# Topic 8 Part 2 *[571 marks]*

1a. *[2 marks]*

## Markscheme

non-S: for example –2 does not belong to the range of $g$    *R1*
non-I: for example
$g(1) = g(-1) = 0$    *R1*
**Note**: Graphical arguments have to recognize that we are dealing with sets of integers and not all real numbers

*[2 marks]*

## Examiners report

Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

1b. *[2 marks]*

## Markscheme

as $f$ is injective
$n(f(S)) = n(S)$    *A1*



*R1*
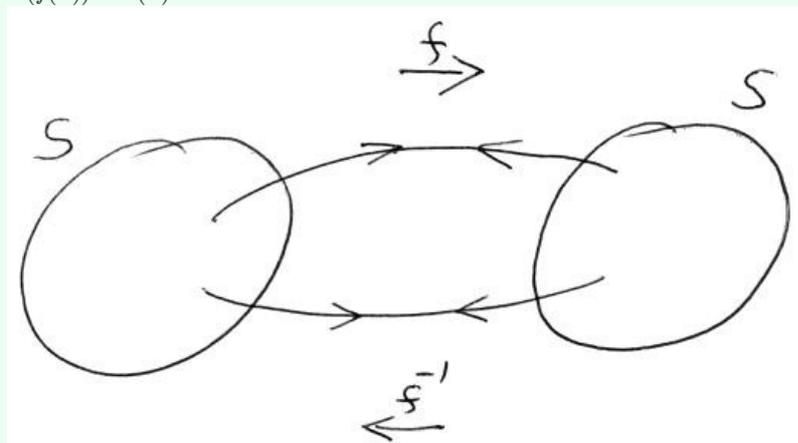
**Note:** Accept alternative explanations.

$f$ is surjective    *AG*
*[2 marks]*

## Examiners report

Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

## Markscheme

for example,
$h(n) = n + 1$    **A1**
**Note:** Only award the **A1** if the function works.

I:
$n + 1 = m + 1 \Rightarrow n = m$    **R1**
non-S: 1 has no pre-image as
$0 \notin \mathbb{Z}^+$    **R1**
*[3 marks]*

## Examiners report

Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

2.

## Markscheme

(i)    consider
$(ghg^{-1})^2$    **M1**
$= ghg^{-1}ghg^{-1} = gh^2g^{-1} = gg^{-1} = e$    **A1**
$ghg^{-1}$ cannot be order 1 $(= e)$ since $h$ is order 2    **R1**
so
$ghg^{-1}$ has order 2    **AG**

(ii)    but $h$ is the unique element of order 2    **R1**
hence
$ghg^{-1} = h \Rightarrow gh = hg$    **A1AG**
*[5 marks]*

## Examiners report

This question was by far the problem to be found most challenging by the candidates. Many were able to show that
$ghg^{-1}$ had order one or two although hardly any candidates also showed that the order was not one thus losing a mark. Part a (ii) was answered correctly by a few candidates who noticed the equality of $h$ and
$ghg^{-1}$. However, many candidates went into algebraic manipulations that led them nowhere and did not justify any marks. Part (b) (i) was well answered by a small number of students who appreciated the nature of the identity and element $h$ thus forcing the other two elements to have order four. However, (ii) was only occasionally answered correctly and even in these cases not systematically. It is possible that candidates lacked time to fully explore the problem. A small number of candidates "guessed" the correct answer.

3a.

## Markscheme

| *  | 2  | 4  | 6  | 8  | 10 | 12 |
|----|----|----|----|----|----|----|
| 2  | 4  | 8  | 12 | 2  | 6  | 10 |
| 4  | 8  | 2  | 10 | 4  | 12 | 6  |
| 6  | 12 | 10 | 8  | 6  | 4  | 2  |
| 8  | 2  | 4  | 6  | 8  | 10 | 12 |
| 10 | 6  | 12 | 4  | 10 | 2  | 8  |
| 12 | 10 | 6  | 2  | 12 | 8  | 4  |

**A4**

**Note:** Award **A4** for all correct, **A3** for one error, **A2** for two errors, **A1** for three errors and **A0** for four or more errors.

*[4 marks]*

3b.

[11 marks]

# Markscheme

(i)    closure: there are no new elements in the table    *A1*

identity: 8 is the identity element    *A1*

inverse: every element has an inverse because there is an 8 in every row and column    *A1*

associativity: (modulo) multiplication is associative    *A1*

therefore $\{S$,

$*\}$ is a group    *AG*

(ii)    the orders of the elements are as follows

| element | order |
|---------|-------|
| 2 | 3 |
| 4 | 3 |
| 6 | 2 |
| 8 | 1 |
| 10 | 6 |
| 12 | 6 |

*A4*

**Note:** Award *A4* for all correct, *A3* for one error, *A2* for two errors, *A1* for three errors and *A0* for four or more errors.

(iii)    **EITHER**

the group is cyclic because there are elements of order 6    *R1*

**OR**

the group is cyclic because there are generators    *R1*

**THEN**

10 and 12 are the generators    *A1A1*

*[11 marks]*

## Markscheme

looking at the Cayley table, we see that

$T = \{2, 4, 8\}$    *A1*

this is a subgroup because it contains the identity element 8, no new elements are formed and 2 and 4 form an inverse pair    *R2*

**Note:** Award *R1* for any two conditions

*[3 marks]*

## Examiners report

Parts (a) and (b) were well done in general. Some candidates, however, when considering closure and associativity simply wrote 'closed' and 'associativity' without justification. Here, candidates were expected to make reference to their Cayley table to justify closure and to state that multiplication is associative to justify associativity. In (c), some candidates tried to show the required result without actually identifying the elements of $T$. This approach was invariably unsuccessful.

## Markscheme

$A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$    *(A1)*

$B = \{3, 8, 13, 18, 23, 28\}$    *(A1)*

**Note:** *FT* on their $A$ and $B$

$A \setminus B = \{$elements in $A$ that are not in $B\}$    *(M1)*

$= \{2, 5, 7, 11, 17, 19, 29\}$    *A1*

*[4 marks]*

## Examiners report

It was disappointing to find that many candidates wrote the elements of A and B incorrectly. The most common errors were the inclusion of 1 as a prime number and the exclusion of 3 in B. It has been suggested that some candidates use $N$ to denote the positive integers. If this is the case, then it is important to emphasise that the IB notation is that $N$ denotes the positive integers and zero and IB candidates should all be aware of that. Most candidates solved the remaining parts of the question correctly and follow through ensured that those candidates with incorrect A and/or B were not penalised any further.

