Topic 1 Part 1 *[440 marks]*

1.

[4 marks]

Markscheme

$u_1 = \frac{1}{3}k$, $r = \frac{1}{3}$ *(A1)*   *(A1)*

$7 = \frac{\frac{1}{3}k}{1-\frac{1}{3}}$ *M1*

$k = 14$ *A1*

[4 marks]

Examiners report

The question was well done generally. Those that did make mistakes on the question usually had the first term wrong, but did understand to use the formula for an infinite geometric series.

2.

[7 marks]

Markscheme

$z_1 = 2a\text{cis}\left(\frac{\pi}{3}\right)$, $z_2 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ *M1*   *A1*   *A1*

EITHER

$\left(\frac{z_1}{z_2}\right)^6 = \frac{2^6 a^6 \text{cis}(0)}{\sqrt{2}^6 \text{cis}\left(\frac{\pi}{2}\right)}$ $\left(= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)\right)$ *M1*   *A1*   *A1*

OR

$\left(\frac{z_1}{z_2}\right)^6 = \left(\frac{2a}{\sqrt{2}}\,\text{cis}\left(\frac{7\pi}{12}\right)\right)^6$ *M1*   *A1*

$= 8a^6 \text{cis}\left(-\frac{\pi}{2}\right)$ *A1*

THEN

$= -8a^6 i$ *A1*

Note: Accept equivalent angles, in radians or degrees.

Accept alternate answers without cis e.g. $= \frac{8a^6}{i}$

[7 marks]

Examiners report

Most students had an idea of what to do but were frequently let down in their calculations of the modulus and argument. The most common error was to give the argument of z_2 as $\frac{3\pi}{4}$, failing to recognise that it should be in the fourth quadrant. There were also errors seen in the algebraic manipulation, in particular forgetting to raise the modulus to the power 6.

3.

[6 marks]

Markscheme

$\sqrt{x^2 + y^2} + x + y\mathrm{i} = 6 - 2\mathrm{i}$ *(A1)*

equating real and imaginary parts   *M1*

$y = -2$ *A1*

$\sqrt{x^2 + 4} + x = 6$ *A1*

$x^2 + 4 = (6 - x)^2$ *M1*

$-32 = -12x \Rightarrow x = \frac{8}{3}$ *A1*

[6 marks]

Examiners report

There were some good solutions to this question, but those who failed to complete the question failed at a variety of different points. Many did not know the definition of the modulus of a complex number and so could not get started at all. Many then did not think to equate real and imaginary parts, and then many failed to solve the resulting irrational equation to be able to find x.

4.   *[5 marks]*

Markscheme

$\log_3 \left(\frac{9}{x+7} \right) = \log_3 \frac{1}{2x}$ *M1M1A1*

Note: Award *M1* for changing to single base, *M1* for incorporating the 2 into a log and *A1* for a correct equation with maximum one log expression each side.

$x + 7 = 18x$ *M1*

$x = \frac{7}{17}$ *A1*

[5 marks]

Examiners report

Some good solutions to this question and few candidates failed to earn marks on the question. Many were able to change the base of the logs, and many were able to deal with the 2, but of those who managed both, poor algebraic skills were often evident. Many students attempted to change the base into base 10, resulting in some complicated algebra, few of which managed to complete successfully.

5a.   *[3 marks]*

Markscheme

$\left(x - \frac{2}{x} \right)^4 = x^4 + 4x^3 \left(-\frac{2}{x} \right) + 6x^2 \left(-\frac{2}{x} \right)^2 + 4x \left(-\frac{2}{x} \right)^3 + \left(-\frac{2}{x} \right)^4$ *(A2)*

Note: Award *(A1)* for 3 or 4 correct terms.

Note: Accept combinatorial expressions, *e.g.* $\binom{4}{2}$ for 6.

$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$ *A1*

[3 marks]

Examiners report

5b.   *[2 marks]*

Markscheme

constant term from expansion of $\left(2x^2 + 1\right)\left(x - \frac{2}{x}\right)^4 = -64 + 24 = -40$ ***A2***

Note: Award ***A1*** for –64 or 24 seen.
[2 marks]

Examiners report

It was disappointing to see many candidates expanding $\left(x - \frac{2}{x}\right)^4$ by first expanding $\left(x - \frac{2}{x}\right)^2$ and then either squaring the result or multiplying twice by $\left(x - \frac{2}{x}\right)$, processes which often resulted in arithmetic errors being made. Candidates at this level are expected to be sufficiently familiar with Pascal's Triangle to use it in this kind of problem. In (b), some candidates appeared not to understand the phrase "constant term".

6a.   *[3 marks]*

Markscheme

attempt to equate real and imaginary parts   ***M1***
equate real parts: $4m + 4n = 16$; equate imaginary parts: $-5m = 15$ ***A1***
$\Rightarrow m = -3, \ n = 7$ ***A1***
[3 marks]

Examiners report

Part (a) was generally well answered. In (b), however, some candidates put $m = a + ib$ and $n = c + id$ which gave four equations for two unknowns so that no further progress could be made.

6b.   *[4 marks]*

Markscheme

let $m = x + iy, \ n = x - iy$ ***M1***
$\Rightarrow (4 - 5i)(x + iy) + 4(x - iy) = 16 + 15i$
$\Rightarrow 4x - 5ix + 4iy + 5y + 4x - 4iy = 16 + 15i$
attempt to equate real and imaginary parts   ***M1***
$8x + 5y = 16, \ -5x = 15$ ***A1***
$\Rightarrow x = -3, \ y = 8$ ***A1***
$(\Rightarrow m = -3 + 8i, \ n = -3 - 8i)$
[4 marks]

Examiners report

Part (a) was generally well answered. In (b), however, some candidates put $m = a + ib$ and $n = c + id$ which gave four equations for two unknowns so that no further progress could be made.

Markscheme

$$\left(\frac{x}{y} - \frac{y}{x}\right)^4 = \left(\frac{x}{y}\right)^4 + 4\left(\frac{x}{y}\right)^3\left(-\frac{y}{x}\right) + 6\left(\frac{x}{y}\right)^2\left(-\frac{y}{x}\right)^2 + 4\left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^3 + \left(-\frac{y}{x}\right)^4$$ *(M1)(A1)*

Note: Award *M1* for attempt to expand and *A1* for correct unsimplified expansion.

$$= \frac{x^4}{y^4} - 4\frac{x^2}{y^2} + 6 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4} \quad \left(= \frac{x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8}{x^4y^4}\right)$$ *A1A1*

Note: Award *A1* for powers, *A1* for coefficients and signs.

Note: Final two *A* marks are independent of first *A* mark.

[4 marks]

Examiners report

This was generally very well answered. Those who failed to gain full marks often made minor sign slips. A surprising number obtained the correct simplified expression, but continued to rearrange their expressions, often doing so incorrectly. Fortunately, there were no penalties for doing so.

Markscheme

METHOD 1

$$\det \begin{pmatrix} 1 & 3 & a-1 \\ 2 & 2 & a-2 \\ 3 & 1 & a-3 \end{pmatrix}$$ *M1*

$$= 1\left(2(a-3) - (a-2)\right) - 3\left(2(a-3) - 3(a-2)\right) + (a-1)(2-6)$$
(or equivalent) *A1*

= 0 (therefore there is no unique solution) *A1*

[3 marks]

METHOD 2

$$\begin{pmatrix} 1 & 3 & a-1 & \big| & 1 \\ 2 & 2 & a-2 & \big| & 1 \\ 3 & 1 & a-3 & \big| & b \end{pmatrix} : \begin{pmatrix} 1 & 3 & a-1 & \big| & 1 \\ 0 & -4 & -a & \big| & -1 \\ 0 & -8 & -2a & \big| & b-3 \end{pmatrix}$$ *M1A1*

$$: \begin{pmatrix} 1 & 3 & a-1 & \big| & 1 \\ 0 & -4 & -a & \big| & -1 \\ 0 & 0 & 0 & \big| & b-1 \end{pmatrix}$$ (and 3 zeros imply no unique solution) *A1*

[3 marks]

Examiners report

The best candidates used row reduction correctly in part a) and were hence able to deduce $b = 1$ in part b) for an easy final 4 marks. The determinant method was often usefully employed in part a).

Markscheme

METHOD 1

$$\left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 2 & 2 & a-2 & 1 \\ 3 & 1 & a-3 & b \end{array}\right) : \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & -8 & -2a & b-3 \end{array}\right) \quad \textit{M1A1}$$

$$: \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & 0 & 0 & b-1 \end{array}\right) \quad \textit{A1}$$

$b = 1$ *A1*   *N2*

Note: Award *M1* for an attempt to use row operations.

[4 marks]

METHOD 2

$b = 1$ *A4*

Note: Award *A4* only if " $b-1$ " seen in (a).

[4 marks]

Examiners report

The best candidates used row reduction correctly in part a) and were hence able to deduce $b = 1$ in part b) for an easy final 4 marks. The determinant method was often usefully employed in part a).

[3 marks]

Markscheme

(i) $z_1 = 2\sqrt{3}\text{cis}\,\frac{3\pi}{2} \Rightarrow z_1 = -2\sqrt{3}\text{i}$ *A1*

(ii) $z_1 + z_2 = -2\sqrt{3}\text{i} - 1 + \sqrt{3}\text{i} = -1 - \sqrt{3}\text{i}$ *A1*

$(z_1 + z_2)^* = -1 + \sqrt{3}\text{i}$ *A1*

[3 marks]

Examiners report

Many candidates were perhaps fortunate in this question due to there being several follow through marks available. Part a) was often done correctly. In part b), incorrect answers of $2\,\text{cis}\left(-\frac{\pi}{3}\right)$ were common, though many of these candidates often applied De Moivre's Theorem correctly to their answers. In c) the majority found $z^2 = -3$ but could then get no further. The second part was often poorly done, with those rationalising the Cartesian form of z_1 having the most success. Part d) posed problems for a great many, and correct solutions were rarely seen. $\text{cis}\left(\frac{5n\pi}{6}\right)$ was often seen, but then finding $n = 12$ proved to be a step too far for many. In general, the manipulation of complex numbers in polar form is not well understood.

[6 marks]

Markscheme

(i) $|z_2| = 2$

$\tan \theta = -\sqrt{3}$ **(M1)**

z_2 lies on the second quadrant

$\theta = \arg z_2 = \frac{2\pi}{3}$

$z_2 = 2\operatorname{cis}\frac{2\pi}{3}$ **A1A1**

(ii)   attempt to use De Moivre's theorem   **M1**

$z = \sqrt[3]{2}\operatorname{cis}\frac{\frac{2\pi}{3}+2k\pi}{3}$, $k = 0, 1$ and 2

$z = \sqrt[3]{2}\operatorname{cis}\frac{2\pi}{9}$, $\sqrt[3]{2}\operatorname{cis}\frac{8\pi}{9}$, $\sqrt[3]{2}\operatorname{cis}\frac{14\pi}{9}$ $\left(= \sqrt[3]{2}\operatorname{cis}\left(\frac{-4\pi}{9}\right)\right)$ **A1A1**

Note: Award **A1** for modulus, **A1** for arguments.

Note: Allow equivalent forms for z .

