Topic 3 Part 2 *[311 marks]*

Let
$f(x) = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

1a. For what values of x does $f(x)$ not exist? *[2 marks]*

1b. Simplify the expression $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$. *[5 marks]*

In the triangle ABC,
$\mathrm{A\hat{B}C} = 90^\circ$,
$\mathrm{AC} = \sqrt{2}$ and AB = BC + 1.

2a. Show that cos $\hat{A} - \sin \hat{A} = \frac{1}{\sqrt{2}}$. *[3 marks]*

2b. By squaring both sides of the equation in part (a), solve the equation to find the angles in the triangle. *[8 marks]*

2c. Apply Pythagoras' theorem in the triangle ABC to find BC, and hence show that $\sin \hat{A} = \frac{\sqrt{6}-\sqrt{2}}{4}$. *[6 marks]*

2d. Hence, or otherwise, calculate the length of the perpendicular from B to [AC]. *[4 marks]*

The function
$f(x) = 3\sin x + 4\cos x$ is defined for
$0 < x < 2\pi$.

3a. Write down the coordinates of the minimum point on the graph of f . *[1 mark]*

3b. The points $\mathrm{P}(p,\ 3)$ and $\mathrm{Q}(q,\ 3)$, $q > p$, lie on the graph of $y = f(x)$. *[2 marks]*

Find p and q .

3c. Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3. *[4 marks]*

3d. Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. *[7 marks]*

4. Find the values of x for which the vectors [5 marks]
$\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix}$ and
$\begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix}$ are perpendicular,
$0 \leqslant x \leqslant \frac{\pi}{2}$.

5. Show that [6 marks]
$\frac{\cos A+\sin A}{\cos A-\sin A} = \sec 2A + \tan 2A$.

The function f is defined on the domain
$\left[0, \frac{3\pi}{2}\right]$ by
$f(x) = e^{-x}\cos x$.

6a. State the two zeros of f . [1 mark]

6b. Sketch the graph of f . [1 mark]

6c. The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B . Show that the ratio of the area of A to the area of B is [7 marks]

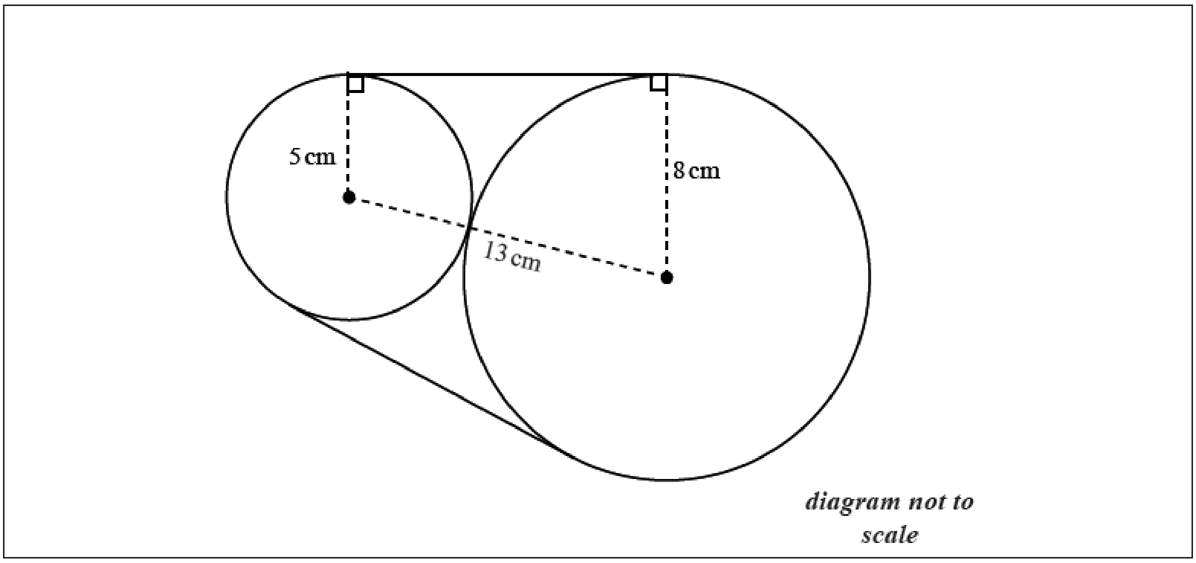
$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