## Markscheme

$B \setminus A = \{8, 18, 28\}$    *(A1)*

$A \Delta B = (A \setminus B) \cup (B \setminus A)$    *(M1)*

$= \{2, 5, 7, 8, 11, 17, 18, 19, 28, 29\}$    *A1*

*[3 marks]*

5a.

# Markscheme

reflexive: $aRa$ because

$a^2 - a^2 = 0$ (which is divisible by 5)   *A1*

symmetric: let $aRb$ so that

$a^2 - b^2 = 5M$   *M1*

it follows that

$a^2 - b^2 = -5M$ which is divisible by 5 so $bRa$   *A1*

transitive: let $aRb$ and $bRc$ so that

$a^2 - b^2 = 5M$ and

$b^2 - c^2 = 5N$   *M1*

$a^2 - b^2 + b^2 - c^2 = 5M + 5N$   *A1*

$a^2 - c^2 = 5M + 5N$ which is divisible by 5 so $aRc$   *A1*

$\Rightarrow R$ is an equivalence relation   *AG*

*[6 marks]*

5b.

# Markscheme

the equivalence classes are

$\{1, 4, 6, 9, \ldots\}$   *A2*

$\{2, 3, 7, 8, \ldots\}$   *A1*

$\{5, 10, \ldots\}$   *A1*

**Note:** Do not award any marks for classes containing fewer elements than shown above.

*[4 marks]*

## Examiners report

Many candidates solved (a) correctly but solutions to (b) were generally poor. Most candidates seemed to have a weak understanding of the concept of equivalence classes and were unaware of any systematic method for finding the equivalence classes. If all else fails, a trial and error approach can be used. Here, starting with 1, it is easily seen that 4, 6, … belong to the same class and the pattern can be established.

6.

## Markscheme

for $f$ to be a bijection it must be both an injection and a surjection     *R1*

**Note:** Award this *R1* for stating this anywhere.

injection:

let

$f(a, b) = f(c, d)$ so that     *(M1)*

$ab^2 = cd^2$ and

$\frac{a}{b} = \frac{c}{d}$     *A1*

dividing the equations,

$b^3 = d^3$ so

$b = d$     *A1*

substituting,

$a = c$     *A1*

it follows that $f$ is an injection because

$f(a, b) = f(c, d) \Rightarrow (a, b) = (c, d)$     *R1*

surjection:

let

$f(a, b) = (c, d)$ where

$(c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$     *(M1)*

then

$c = ab^2$ and

$d = \frac{a}{b}$     *A1*

dividing,

$b^3 = \frac{c}{d}$ so

$b = \sqrt[3]{\frac{c}{d}}$     *A1*

substituting,

$a = d \times \sqrt[3]{\frac{c}{d}}$     *A1*

it follows that $f$ is a surjection because

given

$(c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$ , there exists

$(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$f(a, b) = (c, d)$     *R1*

therefore $f$ is a bijection     *AG*

*[11 marks]*

7a. [4 marks]

# Markscheme

$pq = pr$

$p^{-1}(pq) = p^{-1}(pr)$, every element has an inverse    **A1**

$(p^{-1}p)q = (p^{-1}p)r$, Associativity    **A1**

**Note:** Brackets in lines 2 and 3 must be seen.

$eq = er,$

$p^{-1}p = e$, the identity    **A1**

$q = r,$

$ea = a$ for all elements $a$ of the group    **A1**

*[4 marks]*

# Markscheme

(i)   let $ab = a$ so $b = e$ be which is a contradiction   *R1*

let $ab = b$ so $a = e$ which is a contradiction   *R1*

therefore $ab$ cannot equal either $a$ or $b$   *AG*

(ii)   the two possible Cayley tables are

table 1

|   | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $c$ | $e$ | $b$ |
| $b$ | $b$ | $e$ | $c$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

*A2*

table 2

|   | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

*A2*

(iii)   the group defined by table 1 is isomorphic to the given group   *R1*

because

**EITHER**

both contain one self-inverse element (other than the identity)   *R1*

**OR**

both contain an inverse pair   *R1*

**OR**

both are cyclic   *R1*

**THEN**

the correspondence is

$e \rightarrow 1,$

$c \rightarrow -1,$

$a \rightarrow i,$

$b \rightarrow -i$

(or vice versa for the last two)   *A2*

**Note:** Award the final *A2* only if the correct group table has been identified.

*[10 marks]*

# Examiners report

Solutions to (a) were often poor with inadequate explanations often seen. It was not uncommon to see

$pq = pr$

$p^{-1}pq = p^{-1}pr$

$q = r$

without any mention of associativity. Many candidates understood what was required in (b)(i), but solutions to (b)(ii) were often poor with the tables containing elements such as $ab$ and $bc$ without simplification. In (b)(iii), candidates were expected to determine the isomorphism by noting that the group defined by $\{1, -1, i, -i\}$ under multiplication is cyclic or that $-1$ is the only self-inverse element apart from the identity, without necessarily writing down the Cayley table in full which many candidates did. Many candidates just stated that there was a bijection between the two groups without giving any justification for this.

$pq = pr$

$p^{-1}pq = p^{-1}pr$

$q = r$

8.

# Markscheme

(a)  **EITHER**

consider

$f'(x) = 2e^x - e^{-x} > 0$ for all $x$     *M1A1*

so $f$ is an injection     *A1*

**OR**

let

$2e^x - e^{-x} = 2e^y - e^{-y}$     *M1*

$2(e^x - e^y) + e^{-y} - e^{-x} = 0$

$2(e^x - e^y) + e^{-(x+y)}(e^x - e^y) = 0$

$\left(2 + e^{-(x+y)}\right)(e^x - e^y) = 0$

$e^x = e^y$

$x = y$     *A1*

**Note:** Sufficient working must be shown to gain the above *A1*.


so $f$ is an injection     *A1*

**Note:** Accept a graphical justification *i.e.* horizontal line test.


**THEN**

it is also a surjection (accept any justification including graphical)     *R1*

therefore it is a bijection     *AG*

*[4 marks]*


(b)   let

$y = 2e^x - e^{-x}$     *M1*

$2e^{2x} - ye^x - 1 = 0$     *A1*

$e^x = \frac{y \pm \sqrt{y^2 + 8}}{4}$     *M1A1*

since

$e^x$ is never negative, we take the + sign     *R1*

$f^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2 + 8}}{4}\right)$     *A1*

*[6 marks]*

*Total [10 marks]*


# Examiners report

Solutions to (a) were often disappointing. Many candidates tried to use the result that, for an injection, $f(a) = f(b) \Rightarrow a = b$ – although this is the definition, it is often much easier to proceed by showing that the derivative is everywhere positive or everywhere negative or even to use a horizontal line test. Although (b) is based on core material, solutions were often disappointing with some very poor use of algebra seen.