[6 marks]

Examiners report

Many candidates were perhaps fortunate in this question due to there being several follow through marks available. Part a) was often done correctly. In part b), incorrect answers of $2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ were common, though many of these candidates often applied De Moivre's Theorem correctly to their answers. In c) the majority found $z^2 = -3$ but could then get no further. The second part was often poorly done, with those rationalising the Cartesian form of z_1 having the most success. Part d) posed problems for a great many, and correct solutions were rarely seen. $\operatorname{cis}\left(\frac{5n\pi}{6}\right)$ was often seen, but then finding $n = 12$ proved to be a step too far for many. In general, the manipulation of complex numbers in polar form is not well understood.

9c.   *[6 marks]*

Markscheme

(i)   **METHOD 1**

$z^2 = \left(1 - 1 + \sqrt{3}i\right)^2 = -3 \left(\Rightarrow z = \pm\sqrt{3}i\right)$ **M1**

$z = \sqrt{3}\operatorname{cis}\frac{\pi}{2}$ or $z_1 = \sqrt{3}\operatorname{cis}\frac{3\pi}{2}$ $\left(= \sqrt{3}\operatorname{cis}\left(\frac{-\pi}{2}\right)\right)$ **A1A1**

so $r = \sqrt{3}$ and $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ $\left(= \frac{-\pi}{2}\right)$

Note: Accept $r\operatorname{cis}(\theta)$ form.

METHOD 2

$z^2 = \left(1 - 1 + \sqrt{3}i\right)^2 = -3 \Rightarrow z^2 = 3\operatorname{cis}\left((2n+1)\pi\right)$ **M1**

$r^2 = 3 \Rightarrow r = \sqrt{3}$ **A1**

$2\theta = (2n+1)\pi \Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ $\left(\text{as } 0 \leqslant \theta < 2\pi\right)$ **A1**

Note: Accept $r\operatorname{cis}(\theta)$ form.

(ii)   **METHOD 1**

$z = -\frac{1}{2\operatorname{cis}\frac{2\pi}{3}} \Rightarrow z = \frac{\operatorname{cis}\pi}{2\operatorname{cis}\frac{2\pi}{3}}$ **M1**

$\Rightarrow z = \frac{1}{2}\operatorname{cis}\frac{\pi}{3}$

so $r = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$ **A1A1**

METHOD 2

$z_1 = -\frac{1}{-1+\sqrt{3}i} \Rightarrow z_1 = -\frac{-1-\sqrt{3}i}{\left(-1+\sqrt{3}i\right)\left(-1-\sqrt{3}i\right)}$ **M1**

$z = \frac{1+\sqrt{3}i}{4} \Rightarrow z = \frac{1}{2}\operatorname{cis}\frac{\pi}{3}$

so $r = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$ **A1A1**

[6 marks]

9d.   *[4 marks]*

Markscheme

$\frac{z_1}{z_2} = \sqrt{3} \operatorname{cis} \frac{5\pi}{6}$ *(A1)*

$\left(\frac{z_1}{z_2}\right)^n = \sqrt{3}^n \operatorname{cis} \frac{5n\pi}{6}$ *A1*

equating imaginary part to zero and attempting to solve   *M1*
obtain $n = 12$ *A1*
Note: Working which only includes the argument is valid.

[4 marks]

10a.   *[3 marks]*

Markscheme

$(f \circ f)(x) = f\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$ *M1A1*

$(f \circ f)(x) = \frac{x}{4-3x}$ *A1*

[3 marks]

Examiners report

Part a) proved to be an easy 3 marks for most candidates.

10b.   *[8 marks]*

Markscheme

$P(n) : \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$

$P(1) : f(x) = F_1(x)$

$LHS = f(x) = \frac{x}{2-x}$ and $RHS = F_1(x) = \frac{x}{2^1 - (2^1 - 1)x} = \frac{x}{2-x}$ *A1A1*

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{k \text{ times}}(x) = F_k(x)$ *M1*

consider $P(k+1)$

EITHER

$\underbrace{(f \circ f \circ \ldots \circ f)}_{k+1 \text{ times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}} \right)(x) = f(F_k(x))$ *(M1)*

$= f\left(\frac{x}{2^k - (2^k - 1)x} \right) = \frac{\frac{x}{2^k - (2^k - 1)x}}{2 - \frac{x}{2^k - (2^k - 1)x}}$ *A1*

$= \frac{x}{2\left(2^k - (2^k - 1)x\right) - x} = \frac{x}{2^{k+1} - (2^{k+1} - 2)x - x}$ *A1*

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{k+1 \text{ times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}} \right)(x) = F_k(f(x))$ *(M1)*

$= F_k\left(\frac{x}{2-x} \right) = \frac{\frac{x}{2-x}}{2^k - (2^k - 1)\frac{x}{2-x}}$ *A1*

$= \frac{x}{2^{k+1} - 2^k x - 2^k x + x}$ *A1*

THEN

$= \frac{x}{2^{k+1} - (2^{k+1} - 1)x} = F_{k+1}(x)$ *A1*

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$ *R1*

[8 marks]

Examiners report

Part b) was often answered well, and candidates were well prepared in this session for this type of question. Candidates still need to take care when showing explicitly that P(1) is true, and some are still writing 'Let $n = k$' which gains no marks. The inductive step was often well argued, and given in clear detail, though the final inductive reasoning step was incorrect, or appeared rushed, even from the better candidates. 'True for $n =1$, $n = k$ and $n = k + 1$' is still disappointingly seen, as were some even more unconvincing variations.

10c.

Markscheme

METHOD 1

$x = \frac{y}{2^n - (2^n - 1)y} \Rightarrow 2^n x - (2^n - 1)xy = y$ *M1A1*

$\Rightarrow 2^n x = ((2^n - 1)x + 1)y \Rightarrow y = \frac{2^n x}{(2^n - 1)x + 1}$ *A1*

$F_n^{-1}(x) = \frac{2^n x}{(2^n - 1)x + 1}$ *A1*

$F_n^{-1}(x) = \frac{x}{\frac{2^n - 1}{2^n}x + \frac{1}{2^n}}$ *M1*

$F_n^{-1}(x) = \frac{x}{(1 - 2^{-n})x + 2^{-n}}$ *A1*

$F_n^{-1}(x) = \frac{x}{2^{-n} - (2^{-n} - 1)x}$ *AG*

METHOD 2

attempt $F_{-n}(F_n(x))$ *M1*

$= F_{-n}\left(\frac{x}{2^n - (2^n - 1)x}\right) = \frac{\frac{x}{2^n - (2^n - 1)x}}{2^{-n} - (2^{-n} - 1)\frac{x}{2^n - (2^n - 1)x}}$ *A1A1*

$= \frac{x}{2^{-n}(2^n - (2^n - 1)x) - (2^{-n} - 1)x}$ *A1A1*

Note: Award *A1* marks for numerators and denominators.

$= \frac{x}{1} = x$ *A1AG*

METHOD 3

attempt $F_n(F_{-n}(x))$ *M1*

$= F_n\left(\frac{x}{2^{-n} - (2^{-n} - 1)x}\right) = \frac{\frac{x}{2^{-n} - (2^{-n} - 1)x}}{2^n - (2^n - 1)\frac{x}{2^{-n} - (2^{-n} - 1)x}}$ *A1A1*

$= \frac{x}{2^n(2^{-n} - (2^{-n} - 1)x) - (2^n - 1)x}$ *A1A1*

Note: Award *A1* marks for numerators and denominators.

$= \frac{x}{1} = x$ *A1AG*

[6 marks]

Examiners report

Part c) was again very well answered by the majority. A few weaker candidates attempted to find an inverse for the individual case n = 1 , but gained no credit for this.

10d. *[6 marks]*

Markscheme

(i) $F_n(0) = 0, \ F_n(1) = 1$ *A1*

(ii)   **METHOD 1**

$2^n - (2^n - 1)x - 1 = (2^n - 1)(1 - x)$ *(M1)*

> 0 if $0 < x < 1$ and $n \in \mathbb{Z}^+$ *A1*

so $2^n - (2^n - 1)x > 1$ and $F_n(x) = \frac{x}{2^n - (2^n - 1)x} < \frac{x}{1} \ (< x)$ *R1*

$F_n(x) = \frac{x}{2^n - (2^n - 1)x} < x$ for $0 < x < 1$ and $n \in \mathbb{Z}^+$ *AG*

METHOD 2

$\frac{x}{2^n - (2^n - 1)x} < x \Leftrightarrow 2^n - (2^n - 1)x > 1$ *(M1)*

$\Leftrightarrow (2^n - 1)x < 2^n - 1$ *A1*

$\Leftrightarrow x < \frac{2^n - 1}{2^n - 1} = 1$ true in the interval $]0, 1[$ *R1*

(iii) $B_n = 2\left(A_n - \frac{1}{2}\right) \ (= 2A_n - 1)$ *(M1)A1*

[6 marks]

Examiners report

Part d) was not at all well understood, with virtually no candidates able to tie together the hints given by connecting the different parts of the question. Rash, and often thoughtless attempts were made at each part, though by this stage some seemed to be struggling through lack of time. The inequality part of the question tended to be 'fudged', with arguments seen by examiners being largely unconvincing and lacking clarity. A tiny number of candidates provided the correct answer to the final part, though a surprising number persisted with what should have been recognised as fruitless working – usually in the form of long-winded integration attempts.

11.

[4 marks]

Markscheme

METHOD 1

$102 + 105 + \ldots + 498$ *(M1)*

so number of terms $= 133$ *(A1)*

EITHER

$= \frac{133}{2}\left(2 \times 102 + 132 \times 3\right)$ *(M1)*

$= 39900$ *A1*

OR

$= (102 + 498) \times \frac{133}{2}$ *(M1)*

$= 39900$ *A1*

OR

$\sum\limits_{n=34}^{166} 3n$ *(M1)*

$= 39900$ *A1*

METHOD 2

$500 \div 3 = 166.666\ldots$ and $100 \div 3 = 33.333\ldots$

$102 + 105 + \ldots + 498 = \sum\limits_{n=1}^{166} 3n - \sum\limits_{n=1}^{33} 3n$ *(M1)*

$\sum\limits_{n=1}^{166} 3n = 41583$ *(A1)*

$\sum\limits_{n=1}^{33} 3n = 1683$ *(A1)*

the sum is 39900   *A1*

[4 marks]

12.

Markscheme

the pieces have lengths $a, \ ar, \ ..., \ ar^9$ *(M1)*

$8a = ar^9 \ (\text{or } 8 = r^9)$ *A1*

$r = \sqrt[9]{8} = 1.259922...$ *A1*

$a \frac{r^{10}-1}{r-1} = 1 \quad \left(\text{or } a \frac{r^{10}-1}{r-1} = 1000\right)$ *M1*

$a = \frac{r-1}{r^{10}-1} = 0.0286... \quad \left(\text{or } a = \frac{r-1}{r^{10}-1} = 28.6...\right)$ *(A1)*

a = 29 mm (accept 0.029 m or any correct answer regardless the units)   *A1*

[6 marks]

13.