Consider a triangle ABC with
$\mathrm{B\hat{A}C} = 45.7^{\circ}$, AB = 9.63 cm and BC = 7.5 cm .

7a. By drawing a diagram, show why there are two triangles consistent with this information. [2 marks]

7b. Find the possible values of AC . [6 marks]

8. Two discs, one of radius 8 cm and one of radius 5 cm, are placed such that they touch each other. A piece of string is wrapped around the discs. This is shown in the diagram below. [8 marks]



Calculate the length of string needed to go around the discs.

9. Given that $\frac{\pi}{2} < \alpha < \pi$ and $\cos\alpha = -\frac{3}{4}$, find the value of sin 2α . *[4 marks]*

In the triangle PQR, PQ = 6 , PR = k and $\hat{PQR} = 30^\circ$.

10a. For the case $k = 4$, find the two possible values of QR. *[4 marks]*

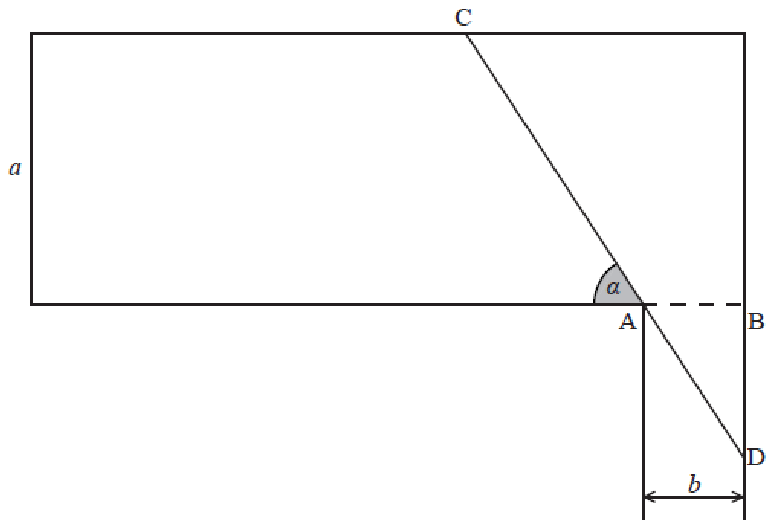
10b. Determine the values of k for which the conditions above define a unique triangle. *[3 marks]*

Consider the curve defined by the equation $x^2 + \sin y - xy = 0$.

11a. Find the gradient of the tangent to the curve at the point (π, π) . *[6 marks]*

11b. Hence, show that $\tan\theta = \frac{1}{1+2\pi}$, where θ is the acute angle between the tangent to the curve at (π, π) and the line $y = x$. *[3 marks]*

The diagram shows the plan of an art gallery a metres wide. [AB] represents a doorway, leading to an exit corridor b metres wide. In order to remove a painting from the art gallery, CD (denoted by L) is measured for various values of α , as represented in the diagram.



12a. If α is the angle between [CD] and the wall, show that $L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}, 0 < \alpha < \frac{\pi}{2}$. *[3 marks]*

12b. If $a = 5$ and $b = 1$, find the maximum length of a painting that can be removed through this doorway. *[4 marks]*

12c. Let $a = 3k$ and $b = k$. Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$. *[3 marks]*

12d. Let $a = 3k$ and $b = k$. Find, in terms of k , the maximum length of a painting that can be removed from the gallery through this doorway. *[6 marks]*

12e. Let $a = 3k$ and $b = k$. [2 marks]

Find the minimum value of k if a painting 8 metres long is to be removed through this doorway.