9.

# Markscheme

(a)   (i)

$a^2 \geqslant 0$ for all

$a \in \mathbb{Z}$, hence $R$ is reflexive   ***R1***

(ii)

$aRb \Rightarrow ab \geqslant 0$   ***M1***

$\Rightarrow ba \geqslant 0$   ***R1***

$\Rightarrow bRa$, hence $R$ is symmetric   ***A1***

(iii)

$aRb$ and $bRc \Rightarrow ab \geqslant 0$ and $bc \geqslant 0$, is $aRc$?   ***M1***

no, for example,

$-3R0$ and

$0R5$, but

$-3R5$ is not true   ***A1***

$aRc$ is not generally true, hence $R$ is not transitive   ***A1***

***[7 marks]***

(b)   $R$ does not satisfy all three properties, hence $R$ is not an equivalence relation   ***R1***

***[1 mark]***

***Total [8 marks]***

# Examiners report

Although the properties of an equivalence relation were well known, few candidates provided a counter-example to show that the relation is not transitive. Some candidates interchanged the definitions of the reflexive and symmetric properties.

10a.

# Markscheme

(i)   let

$x \in \mathbb{R}$

for example,

$f(0,\ x) = x$,   ***M1***

hence $f$ is surjective   ***A1***

(ii)   for example,

$f(2,\ 3) = f(4,\ 3) = 3$, but $(2,\ 3) \neq (4,\ 3)$   ***M1***

hence $f$ is not injective   ***A1***

***[4 marks]***

10b.                                                                                      *[4 marks]*

# Markscheme

(i)    there is no element of $P$ such that

$g(p) = 7$, for example    **R1**

hence $g$ is not surjective    **A1**

(ii)

$g(p) = g(q) \Rightarrow xp = xq \Rightarrow p = q$, hence $g$ is injective    **M1A1**

*[4 marks]*

10c.                                                                                      *[7 marks]*

# Markscheme

(i)    for

$x > 0$, $h(x) = 2, \, 4, \, 6, \, 8 \ldots$    **A1**

for

$x \leqslant 0$, $h(x) = 1, \, 3, \, 5, \, 7 \ldots$    **A1**

therefore $h$ is surjective    **A1**

(ii)    for

$h(x) = h(y)$, since an odd number cannot equal an even number, there are only two possibilities:    **R1**

$x, \, y > 0$, $2x = 2y \Rightarrow x = y$;    **A1**

$x, \, y \leqslant 0$, $1 - 2x = 1 - 2y \Rightarrow x = y$    **A1**

therefore $h$ is injective    **A1**

**Note:** This can be demonstrated in a variety of ways.

*[7 marks]*

# Examiners report

This was the least successfully answered question on the paper. Candidates often could quote the definitions of surjective and injective, but often could not apply the definitions in the examples.

c) Many candidates failed to appreciate that the function is discrete, and hence erroneously attempted to differentiate the function to show that it is monotonic increasing, hence injective. Others who provided a graph again showed a continuous rather than discrete function.

11.

# Markscheme

(a)  (i)  Cayley table for

$\{S, \circ\}$

| $\circ$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---------|-------|-------|-------|-------|-------|-------|
| $x_0$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
| $x_1$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_0$ |
| $x_2$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_0$ | $x_1$ |
| $x_3$ | $x_3$ | $x_4$ | $x_5$ | $x_0$ | $x_1$ | $x_2$ |
| $x_4$ | $x_4$ | $x_5$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
| $x_5$ | $x_5$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |

*A4*

**Note:** Award *A4* for no errors, *A3* for one error, *A2* for two errors, *A1* for three errors and *A0* for four or more errors.

$S$ is closed under

$\circ$   *A1*

$x_0$ is the identity   *A1*

$x_0$ and

$x_3$ are self-inverses,   *A1*

$x_2$ and

$x_4$ are mutual inverses and so are

$x_1$ and

$x_5$   *A1*

modular addition is associative   *A1*

hence,

$\{S, \circ\}$ is a group   *AG*

(ii)   the order of

$x_1$ (or

$x_5$) is 6, hence there exists a generator, and

$\{S, \circ\}$ is a cyclic group   *A1R1*

*[11 marks]*

(b)  (i)   $e, a, b, ab$   *A1*

and

$b^2, ab^2$   *A1A1*

**Note:** Accept

$ba$ and

$b^2 a.$

(ii)

$(ab)^2 = b^2$   *M1A1*

$(ab)^3 = a$   *A1*

$(ab)^4 = b$    **A1**

hence order is 6    **A1**

groups $G$ and $S$ have the same orders and both are cyclic    **R1**

hence isomorphic    **AG**

*[9 marks]*

*Total [20 marks]*

## Examiners report

a) Most candidates had the correct Cayley table and were able to show successfully that the group axioms were satisfied. Some candidates, however, simply stated that an inverse exists for each element without stating the elements and their inverses. Most candidates were able to find a generator and hence show that the group is cyclic.

b) This part was answered less successfully by many candidates. Some failed to find all the elements. Some stated that the order of $ab$ is 6 without showing any working.

12.

## Markscheme

since $G$ is closed, $H$ will be a subset of $G$

closure:

$p, q \in H \Rightarrow p = a^r, q = a^s, r, s \in \mathbb{Z}^+$  **A1**

$p * q = a^r * a^s = a^{r+s}$  **A1**

$r + s \in \mathbb{Z}^+ \Rightarrow p * q \in H$ hence $H$ is closed  **R1**

associativity follows since

$*$ is associative on $G$  **(R1)**

**EITHER**

identity: let the order of $a$ in $G$ be

$m \in \mathbb{Z}^+, m \geqslant 2$  **M1**

then

$a^m = e \in H$  **R1**

inverses:

$a^{m-1} * a = e \Rightarrow a^{m-1}$ is the inverse of $a$  **A1**

$(a^{m-1})^n * a^n = e$, showing that

$a^n$ has an inverse in $H$  **R1**

hence $H$ is a subgroup of $G$  **AG**

**OR**

since

$(G, *)$ is a finite group, and $H$ is a non-empty closed subset of $G$, then

$(H, *)$ is

a subgroup of

$(G, *)$  **R4**

**Note:** To receive the **R4**, the candidate must explicitly state the theorem, *i.e.* the three given conditions, and conclusion.