Markscheme

METHOD 1

$(1 - \omega^2)^* = (1 - \operatorname{cis} 2\theta)^* = ((1 - \cos 2\theta) - \mathrm{i} \sin 2\theta)^*$ *M1A1*

$= (1 - \cos 2\theta) + \mathrm{i} \sin 2\theta$ *A1*

$\left|(1 - \omega^2)^*\right| = \sqrt{(1 - \cos 2\theta)^2 + \sin^2 2\theta} \left(= \sqrt{(2\sin^2\theta)^2 + (2\sin\theta\cos\theta)^2}\right)$ *M1*

$= |2\sin\theta|$ *A1*

$\arg\left((1 - \omega^2)^*\right) = \alpha \Rightarrow \tan\alpha = \cot(\theta)$ *M1*

$\alpha = \frac{\pi}{2} - \theta$ *A1*

therefore:

modulus is $2|\sin\theta|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$

Note: Accept modulus is $2\sin\theta$ and argument is $\frac{\pi}{2} - \theta$

METHOD 2

EITHER

$(1 - \varpi^2)^* = (1 - \operatorname{cis} 2\theta)^* = ((1 - \cos 2\theta) - \mathrm{i} \sin 2\theta)^*$ *M1A1*

$= (1 - \cos 2\theta) + \mathrm{i} \sin 2\theta$ *A1*

$= (1 - 1 + 2\sin^2\theta) + 2\mathrm{i} \sin\theta\cos\theta$ *M1*

OR

$(1 - \varpi^2)^* = \left(1 - (\cos\theta + \mathrm{i}\sin\theta)^2\right)^*$ *M1A1*

$= (1 - \cos^2\theta + \sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta)^*$ *A1*

$= 2\sin^2\theta + 2\mathrm{i}\sin\theta\cos\theta$ *M1*

THEN

$= 2\sin\theta(\sin\theta + \mathrm{i}\cos\theta)$ *(M1)*

$= 2\sin\theta\left(\cos\left(\frac{\pi}{2} - \theta\right) + \mathrm{i}\sin\left(\frac{\pi}{2} - \theta\right)\right)$ *A1A1*

$= 2\sin\theta \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$

therefore:

modulus is $2|\sin\theta|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$

Note: Accept modulus is $2\sin\theta$ and argument is $\frac{\pi}{2} - \theta$.

[7 marks]

Examiners report

This was the most challenging question in part A with just a few candidates scoring full marks. This question showed that many candidates have difficulties with algebraic manipulations, application of De Moivre's theorem and use of trigonometric identities. Although some candidates managed to calculate the square of a complex number, many failed to write down its conjugate or made algebraic errors which lead to wrong results in many cases. Just a few candidates were able to calculate the modulus and the argument of the complex number.

14.   *[5 marks]*

Markscheme

METHOD 1

determinant $= 0$ *M1*

$k(-2 - 16) - (0 - 12) + 2(0 + 3) = 0$ *(M1)(A1)*

$-18k + 18 = 0$ *(A1)*

$k = 1$ *A1*

METHOD 2

writes in the form

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$ (or attempts to solve simultaneous equations)   *(M1)*

Having two 0's in first column (obtaining two equations in the same two variables)   *M1*

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 18k - 18 & 21k - 27 \end{pmatrix}$$ (or isolating one variable)   *A1*

Note: The *A1* is to be awarded for the $18k - 18$. The final column may not be seen.

$k = 1$ *(M1)A1*

[5 marks]

Examiners report

Candidates who used the determinant method usually obtained full marks. Few students used row reduction and of those the success was varied. However, many candidates attempted long algebraic methods, which frequently went wrong at some stage. Of those who did work through to correctly isolate one variable, few were able to interpret the resultant value of k .

15a.   *[3 marks]*
Markscheme

the three girls can sit together in $3! = 6$ ways   *(A1)*

this leaves 4 'objects' to arrange so the number of ways this can be done is 4!   *(M1)*

so the number of arrangements is $6 \times 4! = 144$ *A1*

[3 marks]

Examiners report

Some good solutions to part (a) and certainly fewer completely correct answers to part (b). Many candidates were able to access at least partial credit, if they were showing their reasoning.

15b.   *[4 marks]*

Markscheme

Finding more than one position that the girls can sit   *(M1)*

Counting exactly four positions   *(A1)*

number of ways $= 4 \times 3! \times 3! = 144$ *M1A1   N2*

[4 marks]

Examiners report

Some good solutions to part (a) and certainly fewer completely correct answers to part (b). Many candidates were able to access at least partial credit, if they were showing their reasoning.

16a.

[9 marks]

Markscheme

(i) $z_1 = 2\mathrm{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 2\mathrm{cis}\left(\frac{5\pi}{6}\right)$, $z_3 = 2\mathrm{cis}\left(-\frac{\pi}{2}\right)$ or $2\mathrm{cis}\left(\frac{3\pi}{2}\right)$ *A1A1A1*

Note: Accept modulus and argument given separately, or the use of exponential (Euler) form.

Note: Accept arguments given in rational degrees, except where exponential form is used.

(ii)   the points lie on a circle of radius 2 centre the origin   *A1*

differences are all $\frac{2\pi}{3}$ (mod 2π)   *A1*

$\Rightarrow$ points equally spaced $\Rightarrow$ triangle is equilateral   *R1AG*

Note: Accept an approach based on a clearly marked diagram.

(iii) $z_1^{3n} + z_2^{3n} = 2^{3n}\mathrm{cis}\left(\frac{n\pi}{2}\right) + 2^{3n}\mathrm{cis}\left(\frac{5n\pi}{2}\right)$ *M1*

$= 2 \times 2^{3n}\mathrm{cis}\left(\frac{n\pi}{2}\right)$ *A1*

$2z_3^{3n} = 2 \times 2^{3n}\mathrm{cis}\left(\frac{9n\pi}{2}\right) = 2 \times 2^{3n}\mathrm{cis}\left(\frac{n\pi}{2}\right)$ *A1AG*

[9 marks]

Examiners report

(i) A disappointingly large number of candidates were unable to give the correct arguments for the three complex numbers. Such errors undermined their efforts to tackle parts (ii) and (iii).

16b.

[9 marks]

Markscheme

(i)   attempt to obtain **seven** solutions in modulus argument form   *M1*

$z = \text{cis}\left(\frac{2k\pi}{7}\right), \ k = 0, \ 1 \ldots 6$ *A1*

(ii) w has argument $\frac{2\pi}{7}$ and $1 + w$ has argument ϕ,

then $\tan(\phi) = \left|\frac{\sin\left(\frac{2\pi}{7}\right)}{1+\cos\left(\frac{2\pi}{7}\right)}\right|$ *M1*

$= \left|\frac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}\right|$ *A1*

$= \tan\left(\frac{\pi}{7}\right) \Rightarrow \phi = \frac{\pi}{7}$ *A1*

Note: Accept alternative approaches.

(iii)   since roots occur in conjugate pairs,   *(R1)*

$z^7 - 1$ has a quadratic factor $\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right) \times \left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)$ *A1*

$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ *AG*

other quadratic factors are $z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1$ *A1*

and $z^2 - 2z\cos\left(\frac{6\pi}{7}\right) + 1$ *A1*

[9 marks]

Examiners report

Many candidates were successful in part (i), but failed to capitalise on that – in particular, few used the fact that roots of $z^7 - 1 = 0$ come in complex conjugate pairs.

Markscheme

attempting to express the system in matrix form   *M1*

$\begin{pmatrix} 0.1 & -1.7 & 0.9 \\ -2.4 & 0.3 & 3.2 \\ 2.5 & 0.6 & -3.7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4.4 \\ 1.2 \\ 0.8 \end{pmatrix}$ *A1*

Note: Award *M1A1* for a correct augmented matrix.

[2 marks]

Examiners report

This was generally well done. In part (a), some candidates expressed the system of equations in the form $XA = B$. In part (b), the overwhelming majority of candidates who used a direct GDC approach obtained the correct solution. Candidates who attempted matrix methods such as row reduction without a GDC were generally unsuccessful.

17b.

Markscheme

either direct GDC use, attempting elimination or using an inverse matrix.   *(M1)*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2.4 \\ 1.6 \\ -1.6 \end{pmatrix} \text{(correct to 2sf) or} \begin{pmatrix} -2.40 \\ 1.61 \\ -1.57 \end{pmatrix} \text{(correct to 3sf) or} \begin{pmatrix} -\frac{932}{389} \\ \frac{628}{389} \\ -\frac{612}{389} \end{pmatrix} \text{(exact)}   \textbf{\textit{A2}}$$

[3 marks]

Examiners report

This was generally well done. In part (a), some candidates expressed the system of equations in the form $XA = B$. In part (b), the overwhelming majority of candidates who used a direct GDC approach obtained the correct solution. Candidates who attempted matrix methods such as row reduction without a GDC were generally unsuccessful.

18a.

Markscheme

$u_n - v_n = 1.6 + (n-1) \times 1.5 - 3 \times 1.2^{n-1} \left(= 1.5n + 0.1 - 3 \times 1.2^{n-1} \right)$ *A1A1*

[2 marks]

Examiners report

In part (a), most candidates were able to express u_n and v_n correctly and hence obtain a correct expression for $u_n - v_n$. Some candidates made careless algebraic errors when unnecessarily simplifying u_n while other candidates incorrectly stated v_n as $3(1.2)^n$.

18b.

Markscheme

attempting to solve $u_n > v_n$ numerically or graphically.   *(M1)*

$n = 2.621\ldots, 9.695\ldots$ *(A1)*

So $3 \leqslant n \leqslant 9$ *A1*

[3 marks]

Examiners report

In parts (b) and (c), most candidates treated n as a continuous variable rather than as a discrete variable. Candidates should be aware that a GDC's table feature can be extremely useful when attempting such question types.

18c.

Markscheme

The greatest value of $u_n - v_n|$ is 1.642.   *A1*

Note: Do not accept 1.64.

[1 mark]

Examiners report

In parts (b) and (c), most candidates treated n as a continuous variable rather than as a discrete variable. Candidates should be aware that a GDC's table feature can be extremely useful when attempting such question types. In part (c), a number of candidates attempted to find the maximum value of n rather than attempting to find the maximum value of $u_n - v_n|$

19.

Markscheme

$P(n) : f(n) = 5^{2n} - 24n - 1|$ is divisible by 576 for $n \in \mathbb{Z}^+|$

for $n = 1$, $f(1) = 5^2 - 24 - 1 = 0|$

Zero is divisible by 576, (as every non-zero number divides zero), and so P(1) is true.   *R1*

Note: Award *R0* for P(1) = 0 shown and zero is divisible by 576 not specified.

Note: Ignore P(2) = 576 if P(1) = 0 is shown and zero is divisible by 576 is specified.

Assume $P(k)|$ is true for some k $(\Rightarrow f(k) = N \times 576)|$ *M1*

Note: Do not award *M1* for statements such as "let $n = k$".

consider $P(k+1) : f(k+1) = 5^{2(k+1)} - 24(k+1) - 1|$ *M1*

$= 25 \times 5^{2k} - 24k - 25|$ *A1*

EITHER

$= 25 \times (24k + 1 + N \times 576) - 24k - 25|$ *A1*

$= 576k + 25 \times 576N|$ which is a multiple of 576   *A1*

OR

$= 25 \times 5^{2k} - 600k - 25 + 600k - 24k|$ *A1*

$= 25(5^{2k} - 24k - 1) + 576k|$ (or equivalent) which is a multiple of 576   *A1*

THEN

$P(1)$ is true and $P(k)|$ true $\Rightarrow P(k+1)|$ true, so $P(n)|$ is true for all $n \in \mathbb{Z}^+|$ *R1*

Note: Award *R1* only if at least four prior marks have been awarded.