Consider the planes
$\pi_1 : x - 2y - 3z = 2$ and $\pi_2 : 2x - y - z = k$.

13a. Find the angle between the planes π_1 and π_2 . [4 marks]

13b. The planes π_1 and π_2 intersect in the line L_1 . Show that the vector equation of L_1 is [5 marks]

$$r = \begin{pmatrix} 0 \\ 2 - 3k \\ 2k - 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$$

13c. The line L_2 has Cartesian equation $5 - x = y + 3 = 2 - 2z$. The lines L_1 and L_2 intersect at a point X. Find the coordinates of X. [5 marks]

13d. Determine a Cartesian equation of the plane π_3 containing both lines L_1 and L_2 . [5 marks]

13e. Let Y be a point on L_1 and Z be a point on L_2 such that XY is perpendicular to YZ and the area of the triangle XYZ is 3. Find the perimeter of the triangle XYZ. [5 marks]

14. A rectangle is drawn around a sector of a circle as shown. If the angle of the sector is 1 radian and the area of the sector is 7 cm^2, find the dimensions of the rectangle, giving your answers to the nearest millimetre. [6 marks]

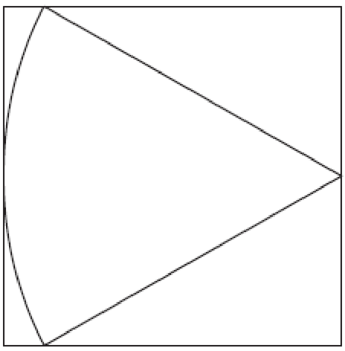
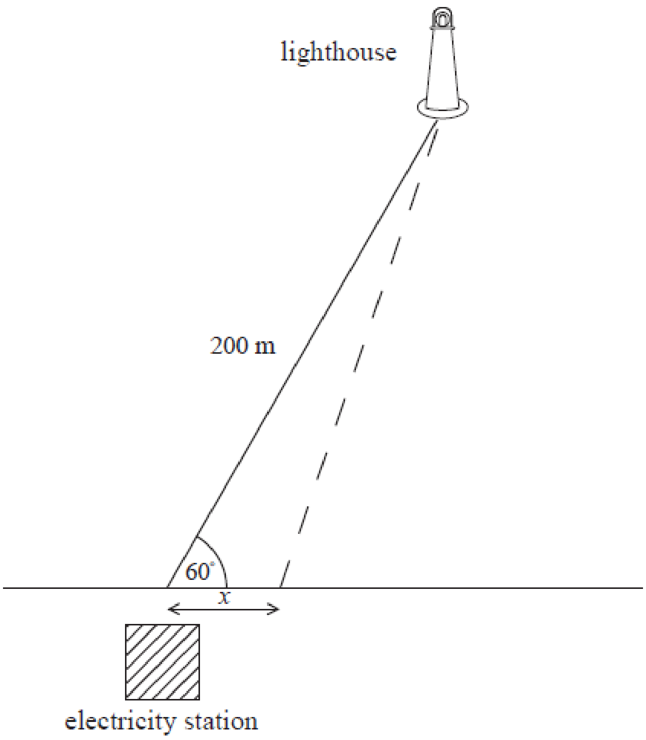


diagram not to scale

An electricity station is on the edge of a straight coastline. A lighthouse is located in the sea 200 m from the electricity station. The angle between the coastline and the line joining the lighthouse with the electricity station is 60°. A cable needs to be laid connecting the lighthouse to the electricity station. It is decided to lay the cable in a straight line to the coast and then along the coast to the electricity station. The length of cable laid along the coastline is x metres. This information is illustrated in the diagram below.



The cost of laying the cable along the sea bed is US$80 per metre, and the cost of laying it on land is US$20 per metre.