*[8 marks]*

## Examiners report

This question was generally answered very poorly, if attempted at all. Candidates failed to realize that the property of closure needed to be properly proved. Others used negative indices when the question specifically states that the indices are positive integers.

13a.

# Markscheme

(i)

$a = 9$, $b = 1$, $c = 13$, $d = 5$, $e = 15$, $f = 11$, $g = 15$, $h = 1$, $i = 15$, $j = 15$   **A3**

**Note:** Award **A2** for one or two errors,

**A1** for three or four errors,

**A0** for five or more errors.

(ii)   since the Cayley table only contains elements of the set $G$, then it is closed   **A1**

there is an identity element which is 1   **A1**

$\{3, 11\}$ and $\{5, 13\}$ are inverse pairs and all other elements are self inverse   **A1**

hence every element has an inverse   **R1**

**Note:** Award **A0R0** if no justification given for every element having an inverse.

since the set is closed, has an identity element, every element has an inverse and it is associative, it is a group   **AG**

*[7 marks]*

# Examiners report

Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

13b.

# Markscheme

(i)   since the Cayley table only contains elements of the set $H$, then it is closed   **A1**

there is an identity element which is $e$   **A1**

$\{a_1, a_3\}$ form an inverse pair and all other elements are self inverse   **A1**

hence every element has an inverse   **R1**

**Note:** Award **A0R0** if no justification given for every element having an inverse.

since the set is closed, has an identity element, every element has an inverse and it is associative, it is a group   **AG**

(ii)   any 2 of

$\{e, a_1, a_2, a_3\}$, $\{e, a_2, b_1, b_2\}$, $\{e, a_2, b_3, b_4\}$   **A2A2**

*[8 marks]*

13c.                                                                                          *[2 marks]*

## Markscheme

the groups are not isomorphic because

$\{H, *\}$ has one inverse pair whereas

$\{G, \times_{16}\}$ has two inverse pairs     ***A2***

**Note:** Accept any other valid reason:

*e.g.* the fact that

$\{G, \times_{16}\}$ is commutative and

$\{H, *\}$ is not.


*[2 marks]*

## Examiners report

Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

13d.
# Markscheme

**EITHER**

a group is not cyclic if it has no generators    *R1*

for the group to have a generator there must be an element in the group of order eight    *A1*

| element | order |
|---------|-------|
| $e$ | 1 |
| $a_1$ | 4 |
| $a_2$ | 2 |
| $a_3$ | 4 |
| $b_1$ | 2 |
| $b_2$ | 2 |
| $b_3$ | 2 |
| $b_4$ | 2 |

since there is no element of order eight in the group, it is not cyclic    *A1*

**OR**

a group is not cyclic if it has no generators    *R1*

only possibilities are

$a_1$,

$a_3$ since all other elements are self inverse    *A1*

this is not possible since it is not possible to generate any of the "$b$" elements from the "$a$" elements – the elements

$a_1$, $a_2$, $a_3$, $a_4$ form a closed set    *A1*

*[3 marks]*

# Examiners report

Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

# Markscheme

(a) (i)



*A1    A1*
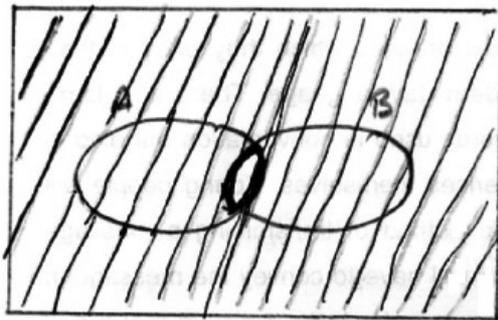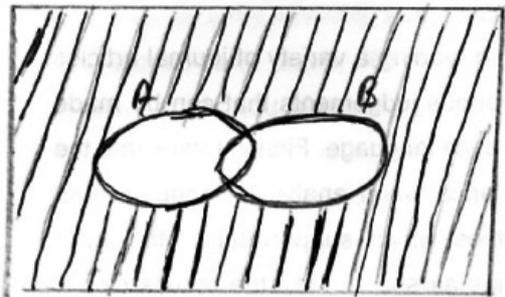


since the shaded regions are different,

$A' \cup B' \neq (A \cup B)'$    *R1*

$\Rightarrow$ not true

(ii)



