is also a right coset. *[6 marks]*

19. Consider the following functions *[14 marks]*

$f :]1,\ +\infty[\rightarrow \mathbb{R}^{+}$ where

$f(x) = (x-1)(x+2)$

$g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ where

$g(x,\ y) = (\sin(x+y),\ x+y)$

$h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ where

$h(x,\ y) = (x+3y,\ 2x+y)$

(a) Show that

f is bijective.

(b) Determine, with reasons, whether

(i)

g is injective;

(ii)

g is surjective.

(c) Find an expression for

$h^{-1}(x,\ y)$ and hence justify that

h has an inverse function.

Let

G be a group of order 12 with identity element e.

Let

$a \in G$ such that

$a^{6} \neq e$ and

$a^{4} \neq e$.

20a. (i) Prove that

G is cyclic and state two of its generators.

(ii) Let

H be the subgroup generated by

a^{4}. Construct a Cayley table for

H. *[9 marks]*

20b. State, with a reason, whether or not it is necessary that a group is cyclic given that all its proper subgroups are cyclic. *[2 marks]*

21. Let $(H,\ *)$ be a subgroup of the group $(G,\ *)$. Consider the relation R defined in G by xRy if and only if $y^{-1} * x \in H$. *[9 marks]*

(a) Show that R is an equivalence relation on G.

(b) Determine the equivalence class containing the identity element.

22. (a) Given a set U, and two of its subsets A and B, prove that *[11 marks]*

$$(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B),\text{ where } A \backslash B = A \cap B'.$$

(b) Let $S = \{A,\ B,\ C,\ D\}$ where $A = \emptyset,\ B = \{0\},\ C = \{0,\ 1\}$ and $D = \{0,\,1,\,2\}$.

State, with reasons, whether or not each of the following statements is true.

(i) The operation \ is closed in S.

(ii) The operation $\cap$ has an identity element in S but not all elements have an inverse.

(iii) Given $Y \in S$, the equation $X \cup Y = Y$ always has a unique solution for X in S.

23a. Associativity and commutativity are two of the five conditions for a set S with the binary operation $*$ to be an Abelian group; state the other three conditions. *[2 marks]*

23b. The Cayley table for the binary operation $\odot$ defined on the set $T = \{p, q, r, s, t\}$ is given below. *[15 marks]*

$\odot$	p	q	r	s	t
p	s	r	t	p	q
q	t	s	p	q	r
r	q	t	s	r	p
s	p	q	r	s	t
t	r	p	q	t	s

(i) Show that exactly three of the conditions for $\{T, \odot\}$ to be an Abelian group are satisfied, but that neither associativity nor commutativity are satisfied.
(ii) Find the proper subsets of T that are groups of order 2, and comment on your result in the context of Lagrange's theorem.
(iii) Find the solutions of the equation $(p \odot x) \odot x = x \odot p$.

The elements of sets P and Q are taken from the universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $P = \{1, 2, 3\}$ and $Q = \{2, 4, 6, 8, 10\}$.

24a. Given that $R = (P \cap Q')'$, list the elements of R. *[3 marks]*

24b. For a set S, let S^* denote the set of all subsets of S, *[5 marks]*
(i) find P^*;
(ii) find $n(R^*)$.

The relation R is defined on the set $\mathbb{N}$ such that for $a, b \in \mathbb{N}$, aRb if and only if $a^3 \equiv b^3 (\bmod 7)$.

25a. Show that R is an equivalence relation. *[6 marks]*

25b. Find the equivalence class containing 0. *[2 marks]*

25c. Denote the equivalence class containing n by C_n. List the first six elements of C_1. *[3 marks]*

25d. Denote the equivalence class containing n by C_n. Prove that $C_n = C_{n+7}$ for all $n \in \mathbb{N}$. *[3 marks]*

26a. The function $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $g(n) = |n| - 1$ for $n \in \mathbb{Z}$. Show that g is neither surjective nor injective. *[2 marks]*

26b. The set S is finite. If the function $f : S \to S$ is injective, show that f is surjective. *[2 marks]*

26c. Using the set $\mathbb{Z}^+$ as both domain and codomain, give an example of an injective function that is not surjective. *[3 marks]*

All of the relations in this question are defined on

$\mathbb{Z} \backslash \{0\}$.

27a. Decide, giving a proof or a counter-example, whether *[4 marks]*

$xRy \Leftrightarrow x + y > 7$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive.

27b. Decide, giving a proof or a counter-example, whether *[4 marks]*

$xRy \Leftrightarrow -2 < x - y < 2$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive.

27c. Decide, giving a proof or a counter-example, whether *[4 marks]*

$xRy \Leftrightarrow xy > 0$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive.

27d. Decide, giving a proof or a counter-example, whether *[4 marks]*

$xRy \Leftrightarrow \frac{x}{y} \in \mathbb{Z}$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive.

27e. One of the relations from parts (a), (b), (c) and (d) is an equivalence relation. *[3 marks]*

For this relation, state what the equivalence classes are.

Let

$A = \{a, b\}$.

28a. Write down all four subsets of A . *[1 mark]*

Let the set of all these subsets be denoted by $P(A)$. The binary operation symmetric difference, Δ , is defined on $P(A)$ by $X\Delta Y = (X\backslash Y) \cup (Y\backslash X)$ where X , $Y \in P(A)$.

28b. Construct the Cayley table for $P(A)$ under Δ . [3 marks]

28c. Prove that $\{P(A),\ \Delta\}$ is a group. You are allowed to assume that Δ is associative. [3 marks]

Let $\mathbb{Z}_4 = \{0,\ 1,\ 2,\ 3\}$ and $+_4$ denote addition modulo 4.

28d. Is $\{P(A),\ \Delta\}$ isomorphic to $\{\mathbb{Z}_4,\ +_4\}$? Justify your answer. [2 marks]

Let S be any non-empty set. Let $P(S)$ be the set of all subsets of S . For the following parts, you are allowed to assume that Δ, $\cup$ and $\cap$ are associative.

28e. (i) State the identity element for $\{P(S), \Delta\}$. [4 marks]

(ii) Write down X^{-1} for $X \in P(S)$.

(iii) Hence prove that $\{P(S), \Delta\}$ is a group.

28f. Explain why $\{P(S), \cup\}$ is not a group. [1 mark]

28g. Explain why $\{P(S), \cap\}$ is not a group. [1 mark]

Let c be a positive, real constant. Let G be the set

$\{x \in \mathbb{R} | -c < x < c\}$. The binary operation

$*$ is defined on the set G by

$x * y = \frac{x+y}{1+\frac{xy}{c^2}}$.

29a. Simplify [2 marks]

$\frac{c}{2} * \frac{3c}{4}$.

29b. State the identity element for G under [1 mark]

$*$.

29c. For [1 mark]

$x \in G$ find an expression for

x^{-1} (the inverse of x under

$*$).

29d. Show that the binary operation [2 marks]

$*$ is commutative on G .

29e. Show that the binary operation [4 marks]

$*$ is associative on G .

29f. (i) If [2 marks]

$x,\ y \in G$ explain why

$(c-x)(c-y) > 0$.

(ii) Hence show that

$x + y < c + \frac{xy}{c}$.

29g. Show that G is closed under [2 marks]

$*$.

29h. Explain why [2 marks]

$\{G, *\}$ is an Abelian group.

30. H and K are subgroups of a group G. By considering the four group axioms, prove that [8 marks]

$H \cap K$ is also a subgroup of G.