

Topic 4 Part 2 [222 marks]

1. The points A, B, C have position vectors $i + j + 2k$, $i + 2j + 3k$, $3i + k$ respectively and lie in the plane

[20 marks]

π .

(a) Find

- (i) the area of the triangle ABC;
- (ii) the shortest distance from C to the line AB;
- (iii) the cartesian equation of the plane

π .

The line L passes through the origin and is normal to the plane

π , it intersects

π at the

point D.

(b) Find

- (i) the coordinates of the point D;
- (ii) the distance of

π from the origin.

2. A ray of light coming from the point $(-1, 3, 2)$ is travelling in the direction of vector

[6 marks]

$\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and meets the plane

$$\pi : x + 3y + 2z - 24 = 0.$$

Find the angle that the ray of light makes with the plane.

3. Find the vector equation of the line of intersection of the three planes represented by the following system of equations.

[6 marks]

$$2x - 7y + 5z = 1$$

$$6x + 3y - z = -1$$

$$-14x - 23y + 13z = 5$$

[6 marks]

4. Given any two non-zero vectors \mathbf{a} and \mathbf{b} , show that

$$\begin{aligned} & |\mathbf{a} \\ & \times \mathbf{b} \\ & |^2 = \\ & |\mathbf{a} \\ & |^2 |\mathbf{b} \\ & |^2 - (\mathbf{a} \\ & \cdot \mathbf{b}) \\ & ^2. \end{aligned}$$

[20 marks]

5. Consider the points A(1, -1, 4), B (2, -2, 5) and O(0, 0, 0).

- (a) Calculate the cosine of the angle between

$$\begin{aligned} & \vec{OA} \text{ and} \\ & \vec{AB}. \end{aligned}$$

- (b) Find a vector equation of the line

L_1 which passes through A and B.

The line

L_2 has equation $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 7\mathbf{k} + t(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, where

$$t \in \mathbb{R}.$$

- (c) Show that the lines

L_1 and

L_2 intersect and find the coordinates of their point of intersection.

- (d) Find the Cartesian equation of the plane which contains both the line

L_2 and the point A.

[6 marks]

6. Three distinct non-zero vectors are given by

$$\vec{OA} = \mathbf{a},$$

$$\vec{OB} = \mathbf{b}, \text{ and}$$

$$\vec{OC} = \mathbf{c}.$$

If

\vec{OA} is perpendicular to

\vec{BC} and

\vec{OB} is perpendicular to

\vec{CA} , show that

\vec{OC} is perpendicular to

\vec{AB} .

7. The angle between the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and the vector $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + m\mathbf{k}$ is 30° .

[6 marks]

Find the values of m .

8. (a) Write the vector equations of the following lines in parametric form.

$$r_1 = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + m \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + n \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

- (b) Hence show that these two lines intersect and find the point of intersection, A.

- (c) Find the Cartesian equation of the plane

Π that contains these two lines.

- (d) Let B be the point of intersection of the plane

Π and the line

$$r = \begin{pmatrix} -8 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix}.$$

Find the coordinates of B.

- (e) If C is the mid-point of AB, find the vector equation of the line perpendicular to the plane

Π and passing through C.

9. (a) Show that a Cartesian equation of the line, l_1 , containing points A(1, -1, 2) and B(3, 0, 3) has the form

$$\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{1}.$$

- (b) An equation of a second line, l_2 , has the form

l_2 , has the form

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{1}.$$

- Show that the lines l_1 and l_2 intersect, and find the coordinates of their point of intersection.

- (c) Given that direction vectors of

l_1 and

l_2 are \mathbf{d}_1

and \mathbf{d}_2

respectively, determine

$\mathbf{d}_1 \times \mathbf{d}_2$.

- (d) Show that a Cartesian equation of the plane, Π , that contains

l_1 and

l_2 is

$$-x - y + 3z = 6.$$

- (e) Find a vector equation of the line l_3 which is perpendicular to the plane Π and passes through the point T(3, 1, -4).

l_3 which is perpendicular to the plane

Π and passes through the point T(3, 1, -4).

- (f) (i) Find the point of intersection of the line l_3 and the plane Π .

Π .

- (ii) Find the coordinates of

T' , the reflection of the point T in the plane Π .