*A1*
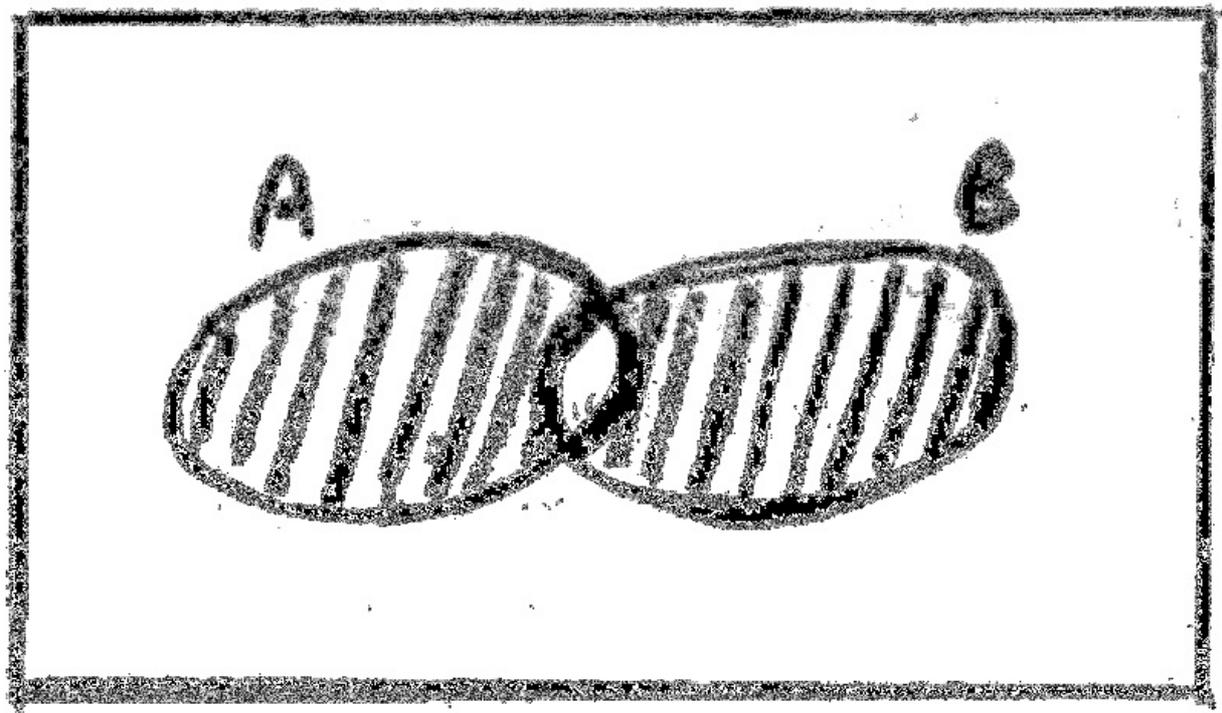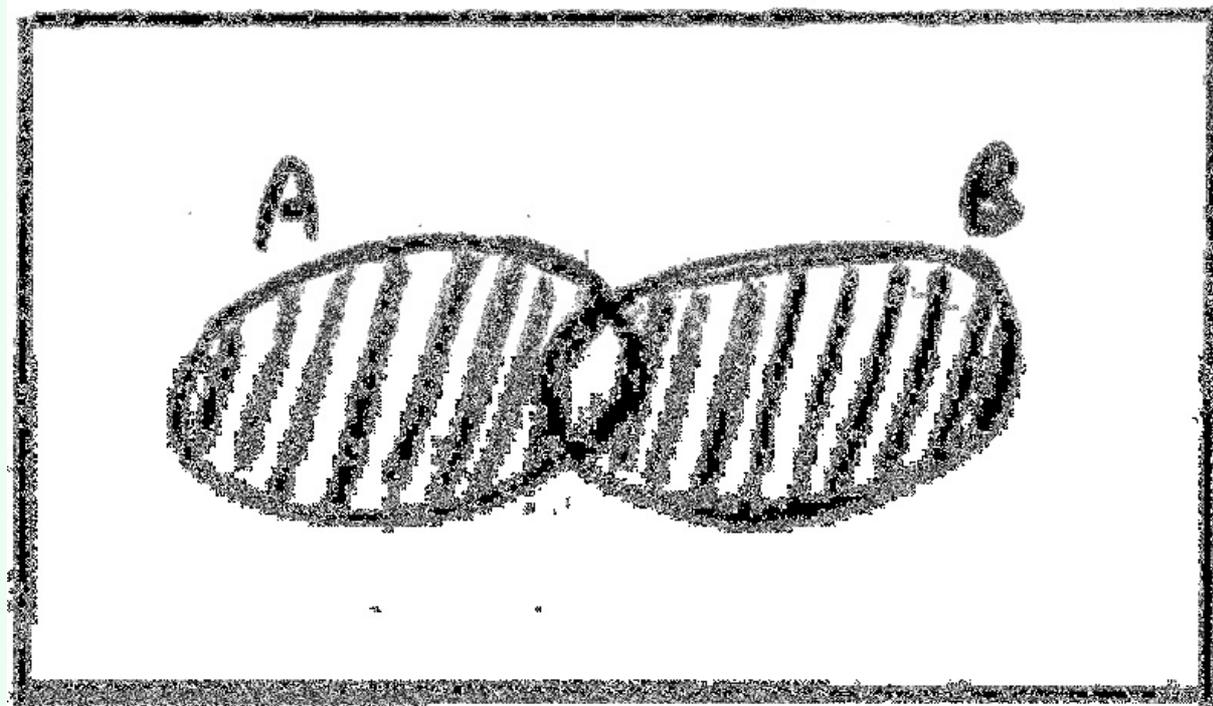
since the shaded regions are the same

$(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$   **R1**

$\Rightarrow$ true

*[6 marks]*

## Examiners report

Part (a) was accessible to most candidates, but a number drew incorrect Venn diagrams. In some cases the clarity of the diagram made it difficult to follow what the candidate intended. Candidates found (b) harder, although the majority made a reasonable start to the proof. Once again a number of candidates were let down by poor explanation.

14b.                                                                                          *[4 marks]*

## Markscheme

$A \backslash B = A \cup B'$ and

$B \backslash A = B \cap A'$   *(A1)*

consider

$A \cap B' \cap B \cap A'$   **M1**

now

$A \cap B' \cap B \cap A' = \emptyset$   **A1**

since this is the empty set, they are disjoint   **R1**

**Note:** Accept alternative valid proofs.

*[4 marks]*

*[8 marks]*

# Markscheme

$xx^{-1} = e \in H$    *M1*

$\Rightarrow xRx$

hence $R$ is reflexive    *A1*

if $xRy$ then

$xy^{-1} \in H$

$\Rightarrow (xy^{-1})^{-1} \in H$    *M1*

now

$(xy^{-1})(xy^{-1})^{-1} = e$ and

$xy^{-1}yx^{-1} = e$

$\Rightarrow (xy^{-1})^{-1} = yx^{-1}$    *A1*

hence

$yx^{-1} \in H \Rightarrow yRx$

hence $R$ is symmetric    *A1*

if $xRy$, $yRz$ then

$xy^{-1} \in H$, $yz^{-1} \in H$    *M1*

$\Rightarrow (xy^{-1})(yz^{-1}) \in H$    *M1*

$\Rightarrow x(y^{-1}y)z^{-1} \in H$

$\Rightarrow x^{-1}z \in H$

hence $R$ is transitive    *A1*

hence $R$ is an equivalence relation    *AG*

*[8 marks]*

# Examiners report

Stronger candidates made a reasonable start to (a), and many were able to demonstrate that the relation was reflexive and transitive. However, the majority of candidates struggled to make a meaningful attempt to show the relation was symmetric, with many making unfounded assumptions. Equivalence classes still cause major problems and few fully correct answers were seen to (b).

15b.

## Markscheme

(i)   for the equivalence class, solving:

**EITHER**

$x(ab)^{-1} = e$ or $x(ab)^{-1} = a^2b$   *(M1)*

$\{ab, a\}$   *A2*

**OR**

$ab(x)^{-1} = e$ or $ab(x)^{-1} = a^2b$   *(M1)*

$\{ab, a\}$   *A2*


(ii)   for the equivalence class, solving:

**EITHER**

$x^{-1}(ab) = e$ or $x^{-1}(ab) = a^2b$   *(M1)*

$\{ab, a^2\}$   *A2*

**OR**

$(ab)^{-1}x = e$ or $(ab)^{-1}x = a^2b$   *(M1)*

$\{ab, a^2\}$   *A2*

*[6 marks]*

## Examiners report

Stronger candidates made a reasonable start to (a), and many were able to demonstrate that the relation was reflexive and transitive. However, the majority of candidates struggled to make a meaningful attempt to show the relation was symmetric, with many making unfounded assumptions. Equivalence classes still cause major problems and few fully correct answers were seen to (b).