15a. Find, in terms of x, an expression for the cost of laying the cable. *[4 marks]*

15b. Find the value of x, to the nearest metre, such that this cost is minimized. *[2 marks]*

16a. Given that *[3 marks]*

$\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right)$, where

$p \in \mathbb{Z}^+$, find p.

16b. Hence find the value of *[3 marks]*

$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$.

A circle of radius 4 cm , centre O , is cut by a chord [AB] of length 6 cm.

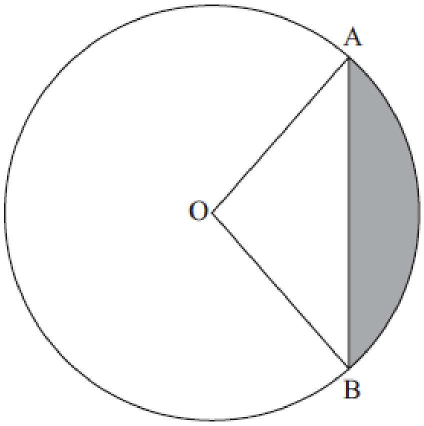


diagram not to scale

17a. Find *[2 marks]*

$\hat{\mathrm{AOB}}$, expressing your answer in radians correct to four significant figures.

17b. Determine the area of the shaded region. *[3 marks]*

18a. Solve the equation *[3 marks]*

$3\cos^2 x - 8\cos x + 4 = 0$, where

$0 \leqslant x \leqslant 180^\circ$, expressing your answer(s) to the nearest degree.

18b. Find the exact values of *[3 marks]*

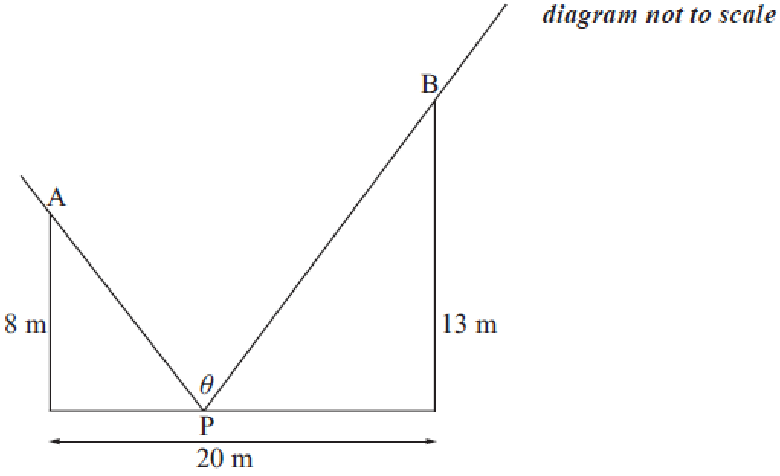
$\sec x$ satisfying the equation

$3\sec^4 x - 8\sec^2 x + 4 = 0$.

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle

θ where

$\theta = \mathrm{A\hat{P}B}$, as shown in the diagram.



19a. Find an expression for [2 marks]

θ in terms of x, where x is the distance of P from the base of the wall of height 8 m.

19b. (i) Calculate the value of [2 marks]

θ when $x = 0$.

(ii) Calculate the value of

θ when $x = 20$.

19c. Sketch the graph of [2 marks]

θ, for

$0 \leqslant x \leqslant 20$.

19d. Show that [6 marks]

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$.

19e. Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. [3 marks]

19f. The point P moves across the street with speed [4 marks]

$0.5\ \mathrm{ms}^{-1}$. Determine the rate of change of

θ with respect to time when P is at the midpoint of the street.

20. From a vertex of an equilateral triangle of side *[6 marks]*

$2x$, a circular arc is drawn to divide the triangle into two regions, as shown in the diagram below.