[7 marks]

Examiners report

This proof by mathematical induction challenged most candidates. While most candidates were able to show that P(1) = 0, a significant number did not state that zero is divisible by 576. A few candidates started their proof by looking at P(2). It was pleasing to see that the inductive step was reasonably well done by most candidates. However many candidates committed simple algebraic errors. The most common error was to state that $5^{2(k+1)} = 5(5)^{2k}$. The concluding statement often omitted the required implication statement and also often omitted that P(1) was found to be true.

Markscheme

(i) $\sum_{k=1}^{n}(2k-1)$ (or equivalent)   *A1*

Note: Award *A0* for $\sum_{n=1}^{n}(2n-1)$ or equivalent.

(ii)   **EITHER**

$2 \times \frac{n(n+1)}{2} - n$ *M1A1*

OR

$\frac{n}{2}(2+(n-1)2)$ $\left(\text{using } S_n = \frac{n}{2}(2u_1+(n-1)d)\right)$ *M1A1*

OR

$\frac{n}{2}(1+2n-1)$ $\left(\text{using } S_n = \frac{n}{2}(u_1+u_n)\right)$ *M1A1*

THEN

$= n^2$ *AG*

(iii) $47^2 - 14^2 = 2013$ *A1*

[4 marks]

Examiners report

In part (a) (i), a large number of candidates were unable to correctly use sigma notation to express the sum of the first n positive odd integers. Common errors included summing $2n-1$ from 1 to n and specifying sums with incorrect limits. Parts (a) (ii) and (iii) were generally well done.

Markscheme

(i) **EITHER**

a pentagon and five diagonals   *A1*

OR

five diagonals (circle optional)   *A1*

(ii)   Each point joins to $n - 3$ other points.   *A1*

a correct argument for $n(n-3)$ *R1*

a correct argument for $\frac{n(n-3)}{2}$ *R1*

(iii)   attempting to solve $\frac{1}{2}n(n-3) > 1\,000\,000$ for n.   *(M1)*

$n > 1415.7$ *(A1)*

$n = 1416$ *A1*

[7 marks]

Examiners report

Parts (b) (i) and (iii) were generally well done. In part (b) (iii), many candidates unnecessarily simplified their quadratic when direct GDC use could have been employed. A few candidates gave $n > 1416$ as their final answer. While some candidates displayed sound reasoning in part (b) (ii), many candidates unfortunately adopted a 'proof by example' approach.

Markscheme

(i) $np = 4$ and $npq = 3$ *(A1)*

attempting to solve for n and p *(M1)*

$n = 16$ and $p = \frac{1}{4}$ *A1*

(ii) $X \sim B(16, 0.25)$ *(A1)*

$P(X = 1) = 0.0534538...\left(= \dbinom{16}{1}(0.25)(0.75)^{15}\right)$ *(A1)*

$P(X = 3) = 0.207876...\left(= \dbinom{16}{3}(0.25)^3(0.75)^{13}\right)$ *(A1)*

$P(X = 1) + P(X = 3)$ *(M1)*

$= 0.261$ *A1*

[8 marks]

Examiners report

Part (c) was generally well done. In part (c) (ii), some candidates multiplied the two probabilities rather than adding the two probabilities.

21.

Markscheme

$h(x) = f(x-3) - 2 = \ln(x-3) - 2$ *(M1)(A1)*

$g(x) = -h(x) = 2 - \ln(x-3)$ *M1*

Note: Award **M1** only if it is clear the effect of the reflection in the *x*-axis:

the expression is correct **OR**
there is a change of signs of the previous expression **OR**
there's a graph or an explanation making it explicit

$= \ln e^2 - \ln(x-3)$ *M1*

$= \ln \left(\frac{e^2}{x-3} \right)$ *A1*

[5 marks]

Examiners report

This question was well attempted but many candidates could have scored better had they written down all the steps to obtain the final expression. In some cases, as the final expression was incorrect and the middle steps were missing, candidates scored just 1 mark. That could be a consequence of a small mistake, but the lack of working prevented them from scoring at least all method marks. Some candidates performed the transformations well but were not able to use logarithms properties to transform the answer and give it as a single logarithm.

22.

Markscheme

$\left(x - \frac{2}{x} \right)^4 = x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$ *(M1)(A1)*

$\left(x^2 + \frac{2}{x} \right)^3 = x^6 + 6x^3 + 12 + \frac{8}{x^3}$ *(M1)(A1)*

Note: Accept unsimplified or uncalculated coefficients in the constant term

$= 24 \times 12$ *(M1)(A1)*

$= 288$ *A1*

[7 marks]

Examiners report

Many correct answers were seen, although most candidates used rather inefficient methods (e.g. expanding the brackets in multiple steps). In a very few cases candidates used the binomial theorem to obtain the answer quickly.

23a.

Markscheme

(i) $(x + iy)^2 = -5 + 12i$
$x^2 + 2ixy + i^2y^2 = -5 + 12i$ *A1*
(ii)   equating real and imaginary parts   *M1*
$x^2 - y^2 = -5$ *AG*
$xy = 6$ *AG*
[2 marks]

Since (a) was a 'show that' question, it was essential for candidates to give a convincing explanation of how the quoted results were obtained. Many candidates just wrote

$$(x + iy)^2 = x^2 - y^2 + 2ixy = -5 + 12i$$

$$\text{Therefore } x^2 - y^2 = -5 \text{ and } xy = 6$$

This was not given full credit since it simply repeated what was given in the question. Candidates were expected to make it clear that they were equating real and imaginary parts. In (b), candidates who attempted to use de Moivre's Theorem to find the square roots were given no credit since the question stated 'hence'.

23b.

[5 marks]

Markscheme

substituting   *M1*
EITHER
$x^2 - \frac{36}{x^2} = -5$
$x^4 + 5x^2 - 36 = 0$ *A1*
$x^2 = 4, -9$ *A1*
$x = \pm 2$ and $y = \pm 3$ *(A1)*
OR
$\frac{36}{y^2} - y^2 = -5$
$y^4 - 5y^2 - 36 = 0$ *A1*
$y^2 = 9, -4$ *A1*
$y^2 = \pm 3$ and $x = \pm 2$ *(A1)*
Note: Accept solution by inspection if completely correct.

THEN
the square roots are $(2 + 3i)$ and $(-2 - 3i)$ *A1*
[5 marks]

Examiners report

Since (a) was a 'show that' question, it was essential for candidates to give a convincing explanation of how the quoted results were obtained. Many candidates just wrote

$$(x + iy)^2 = x^2 - y^2 + 2ixy = -5 + 12i$$

$$\text{Therefore } x^2 - y^2 = -5 \text{ and } xy = 6$$

This was not given full credit since it simply repeated what was given in the question. Candidates were expected to make it clear that they were equating real and imaginary parts. In (b), candidates who attempted to use de Moivre's Theorem to find the square roots were given no credit since the question stated 'hence'.

23c.

[3 marks]

Markscheme

EITHER

consider $z = x + iy$

$z^* = x - iy$

$(z^*)^2 = x^2 - y^2 - 2ixy$ *A1*

$(z^2) = x^2 - y^2 + 2ixy$ *A1*

$(z^2)^* = x^2 - y^2 - 2ixy$ *A1*

$(z^*)^2 = (z^2)^*$ *AG*

OR

$z^* = re^{-i\theta}$

$(z^*)^2 = r^2 e^{-2i\theta}$ *A1*

$z^2 = r^2 e^{2i\theta}$ *A1*

$(z^2)^* = r^2 e^{-2i\theta}$ *A1*

$(z^*)^2 = (z^2)^*$ *AG*

[3 marks]

Examiners report

Since (a) was a 'show that' question, it was essential for candidates to give a convincing explanation of how the quoted results were obtained. Many candidates just wrote

$$(x + iy)^2 = x^2 - y^2 + 2ixy = -5 + 12i$$

$$\text{Therefore } x^2 - y^2 = -5 \text{ and } xy = 6$$

This was not given full credit since it simply repeated what was given in the question. Candidates were expected to make it clear that they were equating real and imaginary parts. In (b), candidates who attempted to use de Moivre's Theorem to find the square roots were given no credit since the question stated 'hence'.

23d.

[2 marks]

Markscheme

$(2 - 3i)$ and $(-2 + 3i)$ *A1A1*

[2 marks]

Examiners report

Since (a) was a 'show that' question, it was essential for candidates to give a convincing explanation of how the quoted results were obtained. Many candidates just wrote

$$(x + iy)^2 = x^2 - y^2 + 2ixy = -5 + 12i$$

$$\text{Therefore } x^2 - y^2 = -5 \text{ and } xy = 6$$

This was not given full credit since it simply repeated what was given in the question. Candidates were expected to make it clear that they were equating real and imaginary parts. In (b), candidates who attempted to use de Moivre's Theorem to find the square roots were given no credit since the question stated 'hence'.

23e.

[2 marks]

Markscheme

the graph crosses the x-axis twice, indicating two real roots   *R1*

since the quartic equation has four roots and only two are real, the other two roots must be complex   *R1*

[2 marks]

Examiners report

23f.   *[5 marks]*

Markscheme

$f(x) = (x+4)(x-2)(x^2+cx+d)$ *A1A1*
$f(0) = -32 \Rightarrow d = 4$ *A1*
Since the curve passes through $(-1, -18)$
$-18 = 3 \times (-3)(5-c)$ *M1*
$c = 3$ *A1*
Hence $f(x) = (x+4)(x-2)(x^2+3x+4)$

[5 marks]

Examiners report

23g.   *[2 marks]*

Markscheme

$x = \frac{-3 \pm \sqrt{9-16}}{2}$ *(M1)*
$\Rightarrow x = -\frac{3}{2} \pm i \frac{\sqrt{7}}{2}$ *A1*

[2 marks]