16a.

## Markscheme

let $s$ and $t$ be in $A$ and

$s \neq t$   *M1*

since $f$ is injective

$f(s) \neq f(t)$   *A1*

since $g$ is injective

$g \circ f(s) \neq g \circ f(t)$   *A1*

hence

$g \circ f$ is injective   *AG*

*[3 marks]*

16b.

## Markscheme

let $z$ be an element of $C$

we must find $x$ in $A$ such that

$g \circ f(x) = z$   *M1*

since $g$ is surjective, there is an element $y$ in $B$ such that

$g(y) = z$   *A1*

since $f$ is surjective, there is an element $x$ in $A$ such that

$f(x) = y$   *A1*

thus

$g \circ f(x) = g(y) = z$   *R1*

hence

$g \circ f$ is surjective   *AG*

*[4 marks]*

## Examiners report

This question was found difficult by a large number of candidates and no fully correct solutions were seen. A number of students made thought-through attempts to show it was surjective, but found more difficulty in showing it was injective. Very few were able to find a single counter example to show that the converses of the earlier results were false. Candidates struggled with the abstract nature of the question.

## Markscheme

converses: if

$g \circ f$ is injective then $g$ and $f$ are injective

if

$g \circ f$ is surjective then $g$ and $f$ are surjective     *(A1)*
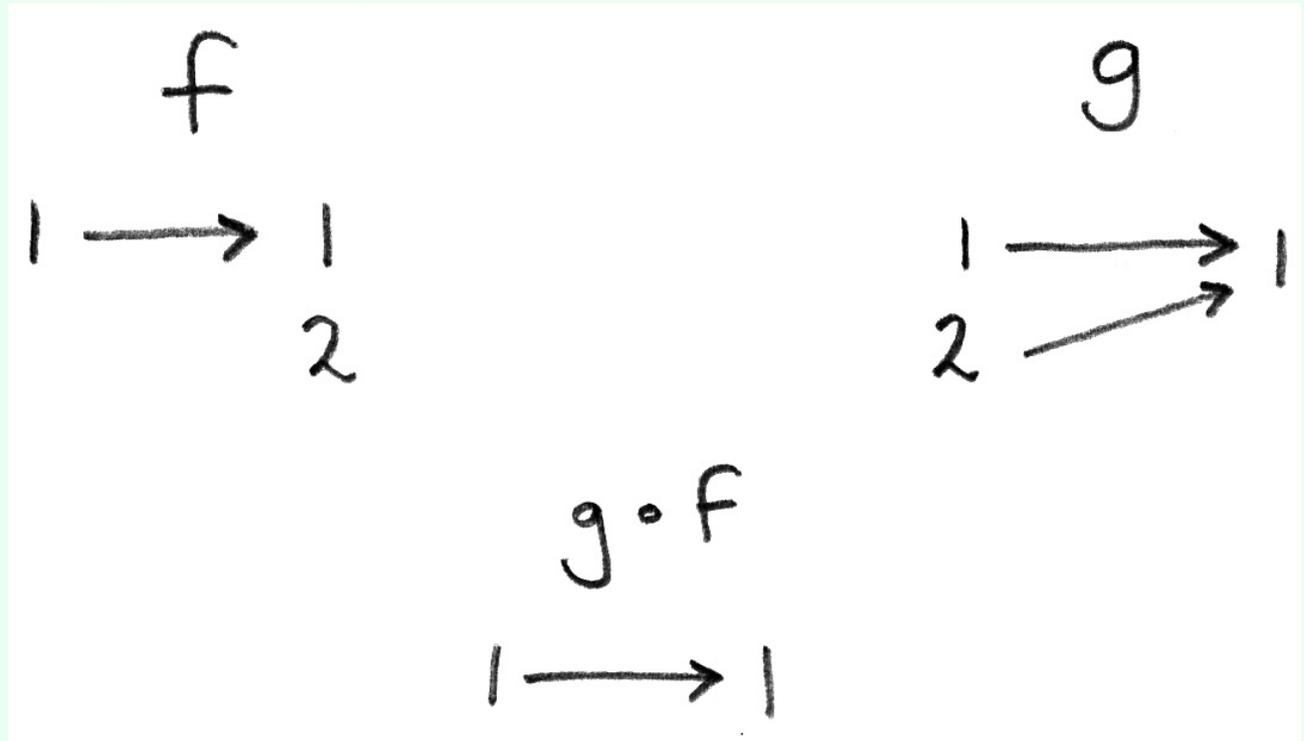


   *A2*

**Note:** There will be many alternative counter-examples.

*[3 marks]*

## Examiners report

This question was found difficult by a large number of candidates and no fully correct solutions were seen. A number of students made thought-through attempts to show it was surjective, but found more difficulty in showing it was injective. Very few were able to find a single counter example to show that the converses of the earlier results were false. Candidates struggled with the abstract nature of the question.

17. 
[12 marks]

## Markscheme

(a)  (i)  It is not closed because

$1 * 1 = 0 \notin \mathbb{Z}^+$ .  *R2*

(ii)

$a * b = a + b - 2$

$b * a = b + a - 2 = a * b$  *M1*

It is commutative.  *A1*

(iii)  It is not associative.  *A1*

Consider

$(1 * 1) * 5$ and

$1 * (1 * 5)$ .

The first is undefined because

$1 * 1 \notin \mathbb{Z}^+$ .

The second equals 3.  *R2*

**Notes:** Award *A1R2* for stating that non-closure implies non-associative.

Award *A1R1* to candidates who show that

$a * (b * c) = (a * b) * c = a + b + c - 4$ and therefore conclude that it is associative, ignoring the non-closure.

*[7 marks]*

(b)  (i)  The identity *e* satisfies

$a * e = a + e - 2 = a$  *M1*

$e = 2$  (and $2 \in \mathbb{Z}^+$)  *A1*

(ii)

$a * a^{-1} = a + a^{-1} - 2 = 2$  *M1*

$a + a^{-1} = 4$  *A1*

So the only elements having an inverse are 1, 2 and 3.  *A1*

**Note:** Due to commutativity there is no need to check two sidedness of identity and inverse.

*[5 marks]*

*Total [12 marks]*

## Examiners report

Almost all the candidates thought that the binary operation was associative, not realising that the non-closure prevented this from being the case. In the circumstances, however, partial credit was given to candidates who 'proved' associativity. Part (b) was well done by many candidates.