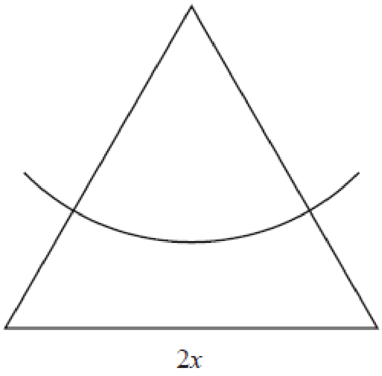


diagram not to scale

Given that the areas of the two regions are equal, find the radius of the arc in terms of x.

21. A circular disc is cut into twelve sectors whose areas are in an arithmetic sequence. *[5 marks]*

The angle of the largest sector is twice the angle of the smallest sector.

Find the size of the angle of the smallest sector.

22. In triangle ABC, AB = 9 cm , AC = 12 cm , and *[5 marks]*

$\hat{B}$ is twice the size of

$\hat{C}$.

Find the cosine of

$\hat{C}$.

23. A system of equations is given by *[6 marks]*

$$\cos x + \cos y = 1.2$$

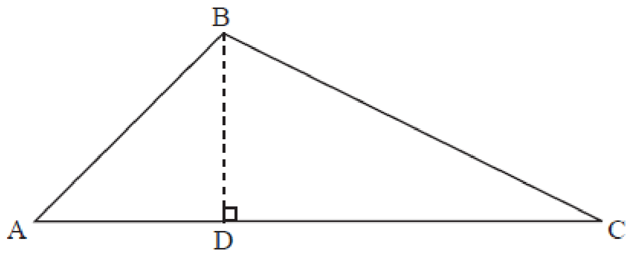
$$\sin x + \sin y = 1.4 .$$

(a) For each equation express y in terms of x.

(b) **Hence** solve the system for

$0 < x < \pi,\ 0 < y < \pi$.

24. In triangle ABC, BC = a , AC = b , AB = c and [BD] is perpendicular to [AC]. *[12 marks]*



(a) Show that

$\mathrm{CD} = b - c\cos A$.

(b) **Hence**, by using Pythagoras' Theorem in the triangle BCD, prove the cosine rule for the triangle ABC.

(c) If

$\mathrm{A\hat{B}C} = 60^\circ$, use the cosine rule to show that

$c = \frac{1}{2}a \pm \sqrt{b^2 - \frac{3}{4}a^2}$.

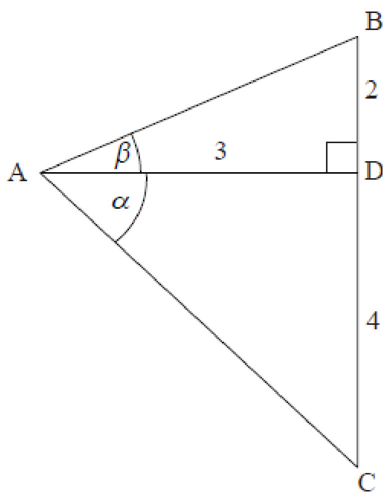
25. *[6 marks]*

In the diagram below, AD is perpendicular to BC.

CD = 4, BD = 2 and AD = 3.

$\mathrm{C\hat{A}D} = \alpha$ and

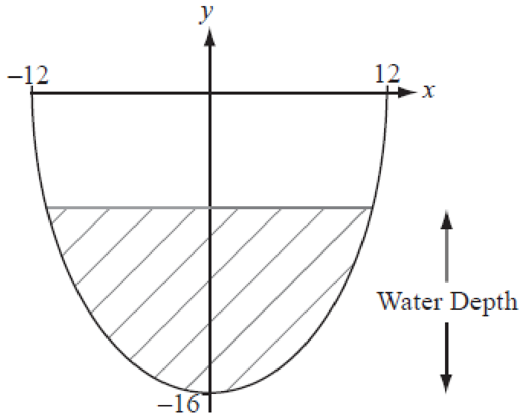
$\mathrm{B\hat{A}D} = \beta$.



Find the exact value of

$\cos(\alpha - \beta)$.