Examiners report

23h.   *[2 marks]*

Markscheme

A1A1

Note: Accept points or vectors on complex plane.
Award *A1* for two real roots and *A1* for two complex roots.
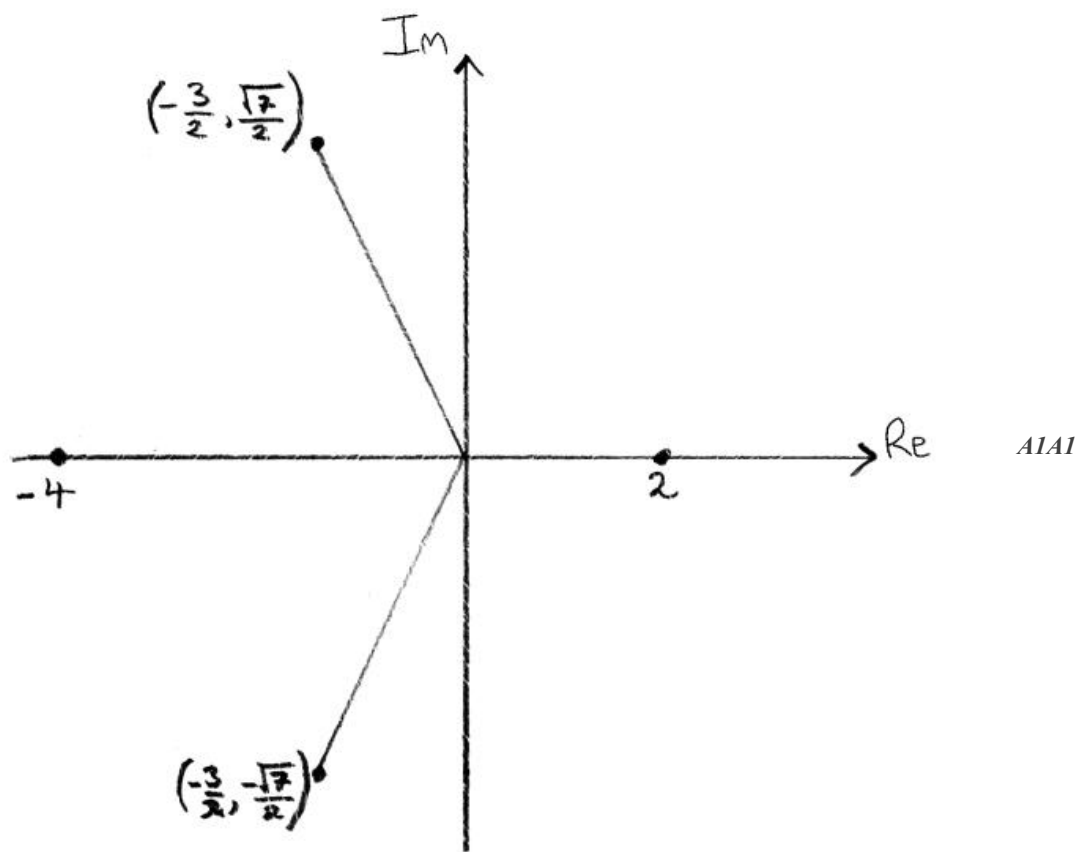
[2 marks]

Examiners report

In (a), the explanations were often unconvincing. Candidates were expected to make it clear that the two intersections with the *x*-axis gave two real roots and, since the polynomial was a quartic and therefore had four zeros, the other two roots must be complex. Candidates who made vague statements such as 'the graph shows two real roots' were not given full credit. In (b), most candidates stated the values of *a* and *b* correctly but algebraic errors often led to incorrect values for the other parameters. Candidates who failed to solve (b) correctly were unable to solve (c), (d) and (e) correctly although follow through was used where possible.

23i.

[6 marks]

Markscheme

real roots are $4e^{i\pi}$ and $2e^{i0}$ **A1A1**

considering $-\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$

$r = \sqrt{\frac{9}{4} + \frac{7}{4}} = 2$ **A1**

finding θ using $\arctan\left(\frac{\sqrt{7}}{3}\right)$ **M1**

$\theta = \arctan\left(\frac{\sqrt{7}}{3}\right) + \pi$ or $\theta = \arctan\left(-\frac{\sqrt{7}}{3}\right) + \pi$ **A1**

$\Rightarrow z = 2e^{i\left(\arctan\left(\frac{\sqrt{7}}{3}\right)+\pi\right)}$ or $\Rightarrow z = 2e^{i\left(\arctan\left(\frac{-\sqrt{7}}{3}\right)+\pi\right)}$ **A1**

Note: Accept arguments in the range $-\pi$ to π or 0 to 2π.
Accept answers in degrees.

[6 marks]

Examiners report

In (a), the explanations were often unconvincing. Candidates were expected to make it clear that the two intersections with the x-axis gave two real roots and, since the polynomial was a quartic and therefore had four zeros, the other two roots must be complex. Candidates who made vague statements such as 'the graph shows two real roots' were not given full credit. In (b), most candidates stated the values of a and b correctly but algebraic errors often led to incorrect values for the other parameters. Candidates who failed to solve (b) correctly were unable to solve (c), (d) and (e) correctly although follow through was used where possible.

24a.   *[4 marks]*

Markscheme

let $f(x) = \frac{1}{2x+1}$ and using the result $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$f'(x) = \lim_{h\to 0}\left(\frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h}\right)$ **M1A1**

$\Rightarrow f'(x) = \lim_{h\to 0}\left(\frac{[2x+1]-[2(x+h)+1]}{h[2(x+h)+1][2x+1]}\right)$ **A1**

$\Rightarrow f'(x) = \lim_{h\to 0}\left(\frac{-2}{[2(x+h)+1][2x+1]}\right)$ **A1**

$\Rightarrow f'(x) = \frac{-2}{(2x+1)^2}$ **AG**

[4 marks]

Examiners report

Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k + 1$. Many candidates just write 'Let $n = k$' which is of course meaningless. The conclusion is often of the form 'True for $n = 1$, $n = k$ and $n = k + 1$ therefore true by induction'. Credit is only given for a conclusion which includes a statement such as 'True for $n = k \Rightarrow$ true for $n = k + 1$'.

24b.   *[9 marks]*

Markscheme

let $y = \frac{1}{2x+1}$

we want to prove that $\frac{d^n y}{dx^n} = (-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$

let $n = 1 \Rightarrow \frac{dy}{dx} = (-1)^1 \frac{2^1 1!}{(2x+1)^{1+1}}$ **M1**

$\Rightarrow \frac{dy}{dx} = \frac{-2}{(2x+1)^2}$ which is the same result as part (a)

hence the result is true for $n = 1$ **R1**

assume the result is true for $n = k$: $\frac{d^k y}{dx^k} = (-1)^k \frac{2^k k!}{(2x+1)^{k+1}}$ **M1**

$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left[(-1)^k \frac{2^k k!}{(2x+1)^{k+1}}\right]$ **M1**

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left[(-1)^k 2^k k!(2x+1)^{-k-1}\right]$ **(A1)**

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = (-1)^k 2^k k!(-k-1)(2x+1)^{-k-2} \times 2$ **A1**

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = (-1)^{k+1} 2^{k+1}(k+1)!(2x+1)^{-k-2}$ **(A1)**

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = (-1)^{k+1} \frac{2^{k+1}(k+1)!}{(2x+1)^{k+2}}$ **A1**

hence if the result is true for $n = k$, it is true for $n = k+1$

since the result is true for $n = 1$, the result is proved by mathematical induction   **R1**

Note: Only award final **R1** if all the **M** marks have been gained.

[9 marks]

Examiners report

Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k+1$. Many candidates just write 'Let $n = k$' which is of course meaningless. The conclusion is often of the form 'True for $n = 1$, $n = k$ and $n = k+1$ therefore true by induction'. Credit is only given for a conclusion which includes a statement such as 'True for $n = k \Rightarrow$ true for $n = k+1$'.

25a.

Markscheme

$S_n = \frac{n}{2}[2a + (n-1)d]$

$212 = \frac{16}{2}(2a + 15d)$ $(= 16a + 120d)$ **A1**

n^{th} term is $a + (n-1)d$

$8 = a + 4d$ **A1**

solving simultaneously: **(M1)**

$d = 1.5, \ a = 2$ **A1**

[4 marks]

Examiners report

This proved to be a good start to the paper for most candidates. The vast majority made a meaningful attempt at this question with many gaining the correct answers. Candidates who lost marks usually did so because of mistakes in the working. In part (b) the most efficient way of gaining the answer was to use the calculator once the initial inequality was set up. A small number of candidates spent valuable time unnecessarily manipulating the algebra before moving to the calculator.

25b.

Markscheme

$\frac{n}{2}[4 + 1.5(n-1)] > 600$ *(M1)*

$\Rightarrow 3n^2 + 5n - 2400 > 0$ *(A1)*

$\Rightarrow n > 27.4..., \ (n < -29.1...)$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ *A1*

[3 marks]

Examiners report

This proved to be a good start to the paper for most candidates. The vast majority made a meaningful attempt at this question with many gaining the correct answers. Candidates who lost marks usually did so because of mistakes in the working. In part (b) the most efficient way of gaining the answer was to use the calculator once the initial inequality was set up. A small number of candidates spent valuable time unnecessarily manipulating the algebra before moving to the calculator.

26a.

Markscheme

number of arrangements of boys is $15!$ and number of arrangements of girls is $10!$ *(A1)*

total number of arrangements is $15! \times 10! \times 2 (= 9.49 \times 10^{18})$ *M1A1*

Note: If 2 is omitted, award *(A1)M1A0*.

[3 marks]

Examiners report

A good number of correct answers were seen to this question, but a significant number of candidates forgot to multiply by 2 in part (a) and in part (b) the most common error was to add the combinations rather than multiply them.

26b.

Markscheme

number of ways of choosing two boys is $\binom{15}{2}$ and the number of ways of choosing three girls is $\binom{10}{3}$ *(A1)*

number of ways of choosing two boys and three girls is $\binom{15}{2} \times \binom{10}{3} = 12600$ *M1A1*

[3 marks]

Examiners report

A good number of correct answers were seen to this question, but a significant number of candidates forgot to multiply by 2 in part (a) and in part (b) the most common error was to add the combinations rather than multiply them.

27a.

Markscheme

height $= 4 \times 0.95^4$ *(A1)*

$= 3.26$ (metres)   *A1*

[2 marks]

Examiners report

The majority of candidates were able to start this question and gain some marks, but only better candidates gained full marks. In part (a) the common error was to assume the wrong number of bounces and in part (b) many candidates lost marks due to rounding the inequality in the wrong direction. Part (c) was found difficult with only a limited number recognising the need for the sum to infinity of a geometric sequence and many of those did not recognise how to link the sum to infinity to the total distance travelled.

27b.   *[3 marks]*

Markscheme

$4 \times 0.95^n < 1$ *(M1)*

$0.95^n < 0.25$

$\Rightarrow n > \frac{\ln 0.25}{\ln 0.95}$ *(A1)*

$\Rightarrow n > 27.0$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ *A1*

Note: If candidates have used $n - 1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

[3 marks]

Examiners report

The majority of candidates were able to start this question and gain some marks, but only better candidates gained full marks. In part (a) the common error was to assume the wrong number of bounces and in part (b) many candidates lost marks due to rounding the inequality in the wrong direction. Part (c) was found difficult with only a limited number recognising the need for the sum to infinity of a geometric sequence and many of those did not recognise how to link the sum to infinity to the total distance travelled.

27c.   *[3 marks]*

Markscheme

METHOD 1

recognition of geometric series with sum to infinity, first term of 4×0.95 and common ratio 0.95   *M1*

recognition of the need to double this series and to add 4   *M1*

total distance travelled is $2\left(\frac{4 \times 0.95}{1 - 0.95}\right) + 4 = 156$ (metres)   *A1*

[3 marks]

Note: If candidates have used $n - 1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

METHOD 2

recognition of a geometric series with sum to infinity, first term of 4 and common ratio 0.95   *M1*

recognition of the need to double this series and to subtract 4   **M1**

total distance travelled is $2\left(\frac{4}{1 - 0.95}\right) - 4 = 156$ (metres)   **A1**

[3 marks]

Examiners report

The majority of candidates were able to start this question and gain some marks, but only better candidates gained full marks. In part (a) the common error was to assume the wrong number of bounces and in part (b) many candidates lost marks due to rounding the inequality in the wrong direction. Part (c) was found difficult with only a limited number recognising the need for the sum to infinity of a geometric sequence and many of those did not recognise how to link the sum to infinity to the total distance travelled.