# Markscheme

(a)

$]-1,1[$    ***A1A1***

**Note:** Award ***A1*** for the values $-1$, $1$ and ***A1*** for the open interval.

*[2 marks]*

(b)    **EITHER**

Let

$\frac{1-e^{-x}}{1+e^{-x}} = \frac{1-e^{-y}}{1+e^{-y}}$    ***M1***

$1 - e^{-x} + e^{-y} - e^{-(x+y)} = 1 + e^{-x} - e^{-y} - e^{-(x+y)}$    ***A1***

$e^{-x} = e^{-y}$

$x = y$    ***A1***

Therefore $f$ is an injection    ***AG***

**OR**

Consider

$f'(x) = \frac{e^{-x}(1+e^{-x})+e^{-x}(1-e^{-x})}{(1+e^{-x})^2}$    ***M1***

$= \frac{2e^{-x}}{(1+e^{-x})^2}$    ***A1***

$> 0$ for all $x$.    ***A1***

Therefore $f$ is an injection.    ***AG***

**Note:** Award ***M1A1A0*** for a graphical solution.

*[3 marks]*

(c)    Let

$y = \frac{1-e^{-x}}{1+e^{-x}}$    ***M1***

$y(1 + e^{-x}) = 1 - e^{-x}$    ***A1***

$e^{-x}(1 + y) = 1 - y$    ***A1***

$e^{-x} = \frac{1-y}{1+y}$

$x = \ln\left(\frac{1+y}{1-y}\right)$    ***A1***

$f^{-1}(x) = \ln\left(\frac{1+x}{1-x}\right)$    ***A1***

*[5 marks]*

*Total [10 marks]*

# Examiners report

Most candidates found the range of $f$ correctly. Two algebraic methods were seen for solving (b), either showing that the derivative of $f$ is everywhere positive or showing that $f(a) = f(b) \Rightarrow a = b$. Candidates who based their 'proof' on a graph produced on their graphical calculators were given only partial credit on the grounds that the whole domain could not be shown and, in any case, it was not clear from the graph that $f$ was an injection.

19.

## Markscheme

(a)

$z^6 = 1 = \operatorname{cis} 2n\pi$   *(M1)*

The six roots are

$\operatorname{cis} 0 (1),\ \operatorname{cis}\frac{\pi}{3},\ \operatorname{cis}\frac{2\pi}{3},\ \operatorname{cis}\pi(-1),\ \operatorname{cis}\frac{4\pi}{3},\ \operatorname{cis}\frac{5\pi}{3}$   *A3*

**Note:** Award *A2* for 4 or 5 correct roots, *A1* for 2 or 3 correct roots.

*[4 marks]*

(b)   (i)   Closure: Consider any two roots

$\operatorname{cis}\frac{m\pi}{3},\ \operatorname{cis}\frac{n\pi}{3}$.   *M1*

$\operatorname{cis}\frac{m\pi}{3} \times \operatorname{cis}\frac{n\pi}{3} = \operatorname{cis}(m+n)(\mathrm{mod}\,6)\frac{\pi}{3} \in G$   *A1*

**Note:** Award *M1A1* for a correct Cayley table showing closure.

Identity: The identity is 1.   *A1*

Inverse: The inverse of

$\operatorname{cis}\frac{m\pi}{3}$ is $\operatorname{cis}\frac{(6-m)\pi}{3} \in G$ .   *A2*

Associative: This follows from the associativity of multiplication.   *R1*

The 4 group axioms are satisfied.   *R1*

(ii)   Successive powers of

$\operatorname{cis}\frac{\pi}{3}\left(\text{or } \operatorname{cis}\frac{5\pi}{3}\right)$

generate the group which is therefore cyclic.   *R2*

The (only) other generator is

$\operatorname{cis}\frac{5\pi}{3}\left(\text{or } \operatorname{cis}\frac{\pi}{3}\right)$ .   *A1*

**Note:** Award *A0* for any additional answers.

(iii)   The group of the integers 0, 1, 2, 3, 4, 5 under addition modulo 6.   *R2*

The correspondence is

$m \to \operatorname{cis}\frac{m\pi}{3}$   *R1*

**Note:** Accept any other cyclic group of order 6.

*[13 marks]*

*Total [17 marks]*

## Examiners report

This question was reasonably well answered by many candidates, although in (b)(iii), some candidates were unable to give another group isomorphic to $G$.

20.
[9 marks]

## Markscheme

(a)    Reflexive:

$(a, b)R(a, b)$ because

$ab = ba$    **R1**

Symmetric:

$(a, b)R(c, d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c, d)R(a, b)$    **M1A1**

Transitive:

$(a, b)R(c, d) \Rightarrow ad = bc$    **M1**

$(c, d)R(e, f) \Rightarrow cf = de$

Therefore

$\frac{ad}{de} = \frac{bc}{cf}$ so $af = be$    **A1**

It follows that

$(a, b)R(e, f)$    **R1**

*[6 marks]*


(b)

$(a, b)R(c, d) \Rightarrow \frac{a}{b} = \frac{c}{d}$    *(M1)*

Equivalence classes are therefore points lying, in the first quadrant, on straight lines through the origin.    *A2*

**Notes:** Accept a correct sketch.

Award *A1* if "in the first quadrant" is omitted.

Do not penalise candidates who fail to exclude the origin.


*[3 marks]*

*Total [9 marks]*

## Examiners report

Part (a) was well answered by many candidates although some misunderstandings of the terminology were seen. Some candidates appeared to believe, incorrectly, that reflexivity was something to do with

$(a, a)R(a, a)$ and some candidates confuse the terms 'reflexive' and 'symmetric'. Many candidates were unable to describe the equivalence classes geometrically.

21.