26. The diagram below shows the boundary of the cross-section of a water channel. *[6 marks]*



The equation that represents this boundary is

$y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where *x* and *y* are both measured in cm.

The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm.

Find the width of the water surface in the channel when the water depth is 10 cm.

Give your answer in the form

$a \arccos b$ where

$a,\ b \in \mathbb{R}$.

27. The depth, $h(t)$ metres, of water at the entrance to a harbour at *t* hours after midnight on a particular day is given by *[6 marks]*

$$h(t) = 8 + 4\sin\left(\frac{\pi t}{6}\right),\ 0 \leqslant t \leqslant 24.$$

(a) Find the maximum depth and the minimum depth of the water.

(b) Find the values of *t* for which

$h(t) \geqslant 8.$

28. Consider triangle ABC with *[7 marks]*

$\mathrm{B\hat{A}C} = 37.8^\circ$, AB = 8.75 and BC = 6 .

Find AC.

29. (a) Sketch the curve *[21 marks]*

$f(x) = \sin 2x$,

$0 \leqslant x \leqslant \pi$.

(b) Hence sketch on a separate diagram the graph of

$g(x) = \csc 2x$,

$0 \leqslant x \leqslant \pi$, clearly stating the coordinates of any local maximum or minimum points and the equations of any asymptotes.

(c) Show that tan

$x + \cot x \equiv 2\csc 2x$.

(d) Hence or otherwise, find the coordinates of the local maximum and local minimum points on the graph of

$y = \tan 2x + \cot 2x$,

$0 \leqslant x \leqslant \frac{\pi}{2}$.

(e) Find the solution of the equation

$\csc 2x = 1.5\tan x - 0.5$,

$0 \leqslant x \leqslant \frac{\pi}{2}$.

30. In a triangle ABC, [7 marks]

$\hat{A} = 35^\circ$, BC = 4 cm and AC = 6.5 cm. Find the possible values of $\hat{B}$ and the corresponding values of AB.

The angle θ lies in the first quadrant and $\cos\theta = \frac{1}{3}$.

31a. Write down the value of $\sin\theta$. [1 mark]

31b. Find the value of $\tan 2\theta$. [2 marks]

31c. Find the value of $\cos\left(\frac{\theta}{2}\right)$, giving your answer in the form $\frac{\sqrt{a}}{b}$ where a , $b \in \mathbb{Z}^+$. [3 marks]

In the triangle ABC, $\mathrm{AB} = 2\sqrt{3}$, AC = 9 and $\mathrm{B\hat{A}C} = 150^\circ$.

32a. Determine BC, giving your answer in the form $k\sqrt{3}$, $k \in \mathbb{Z}^+$. [3 marks]

32b. The point D lies on (BC), and (AD) is perpendicular to (BC). Determine AD. [4 marks]

Consider the following system of equations:

$$x + y + z = 1$$

$$2x + 3y + z = 3$$

$$x + 3y - z = \lambda$$

where

$\lambda \in \mathbb{R}$.

33a. Show that this system does not have a unique solution for any value of *[4 marks]*

λ .

33b. (i) Determine the value of *[4 marks]*

λ for which the system is consistent.

(ii) For this value of

λ , find the general solution of the system.

34. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ satisfy the equation $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$. Show that $\boldsymbol{a}$ *[6 marks]*

$\times \boldsymbol{b} = \boldsymbol{b}$

$\times \boldsymbol{c} = \boldsymbol{c}$

$\times \boldsymbol{a}$.

A ship, S, is 10 km north of a motorboat, M, at 12.00pm. The ship is travelling northeast with a constant velocity of $20\ \mathrm{km\,hr^{-1}}$. The motorboat wishes to intercept the ship and it moves with a constant velocity of $30\ \mathrm{km\,hr^{-1}}$ in a direction θ degrees east of north. In order for the interception to take place, determine

35a. the value of *[4 marks]*

θ.

35b. the time at which the interception occurs, correct to the nearest minute. *[5 marks]*