28a.   *[5 marks]*

Markscheme

in augmented matrix form $\begin{vmatrix} 1 & -3 & 1 & 3 \\ 1 & 5 & -2 & 1 \\ 0 & 16 & -6 & k \end{vmatrix}$

attempt to find a line of zeros   *(M1)*

$r_2 - r_1 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 16 & -6 & k \end{vmatrix}$ *(A1)*

$r_3 - 2r_2 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$ *(A1)*

there is an infinite number of solutions when $k = -4$ *R1*
there is no solution when
$k \neq -4, \ (k \in \mathbb{R})$ *R1*

Note: Approaches other than using the augmented matrix are acceptable.

[5 marks]

Examiners report

Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

Markscheme

using $\begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$ and letting $z = \lambda$ *(M1)*

$8y - 3\lambda = -2$
$\Rightarrow y = \frac{3\lambda - 2}{8}$ *(A1)*
$x - 3y + z = 3$
$\Rightarrow x - \left(\frac{9\lambda - 6}{8} \right) + \lambda = 3$ *(M1)*
$\Rightarrow 8x - 9\lambda + 6 + 8\lambda = 24$
$\Rightarrow x = \frac{18 + \lambda}{8}$ *(A1)*

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{18}{8} \\ -\frac{2}{8} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 1 \end{pmatrix}$ *(M1)(A1)*

$r = \begin{pmatrix} \frac{9}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$ *A1*

Note: Accept equivalent answers.

[7 marks]

Examiners report

Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

28c.

Markscheme

recognition that $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ is parallel to the plane   *(A1)*

direction normal of the plane is given by $\begin{vmatrix} i & j & k \\ 1 & 3 & 8 \\ 3 & -2 & 0 \end{vmatrix}$ *(M1)*

$= 16i + 24j - 11k$ *A1*

Cartesian equation of the plane is given by $16x + 24y - 11z = d$ and a point which fits this equation is $(1, 2, 0)$ *(M1)*

$\Rightarrow 16 + 48 = d$

$d = 64$ *A1*

hence Cartesian equation of plane is $16x + 24y - 11z = 64$ *AG*

Note: Accept alternative methods using dot product.

[5 marks]

Examiners report

Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

28d.

Markscheme

the plane crosses the z-axis when $x = y = 0$ *(M1)*

coordinates of P are $\left(0, 0, -\frac{64}{11}\right)$ *A1*

Note: Award *A1* for stating $z = -\frac{64}{11}$.

Note: Accept. $\begin{pmatrix} 0 \\ 0 \\ -\frac{64}{11} \end{pmatrix}$

[2 marks]

Examiners report

Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

28e.

Markscheme

recognition that the angle between the line and the direction normal is given by:

$\left| \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 16 \\ 24 \\ -11 \end{pmatrix} = \sqrt{29}\sqrt{953}\cos\theta \right|$ where θ is the angle between the line and the normal vector   *M1A1*

$\Rightarrow \left| 122 = \sqrt{29}\sqrt{953}\cos\theta \right|$ *(A1)*

$\Rightarrow \left| \theta = 42.8° \text{ (0.747 radians)} \right|$ *(A1)*

hence the angle between the line and the plane is $90° - 42.8° = 47.2°$ (0.824 radians)   *A1*

[5 marks]

Note: Accept use of the formula $\boldsymbol{a}.\boldsymbol{b} = \left| \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin\theta \right|$.

Examiners report

Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

29.

Markscheme

$\left| i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \right|$ *(A1)*

$z_1 = i^{\frac{1}{3}} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \right)^{\frac{1}{3}} = \left| \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \quad \left(= \frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \right|$ *M1A1*

$z_2 = \left| \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} \quad \left(= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \right|$ *(M1)A1*

$z_3 = \left| \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i \right|$ *A1*

Note: Accept exponential and cis forms for intermediate results, but not the final roots.

Note: Accept the method based on expanding $\left| (a+b)^3 \right|$. *M1* for attempt, *M1* for equating real and imaginary parts, *A1* for finding $a = 0$ and $\left| b = \frac{1}{2} \right|$, then *A1A1A1* for the roots.

[6 marks]

Examiners report

A varied response. Many knew how to solve this standard question in the most efficient way. A few candidates expanded $\left| (a+ib)^3 \right|$ and solved the resulting fairly simple equations. A disappointing minority of candidates did not know how to start.

30.

Markscheme

proposition is true for $n = 1$ since $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \left.\dfrac{1}{(1-x)^2}\right|$ **M1**

$= \left.\dfrac{1!}{(1-x)^2}\right|$ **A1**

Note: Must see the 1! for the **A1**.

assume true for $n = k$, $k \in \mathbb{Z}^+$, *i.e.* $\dfrac{\mathrm{d}^k y}{\mathrm{d}x^k} = \left.\dfrac{k!}{(1-x)^{k+1}}\right|$ **M1**

consider $\dfrac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \left.\dfrac{\mathrm{d}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)}{\mathrm{d}x}\right|$ **(M1)**

$= \left.(k+1)k!(1-x)^{-(k+1)-1}\right|$ **A1**

$= \left.\dfrac{(k+1)!}{(1-x)^{k+2}}\right|$ **A1**

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers   **R1**

Note: The final **R1** is only available if at least 4 of the previous marks have been awarded.

[7 marks]

Examiners report

Most candidates were awarded good marks for this question. A disappointing minority thought that the $(k+1)$th derivative was the (k)th derivative multiplied by the first derivative. Providing an acceptable final statement remains a perennial issue.

31. *[7 marks]*

Markscheme

METHOD 1

$\arg(z_1 z_2) = \frac{5\pi}{6}$ $(150°)$ *(A1)*

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ $(90°)$ *(A1)*

$\Rightarrow \arg(z_1) + \arg(z_2) = \frac{5\pi}{6}$; $\arg(z_1) - \arg(z_2) = \frac{\pi}{2}$ *M1*

solving simultaneously

$\arg(z_1) = \frac{2\pi}{3}$ $(120°)$ and $\arg(z_2) = \frac{\pi}{6}$ $(30°)$ *A1A1*

Note: Accept decimal approximations of the radian measures.

$|z_1 z_2| = 2 \Rightarrow |z_1|\,|z_2| = 2;$ $\left|\frac{z_1}{z_2}\right| = 2 \Rightarrow \frac{|z_1|}{|z_2|} = 2$ *M1*

solving simultaneously

$|z_1| = 2;$ $|z_2| = 1$ *A1*

[7 marks]

METHOD 2

$z_1 = 2iz_2$ $2iz_2^2 = -\sqrt{3} + i$ *(M1)*

$z_2^2 = \frac{-\sqrt{3}+i}{2i}$ *A1*

$z_2 = \sqrt{\frac{-\sqrt{3}+i}{2i}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ or $e^{\frac{\pi}{6}i}$ *(M1)(A1)*

(allow $0.866 + 0.5i$ or $e^{0.524i}$)

$z_1 = -1 + \sqrt{3}i$ or $2e^{\frac{2\pi}{3}i}$ — (allow $-1 + 1.73i$ or $2e^{2.09i}$)   *(A1)*

z_1 modulus = 2, argument = $\frac{2\pi}{3}$ *A1*

z_2 modulus = 1, argument = $\frac{\pi}{6}$ *A1*

Note: Accept degrees and decimal approximations to radian measure.

[7 marks]

Examiners report

Candidates generally found this question challenging. Many candidates had difficulty finding the arguments of $z_1 z_2$ and z_1/z_2. Among candidates who attempted to solve for z_1 and z_2 in Cartesian form, many had difficulty with the algebraic manipulation involved.

32a.

[4 marks]

Markscheme

for the series to have a finite sum, $\left|\frac{2x}{x+1}\right| < 1$ *R1*

(sketch from gdc or algebraic method)   *M1*

S_∞ exists when $-\frac{1}{3} < x < 1$ *A1A1*

Note: Award *A1* for bounds and *A1* for strict inequalities.

[4 marks]

Examiners report

A large number of candidates omitted the absolute value sign in the inequality in (a), or the use of the correct double inequality. Among candidates who had the correct statement, those who used their GDC were the most successful. The algebraic solution of the inequality was difficult for some candidates. In (b), quite a number of candidates found the sum of the first *n* terms of the geometric series, rather than the infinite sum of the series.

[2 marks]

32b.

Markscheme

$S_\infty = \frac{\frac{2x}{x+1}}{1 - \frac{2x}{x+1}} = \frac{2x}{1-x}$ *M1A1*

[2 marks]

Examiners report

A large number of candidates omitted the absolute value sign in the inequality in (a), or the use of the correct double inequality. Among candidates who had the correct statement, those who used their GDC were the most successful. The algebraic solution of the inequality was difficult for some candidates. In (b), quite a number of candidates found the sum of the first *n* terms of the geometric series, rather than the infinite sum of the series.

[7 marks]

33.

Markscheme

METHOD 1

$\frac{2-i}{1+i} = \frac{1-3i}{2}$ *A1*

$\frac{6+8i}{u+i} \times \frac{u-i}{u-i} = \frac{6u+8+(8u-6)i}{u^2+1}$ *M1A1*

$\Rightarrow \frac{2-i}{1+i} - \frac{6+8u}{u+i} = \frac{1}{2} - \frac{6u+8}{u^2+1} - \left(\frac{3}{2} + \frac{8u-6}{u^2+1}\right)i$

$\operatorname{Im} z = \operatorname{Re} z$

$\Rightarrow \frac{1}{2} - \frac{6u+8}{u^2+1} = -\frac{3}{2} - \frac{8u-6}{u^2+1}$ *A1*

(sketch from gdc, or algebraic method)   *(M1)*

$u = -3; u = 2$ *A1A1*   *N2*

[7 marks]

METHOD 2

$\frac{2-i}{1+i} - \frac{6+8i}{u+i} = \frac{(2-i)(u+i)-(1+i)(6+8i)}{(u-1)+i(u+1)}$ *M1A1*

$= \frac{(2-i)(u+i)-(1+i)(6+8i)}{(u-1)+i(u+1)} \cdot \frac{(u-1)-i(u+1)}{(u-1)-i(u+1)}$ *M1*

$= \frac{u^2-12u-15+i(-3u^2-16u+9)}{2(u^2+1)}$ *A1*

$\operatorname{Re} z = \operatorname{Im} z \Rightarrow u^2 - 12u - 15 = -3u^2 - 16u + 9$ *M1*

$u = -3; u = 2$ *A1A1*   *N2*

[7 marks]

Examiners report

Many candidates failed to access their GDC early enough to avoid huge algebraic manipulations, often carried out with many errors. Some candidates failed to separate and equate the real and imaginary parts of the expression obtained.