Π .

- (iii) Hence find the magnitude of the vector $\overrightarrow{TT'}$.

10. The vectors \mathbf{a} , \mathbf{b} , \mathbf{c} satisfy the equation $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Show that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

[6 marks]

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$$

$$= \mathbf{c} \times \mathbf{a}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$$

OABCDE is a regular hexagon and \mathbf{a} , \mathbf{b} denote respectively the position vectors of A, B with respect to O.

- 11a. Show that $OC = 2AB$.

[2 marks]

- 11b. Find the position vectors of C, D and E in terms of \mathbf{a} and \mathbf{b} .

[7 marks]

The points A and B have coordinates (1, 2, 3) and (3, 1, 2) relative to an origin O.

12a. (i) Find [5 marks]
 $\vec{OA} \times \vec{OB}$.

(ii) Determine the area of the triangle OAB.

(iii) Find the Cartesian equation of the plane OAB.

12b. (i) Find the vector equation of the line [7 marks]
 L_1 containing the points A and B.

(ii) The line

L_2 has vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

Determine whether or not

L_1 and

L_2 are skew.

The function f is defined on the domain $[0, 2]$ by

$$f(x) = \ln(x+1) \sin(\pi x).$$

13a. Obtain an expression for [3 marks]
 $f'(x)$.

13b. Sketch the graphs of f and [4 marks]
 f' on the same axes, showing clearly all x -intercepts.

13c. Find the x -coordinates of the two points of inflexion on the graph of f . [2 marks]

13d. Find the equation of the normal to the graph of f where $x = 0.75$, giving your answer in the form $y = mx + c$. [3 marks]

13e. Consider the points [6 marks]

$$A(a, f(a)),$$

$$B(b, f(b)) \text{ and}$$

$$C(c, f(c)) \text{ where } a, b \text{ and } c$$

$(a < b < c)$ are the solutions of the equation

$$f(x) = f'(x). \text{ Find the area of the triangle ABC.}$$

14. The vector equation of line l is given as [6 marks]
- $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$
- Find the Cartesian equation of the plane containing the line l and the point $A(4, -2, 5)$.
15. (a) Find the coordinates of the point [19 marks]
 A on l_1 and the point B on l_2 such that \overrightarrow{AB} is perpendicular to both l_1 and l_2 .
- (b) Find $|\overrightarrow{AB}|$.
- (c) Find the Cartesian equation of the plane Π which contains l_1 and does not intersect l_2 .
16. Find the angle between the lines [6 marks]
 $\frac{x-1}{2} = 1 - y = 2z$ and $x = y = 3z$.
17. (a) If [8 marks]
 $a = 4$ find the coordinates of the point of intersection of the three planes.
- (b) (i) Find the value of a for which the planes do not meet at a unique point.
- (ii) For this value of a show that the three planes do not have any common point.

The position vector at time
 t of a point
 P is given by

$$\vec{OP} = (1+t)i + (2-2t)j + (3t-1)k, t \geq 0.$$

18. (a) Find the coordinates of P when
 $t = 0$.
- (b) Show that P moves along the line
 L with Cartesian equations

[23 marks]

$$x-1 = \frac{y-2}{-2} = \frac{z+1}{3}$$

- (c) (i) Find the value of t when P lies on the plane with equation
 $2x + y + z = 6$.
- (ii) State the coordinates of P at this time.
- (iii) Hence find the total distance travelled by P before it meets the plane.

The position vector at time
 t of another point, Q , is given by

$$\vec{OQ} = \begin{pmatrix} t^2 \\ 1-t \\ 1-t^2 \end{pmatrix}, t \geq 0.$$

- (d) (i) Find the value of t for which the distance from Q to the origin is minimum.
- (ii) Find the coordinates of Q at this time.
- (e) Let
 a ,
 b and
 c be the position vectors of Q at times
 $t = 0$,
 $t = 1$ and
 $t = 2$ respectively.

(i) Show that the equation
 $a - b = k(b - c)$ has no solution for
 k .

(ii) Hence show that the path of Q is not a straight line.

19. Given that
 $a = 2\sin\theta i + (1 - \sin\theta)j$, find the value of the acute angle
 θ , so that
 a is perpendicular to the line
 $x + y = 1$.

[5 marks]