34a.   *[9 marks]*

Markscheme

$S_{2n} = \frac{2n}{2}\left(2(8) + (2n-1)\frac{1}{4}\right)$ *(M1)*

$= n\left(16 + \frac{2n-1}{4}\right)$ *A1*

$S_{3n} = \frac{3n}{2}\left(2 \times 8 + (3n-1)\frac{1}{4}\right)$ *(M1)*

$= \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ *A1*

$S_{2n} = S_{3n} - S_{2n} \Rightarrow 2S_{2n} = S_{3n}$ *M1*

solve $2S_{2n} = S_{3n}$

$\Rightarrow 2n\left(16 + \frac{2n-1}{4}\right) = \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ *A1*

$\left(\Rightarrow 2\left(16 + \frac{2n-1}{4}\right) = \frac{3}{2}\left(16 + \frac{3n-1}{4}\right)\right)$

(gdc or algebraic solution)   *(M1)*

$n = 63$ *A2*

[9 marks]

34b.

[7 marks]

Markscheme

$(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + \ldots$

$= (a_1 - a_1 r)^2 + (a_1 r - a_1 r^2)^2 + (a_1 r^2 - a_1 r^3) + \ldots$ *M1A1*

$= [a_1(1-r)]^2 + [a_1 r(1-r)]^2 + \left[a_1 r^2(1-r)\right]^2 + \ldots + \left[a_1 r^{n-1}(1-r)\right]^2$ *(A1)*

Note: This *A1* is for the expression for the last term.

$= a_1^2(1-r)^2 + a_1^2 r^2(1-r)^2 + a_1^2 r^4(1-r)^2 + \ldots + a_1^2 r^{2n-2}(1-r)^2$ *A1*

$= a_1^2(1-r)^2(1 + r^2 + r^4 + \ldots + r^{2n-2})$ *A1*

$= a_1^2(1-r)^2 \left(\frac{1-r^{2n}}{1-r^2}\right)$ *M1A1*

$= \frac{a_1^2(1-r)(1-r^{2n})}{1+r}$ *AG*

[7 marks]

Examiners report

35a.

[1 mark]

Markscheme

$\left|e^{i\theta}\right| \ (= \ |\cos\theta + i\sin\theta|) = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ *M1AG*

[1 mark]

Examiners report

Parts (b) and (c) were answered fairly well by quite a few candidates. In (a) many candidates failed to write the formula for the modulus of a complex number. (c) proved inaccessible for a large number of candidates. The algebraic manipulation required and the recognition of the imaginary and real parts in order to arrive at the necessary relationship were challenging for many candidates.

35b.

[2 marks]

Markscheme

$z = \frac{1}{3} e^{i\theta}$ *A1*

$|z| = \left| \frac{1}{3} e^{i\theta} \right| = \frac{1}{3}$ *A1AG*

[2 marks]

Examiners report

Parts (b) and (c) were answered fairly well by quite a few candidates. In (a) many candidates failed to write the formula for the modulus of a complex number. (c) proved inaccessible for a large number of candidates. The algebraic manipulation required and the recognition of the imaginary and real parts in order to arrive at the necessary relationship were challenging for many candidates.

35c.

Markscheme

$S_\infty = \frac{a}{1-r} = \frac{1}{1 - \frac{1}{3} e^{i\theta}}$ *(M1)A1*

[2 marks]

Examiners report

Parts (b) and (c) were answered fairly well by quite a few candidates. In (a) many candidates failed to write the formula for the modulus of a complex number. (c) proved inaccessible for a large number of candidates. The algebraic manipulation required and the recognition of the imaginary and real parts in order to arrive at the necessary relationship were challenging for many candidates.

35d.

Markscheme

EITHER

$$S_\infty = \left| \frac{1}{1 - \frac{1}{3}\cos\theta - \frac{1}{3}\mathrm{i}\sin\theta} \right| \quad \textbf{\textit{A1}}$$

$$= \left| \frac{1 - \frac{1}{3}\cos\theta + \frac{1}{3}\mathrm{i}\sin\theta}{\left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\mathrm{i}\sin\theta\right)\left(1 - \frac{1}{3}\cos\theta + \frac{1}{3}\mathrm{i}\sin\theta\right)} \right| \quad \textbf{\textit{M1A1}}$$

$$= \left| \frac{1 - \frac{1}{3}\cos\theta + \frac{1}{3}\mathrm{i}\sin\theta}{\left(1 - \frac{1}{3}\cos\theta\right)^2 + \frac{1}{9}\sin^2\theta} \right| \quad \textbf{\textit{A1}}$$

$$= \left| \frac{1 - \frac{1}{3}\cos\theta + \frac{1}{3}\mathrm{i}\sin\theta}{1 - \frac{2}{3}\cos\theta + \frac{1}{9}} \right| \quad \textbf{\textit{A1}}$$

OR

$$S_\infty = \left| \frac{1}{1 - \frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}} \right|$$

$$= \left| \frac{1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{\left(1 - \frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}\right)\left(1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}\right)} \right| \quad \textbf{\textit{M1A1}}$$

$$= \left| \frac{1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{1 - \frac{1}{3}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right) + \frac{1}{9}} \right| \quad \textbf{\textit{A1}}$$

$$= \left| \frac{1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{\frac{10}{9} - \frac{2}{3}\cos\theta} \right| \quad \textbf{\textit{A1}}$$

$$= \left| \frac{1 - \frac{1}{3}(\cos\theta - \mathrm{i}\sin\theta)}{\frac{10}{9} - \frac{2}{3}\cos\theta} \right| \quad \textbf{\textit{A1}}$$

THEN

taking imaginary parts on both sides

$$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \ldots = \left| \frac{\frac{1}{3}\sin\theta}{\frac{10}{9} - \frac{2}{3}\cos\theta} \right| \quad \textbf{\textit{M1A1A1}}$$

$$= \left| \frac{\sin\theta}{\frac{10}{9} - \frac{2}{3}\cos\theta} \right|$$

$$\Rightarrow \sin\theta + \frac{1}{3}\sin 2\theta + \ldots = \left| \frac{9\sin\theta}{10 - 6\cos\theta} \right| \quad \textbf{\textit{AG}}$$

[8 marks]

Examiners report

Parts (b) and (c) were answered fairly well by quite a few candidates. In (a) many candidates failed to write the formula for the modulus of a complex number. (c) proved inaccessible for a large number of candidates. The algebraic manipulation required and the recognition of the imaginary and real parts in order to arrive at the necessary relationship were challenging for many candidates.

36a. *[2 marks]*

Markscheme

for $n \geqslant 1$, $n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1 \geqslant 2 \times 2 \times 2 \ldots 2 \times 2 \times 1 = 2^{n-1}$ *__M1A1__*

$\Rightarrow n! \geqslant 2^{n-1}$ for $n \geqslant 1$ *__AG__*

[2 marks]

Part (a) of this question was found challenging by the majority of candidates, a fairly common 'solution' being that the result is true for $n = 1, 2, 3$ and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between $\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

36b.

Markscheme

$n! \geqslant 2^{n-1} \Rightarrow \frac{1}{n!} \leqslant \frac{1}{2^{n-1}}$ for $n \geqslant 1$ **A1**

$\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a positive converging geometric series   **R1**

hence $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test   **R1**

[3 marks]

Examiners report

Part (a) of this question was found challenging by the majority of candidates, a fairly common 'solution' being that the result is true for $n = 1, 2, 3$ and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between $\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

37.

Markscheme

METHOD 1

$z = (2 - \mathrm{i})(z + 2)$ *M1*

$= 2z + 4 - \mathrm{i}z - 2\mathrm{i}$

$z(1 - \mathrm{i}) = -4 + 2\mathrm{i}$

$z = \dfrac{-4 + 2\mathrm{i}}{1 - \mathrm{i}}$ *A1*

$z = \dfrac{-4 + 2\mathrm{i}}{1 - \mathrm{i}} \times \dfrac{1 + \mathrm{i}}{1 + \mathrm{i}}$ *M1*

$= -3 - \mathrm{i}$ *A1*

METHOD 2

let $z = a + \mathrm{i}b$

$\dfrac{a + \mathrm{i}b}{a + \mathrm{i}b + 2} = 2 - \mathrm{i}$ *M1*

$a + \mathrm{i}b = (2 - i)((a + 2) + \mathrm{i}b)$

$a + \mathrm{i}b = 2(a + 2) + 2b\mathrm{i} - \mathrm{i}(a + 2) + b$

$a + \mathrm{i}b = 2a + b + 4 + (2b - a - 2)\mathrm{i}$

attempt to equate real and imaginary parts   *M1*

$a = 2a + b + 4 \ (\Rightarrow a + b + 4 = 0)$

and $b = 2b - a - 2 \ (\Rightarrow -a + b - 2 = 0)$ *A1*

Note: Award *A1* for two correct equations.

$b = -1$; $a = -3$ *A1*

$z = -3 - \mathrm{i}$

[4 marks]

Examiners report

A number of different methods were adopted in this question with some candidates working through their method to a correct answer. However many other candidates either stopped with z still expressed as a quotient of two complex numbers or made algebraic mistakes.

38.

[5 marks]

Markscheme

METHOD 1

constant term: $\binom{5}{0}(-2x)^0 \binom{7}{0}x^0 = 1$ *A1*

term in x: $\binom{7}{1}x + \binom{5}{1}(-2x) = -3x$ *(M1)A1*

term in x^2: $\binom{7}{2}x^2 + \binom{5}{2}(-2x)^2 + \binom{7}{1}x\binom{5}{1}(-2x) = -9x^2$ *M1A1*   *N3*

[5 marks]

METHOD 2

$(1-2x)^5(1+x)^7 = \left(1 + 5(-2x) + \frac{5\times4(-2x)^2}{2!} + \dots\right)\left(1 + 7x + \frac{7\times6}{2}x^2 + \dots\right)$ *M1M1*

$= (1 - 10x + 40x^2 + \dots)(1 + 7x + 21x^2 + \dots)$

$= 1 + 7x + 21x^2 - 10x - 70x^2 + 40x^2 + \dots$

$= 1 - 3x - 9x^2 + \dots$ *A1A1A1*   *N3*

[5 marks]

Examiners report

Although the majority of the candidates understood the question and attempted it, excessive time was spent on actually expanding the expression without consideration of the binomial theorem. A fair amount of students confused "ascending order", giving the last three instead of the first three terms.

39.

Markscheme

$(\sin\theta + i(1 - \cos\theta))^2 = \sin^2\theta - (1 - \cos\theta)^2 + i2\sin\theta(1 - \cos\theta)$ *M1A1*

Let α be the required argument.

$\tan\alpha = \dfrac{2\sin\theta(1-\cos\theta)}{\sin^2\theta - (1-\cos\theta)^2}$ *M1*

$= \dfrac{2\sin\theta(1-\cos\theta)}{(1-\cos^2\theta) - (1 - 2\cos\theta + \cos^2\theta)}$ *(M1)*

$= \dfrac{2\sin\theta(1-\cos\theta)}{2\cos\theta(1-\cos\theta)}$ *A1*

$= \tan\theta$ *A1*

$\alpha = \theta$ *A1*

[7 marks]

Examiners report

Very few candidates scored more than the first two marks in this question. Some candidates had difficulty manipulating trigonometric identities. Most candidates did not get as far as defining the argument of the complex expression.

40.

Markscheme

(a) $\left|1+i\sqrt{3}\right|=2$ or $\left|1-i\right|=\sqrt{2}$ *(A1)*

$\arg(1+i\sqrt{3})=\frac{\pi}{3}$ or $\arg(1-i)=-\frac{\pi}{4}$ $\left(\text{accept }\frac{7\pi}{4}\right)$ *(A1)*

$\left|z_1\right|=2^m$ *A1*

$\left|z_2\right|=\sqrt{2}^n$ *A1*

$\arg(z_1)=m\arctan\sqrt{3}=m\frac{\pi}{3}$ *A1*

$\arg(z_2)=n\arctan(-1)=n\frac{-\pi}{4}$ $\left(\text{accept }n\frac{7\pi}{4}\right)$ *A1 N2*

[6 marks]

(b) $2^m=\sqrt{2}^n\Rightarrow n=2m$ *(M1)A1*

$m\frac{\pi}{3}=n\frac{-\pi}{4}+2\pi k$, where k is an integer *M1A1*

$\Rightarrow m\frac{\pi}{3}+n\frac{\pi}{4}=2\pi k$

$\Rightarrow m\frac{\pi}{3}+2m\frac{\pi}{4}=2\pi k$ *(M1)*

$\frac{5}{6}m\pi=2\pi k$

$\Rightarrow m=\frac{12}{5}k$ *A1*

The smallest value of k such that m is an integer is 5, hence

$m=12$ *A1*

$n=24.$ *A1 N2*

[8 marks]

Total [14 marks]

Examiners report

Part (a) of this question was answered fairly well by candidates who attempted this question. The main error was the sign of the argument of z_2. Few candidates attempted part (b), and of those who did, most scored the first two marks for equating the modulii. Only a very small number equated the arguments correctly using $2\pi k$.

41.

Markscheme

METHOD 1

Substituting $z = x + iy$ to obtain $w = \frac{x+yi}{(x+yi)^2+1}$ *(A1)*

$w = \frac{x+yi}{x^2-y^2+1+2xyi}$ *A1*

Use of $(x^2 - y^2 + 1 + 2xyi)$ to make the denominator real.   *M1*

$= \frac{(x+yi)(x^2-y^2+1-2xyi)}{(x^2-y^2+1)^2+4x^2y^2}$ *A1*

$\text{Im } w = \frac{y(x^2-y^2+1)-2x^2y}{(x^2-y^2+1)^2+4x^2y^2}$ *(A1)*

$= \frac{y(1-x^2-y^2)}{(x^2-y^2+1)^2+4x^2y^2}$ *A1*

$\text{Im } w = 0 \Rightarrow 1 - x^2 - y^2 = 0$ *i.e.* $|z| = 1$ as $y \neq 0$ *R1AG*   *N0*

[7 marks]

METHOD 2

$w(z^2 + 1) = z$ *(A1)*

$w(x^2 - y^2 + 1 + 2ixy) = x + yi$ *A1*

Equating real and imaginary parts

$w(x^2 - y^2 + 1) = x$ and $2wx = 1$, $y \neq 0$ *M1A1*

Substituting $w = \frac{1}{2x}$ to give $\frac{x}{2} - \frac{y^2}{2x} + \frac{1}{2x} = x$ *A1*

$-\frac{1}{2x}(y^2 - 1) = \frac{x}{2}$ or equivalent   *(A1)*

$x^2 + y^2 = 1$, *i.e.* $|z| = 1$ as $y \neq 0$ *R1AG*

[7 marks]

Examiners report

This was a difficult question that troubled most candidates. Most candidates were able to substitute $z = x + yi$ into w but were then unable to make any further meaningful progress. Common errors included not expanding $(x + iy)^2$ correctly or not using a correct complex conjugate to make the denominator real. A small number of candidates produced correct solutions by using $w = \frac{1}{z+z^{-1}}$

42a.   *[1 mark]*

Markscheme

$\text{sum} = \frac{45}{9}$, $\text{product} = \frac{40}{9}$ *A1*

[1 mark]

Examiners report

[N/A]

42b.   *[6 marks]*

Markscheme

it follows that $3\alpha = \frac{45}{9}$ and $\alpha(\alpha^2 - \beta^2) = \frac{40}{9}$ ***A1A1***

solving, $\alpha = \frac{5}{3}$ ***A1***

$\frac{5}{3}\left(\frac{25}{9} - \beta^2\right) = \frac{40}{9}$ ***M1***

$\beta = (\pm)\frac{1}{3}$ ***A1***

the other two roots are $2, \frac{4}{3}$ ***A1***

[6 marks]

Examiners report

[N/A]

43a.

Markscheme

using row operations, ***M1***

to obtain 2 equations in the same 2 variables ***A1A1***

for example $y - z = 1$

$2y - 2z = \lambda - 1$

the fact that one of the left hand sides is a multiple of the other left hand side indicates that the equations do not have a unique solution, or equivalent ***R1AG***

[4 marks]

Examiners report

[N/A]

43b.

Markscheme

(i) $\lambda = 3$ ***A1***

(ii)   put $z = \mu$ ***M1***

then $y = 1 + \mu$ ***A1***

and $x = -2\mu$ or equivalent ***A1***

[4 marks]

Examiners report

[N/A]

44a.

Markscheme

in Cartesian form

$z_1 = 2 \times -\frac{\sqrt{3}}{2} + 2 \times \frac{1}{2}i$ *M1*

$= -\sqrt{3} + i$ *A1*

$\frac{z_1}{z_2} = \frac{-\sqrt{3}+i}{-1+i}$

$= \frac{(-\sqrt{3}+i)}{(-1+i)} \times \frac{(-1-i)}{(-1-i)}$ *M1*

$= \frac{1+\sqrt{3}}{2} + \frac{(\sqrt{3}-1)}{2}i$ *A1*

in modulus-argument form

$z_2 = \sqrt{2}\,\mathrm{cis}\,135°$ *A1*

$\frac{z_1}{z_2} = \frac{2\,\mathrm{cis}\,150°}{\sqrt{2}\,\mathrm{cis}\,135°}$

$= \sqrt{2}\,\mathrm{cis}\,15°$ *A1A1*

[7 marks]

Examiners report

[N/A]

44b.

Markscheme

equating the two expressions for $\frac{z_1}{z_2}$

$\cos 15° = \frac{1+\sqrt{3}}{2\sqrt{2}}$ *A1*

$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$ *A1*

$\tan 75° = \frac{\cos 15°}{\sin 15°} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ *M1*

$= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$ *A1*

$= 2 + \sqrt{3}$ *A1*

[5 marks]

Examiners report

[N/A]

45a.

Markscheme

$f'(x) = e^x \sin x + e^x \cos x$ *A1*

$f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$ *A1*

$= 2e^x \cos x$ *A1*

$= 2e^x \sin\left(x + \frac{\pi}{2}\right)$ *AG*

[3 marks]

45b.

Markscheme

$f'''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right) + 2e^x \cos\left(x + \frac{\pi}{2}\right)$ **A1**

$f^{(4)}(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right) + 2e^x \cos\left(x + \frac{\pi}{2}\right) + 2e^x \cos\left(x + \frac{\pi}{2}\right) - 2e^x \sin\left(x + \frac{\pi}{2}\right)$ **A1**

$= 4e^x \cos\left(x + \frac{\pi}{2}\right)$ **A1**

$= 4e^x \sin(x + \pi)$ **A1**

[4 marks]

45c.

Markscheme

the conjecture is that

$f^{(2n)}(x) = 2^n e^x \sin\left(x + \frac{n\pi}{2}\right)$ **A1**

for $n = 1$, this formula gives

$f''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right)$ which is correct   **A1**

let the result be true for $n = k$, $\left(i.\,e.\ f^{(2k)}(x) = 2^k e^x \sin\left(x + \frac{k\pi}{2}\right)\right)$ **M1**

consider $f^{(2k+1)}(x) = 2^k e^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k e^x \cos\left(x + \frac{k\pi}{2}\right)$ **M1**

$f^{(2(k+1))}(x) = 2^k e^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k e^x \cos\left(x + \frac{k\pi}{2}\right) + 2^k e^x \cos\left(x + \frac{k\pi}{2}\right) - 2^k e^x \sin\left(x + \frac{k\pi}{2}\right)$ **A1**

$= 2^{k+1} e^x \cos\left(x + \frac{k\pi}{2}\right)$ **A1**

$= 2^{k+1} e^x \sin\left(x + \frac{(k+1)\pi}{2}\right)$ **A1**

therefore true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 1$

the result is proved by induction.   **R1**

Note: Award the final **R1** only if the two **M** marks have been awarded.

[8 marks]

46.

Markscheme

we are given that $ar^2 = 9$ and $\frac{a}{1-r} = 64$ *A1*

dividing, $r^2(1-r) = \frac{9}{64}$ *M1*

$64r^3 - 64r^2 + 9 = 0$ *A1*

$r = 0.75, \ a = 16$ *A1A1*

[5 marks]

Examiners report

[N/A]

47a.

Markscheme

$\text{mod}\,(z) = 2, \ \arg(z) = 150°$ *A1A1*

[2 marks]

Examiners report

[N/A]

47b.

Markscheme

$z^{\frac{1}{3}} = 2^{\frac{1}{3}}\left(\cos 50° + \mathrm{i}\sin 50°\right)$ *(M1)*

$= 0.810 + 0.965\mathrm{i}$ *A1*

[2 marks]

Examiners report

[N/A]

47c.

Markscheme

we require to find a multiple of 150 that is also a multiple of 360, so by any method,   *M1*

$n = 12$ *A1*

Note: Only award 1 mark for part (c) if $n = 12$ is based on $\arg(z) = -30$.

[2 marks]

Examiners report

[N/A]

Markscheme

(i) $S_1 = P\left(1 + \frac{I}{100}\right) - R$ **A1**

$S_2 = P\left(1 + \frac{I}{100}\right)^2 - R\left(1 + \frac{I}{100}\right) - R$ **M1A1**

$= P\left(1 + \frac{I}{100}\right)^2 - R\left(1 + \left(1 + \frac{I}{100}\right)\right)$ **AG**

(ii) extending this,

$S_n = P\left(1 + \frac{I}{100}\right)^n - R\left(1 + \left(1 + \frac{I}{100}\right) + \ldots + \left(1 + \frac{I}{100}\right)^{n-1}\right)$ **M1A1**

$= P\left(1 + \frac{I}{100}\right)^n - \frac{R\left(\left(1 + \frac{I}{100}\right)^n - 1\right)}{\frac{I}{100}}$ **M1A1**

$= P\left(1 + \frac{I}{100}\right)^n - \frac{100R}{I}\left(\left(1 + \frac{I}{100}\right)^n - 1\right)$ **AG**

[7 marks]

Examiners report

[N/A]

Markscheme

(i) putting $S_{60} = 0$, $P = 5000$, $I = 1$ **M1**

$5000 \times 1.01^{60} = 100R\left(1.01^{60} - 1\right)$ **A1**

$R = (\$)111.22$ **A1**

(ii) $n = 20$, $P = 5000$, $I = 1$, $R = 111.22$ **M1**

$S_{20} = 5000 \times 1.01^{20} - 100 \times 111.22(1.01^{20} - 1)$ **A1**

$= (\$)3652$ **A1**

which is the outstanding amount

[6 marks]

Examiners report

[N/A]