## Markscheme

The identity is 1.   *(R1)*

Consider

$2^1$, $2^2$, $2^3$, ..., $2^k$

$2^k = p - 1$   *R1*

Therefore all the above powers of two are different   *R1*

Now consider

$2^{k+1} \equiv 2p - 2(\bmod p) = p - 2$   *M1A1*

$2^{k+2} \equiv 2p - 4(\bmod p) = p - 4$   *A1*

$2^{k+3} = p - 8$

*etc.*

$2^{2k-1} = p - 2^{k-1}$

$2^{2k} = p - 2^k$   *A1*

$= 1$   *A1*

and this is the first power of 2 equal to 1.   *R2*

The order of 2 is therefore $2k$.   *AG*

Using Lagrange's Theorem, it follows that $2k$ is a factor of

$2^k$, the order of the group, in which case $k$ must be as given.   *R2*

*[12 marks]*

## Examiners report

Few solutions were seen to this question with many candidates unable even to start.

22.

# Markscheme

(a)

| $+_6$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 5 | 0 | 1 | 2 | 3 | 4 |

*A3*

**Note:** Award *A2* for 1 error, *A1* for 2 errors and *A0* for more than 2 errors.

*[3 marks]*

(b)    The table is closed    *A1*

Identity element is 0    *A1*

Each element has a unique inverse (0 appears exactly once in each row and column)    *A1*

Addition mod 6 is associative    *A1*

Hence

$\{G, +_6\}$ forms a group    *AG*

*[4 marks]*

(c)    0 has order 1 (0 = 0),

1 has order 6 (1 + 1 + 1 + 1 + 1 + 1 = 0),

2 has order 3 (2 + 2 + 2 = 0),

3 has order 2 (3 + 3 = 0),

4 has order 3 (4 + 4 + 4 = 0),

5 has order 6 (5 + 5 + 5 + 5 + 5 + 5 = 0).    *A3*

**Note:** Award *A2* for 1 error, *A1* for 2 errors and *A0* for more than 2 errors.

*[3 marks]*

(d)    Since 1 and 5 are of order 6 (the same as the order of the group) every element can be written as sums of either 1 or 5. Hence the group is cyclic.    *R1*

The generators are 1 and 5.    *A1*

*[2 marks]*

(e)    A subgroup of order 3 is

$(\{0, 2, 4\}, +_6)$    *A2*

**Note:** Award *A1* if only $\{0, 2, 4\}$ is seen.

*[2 marks]*

(f)   Other proper subgroups are

$(\{0\}+_6)$, $(\{0, 3\}+_6)$   *A1A1*

**Note:** Award *A1* if only $\{0\}$, $\{0, 3\}$ is seen.

*[2 marks]*

*Total [16 marks]*

## Examiners report

The table was well done as was showing its group properties. The order of the elements in (b) was done well except for the order of 0 which was often not given. Finding the generators did not seem difficult but correctly stating the subgroups was not often done. The notion of a 'proper' subgroup is not well known.

23a.                                                                                              *[4 marks]*

## Markscheme

*f* is surjective because every horizontal line through *Q* meets the graph somewhere   *R1*

*f* is not injective because it is a many-to-one function   *R1*

*g* is injective because it always has a positive gradient   *R1*

(accept horizontal line test reasoning)

*g* is not surjective because a horizontal line through the negative part of *B* would not meet the graph at all   *R1*

*[4 marks]*

## Examiners report

'Using features of the graph' should have been a fairly open hint but too many candidates contented themselves with describing what injective and surjective meant rather than explaining which graph had which properties. Candidates found considerable difficulty with presenting a convincing argument in part (b).

# Markscheme

(i)   **EITHER**

Let

$x_1,\ x_2 \in X$ and $y_1 = h(x_1)$ and $y_2 = h(x_2)$   **M1**

Then

$k \circ (h(x_1)) = k \circ (h(x_2))$

$\Rightarrow k(y_1) = k(y_2)$   **A1**

$\Rightarrow y_1 = y_2$   ($k$ is injective)   **A1**

$\Rightarrow h(x_1) = h(x_2)$   ($h(x_1) = y_1$ and $h(x_2) = y_2$)   **A1**

$\Rightarrow x_1 \equiv x_2$   ($h$ is injective)   **A1**

Hence

$k \circ h$ is injective   **AG**

**OR**

$x_1,\ x_2 \in X,\ x_1 \neq x_2$   **M1**

since $h$ is an injection

$\Rightarrow h(x_1) \neq h(x_2)$   **A1**

$h(x_1),\ h(x_2) \in Y$   **A1**

since $k$ is an injection

$\Rightarrow k(h(x_1)) \neq k(h(x_2))$   **A1**

$k(h(x_1)),\ k(h(x_2)) \in \mathbb{Z}$   **A1**

so

$k \circ h$ is an injection.   **AG**


(ii)   $h$ and $k$ are surjections and let

$z \in \mathbb{Z}$

Since $k$ is surjective there exists

$y \in Y$ such that $k(y) = z$   **R1**

Since $h$ is surjective there exists

$x \in X$ such that $h(x) = y$   **R1**

Therefore there exists

$x \in X$ such that

$k \circ h(x) = k(h(x))$

$= k(y)$   **R1**

$= z$   **A1**

So

$k \circ h$ is surjective   **AG**

*[9 marks]*

24.

# Markscheme

$(A \cap B) \backslash (A \cap C) = (A \cap B) \cap (A \cap C)'$   ***M1***

$= (A \cap B) \cap (A' \cup C')$   ***A1***

$= (A \cap B \cap A') \cup (A \cap B \cap C')$   ***A1***

$= (A \cap A' \cap B) \cup (A \cap B \cap C')$   ***A1***

$= (\emptyset \cap B) \cup (A \cap B \cap C')$   ***(A1)***

$= \emptyset \cup (A \cap B \cap C')$

$= (A \cap (B \cap C'))$   ***A1***

$= A \cap (B \backslash C)$   ***AG***

**Note:** Do not accept proofs by Venn diagram.

*[6 marks]*

**25a.**

## Markscheme

(i)

$aRa \Rightarrow a \cdot a = a^2$ so $R$ is reflexive    ***A1***

$aRb = m^2 \Rightarrow bRa$ so $R$ is symmetric    ***A1***

$aRb = ab = m^2$ and $bRc = bc = n^2$    ***M1A1***

so

$a = \frac{m^2}{b}$ and $c = \frac{n^2}{b}$

$ac = \frac{m^2 n^2}{b^2} = \left(\frac{mn}{b}\right)^2$,    ***A1***

$ac$ is an integer hence

$\left(\frac{mn}{b}\right)^2$ is an integer    ***R1***

so $aRc$, hence $R$ is transitive    ***R1***

$R$ is therefore an equivalence relation    ***AG***

(ii)    $1R4$ and $4R9$ **or** $2R8$    ***M1***

so $\{1, 4, 9\}$ is an equivalence class    ***A1***

and $\{2, 8\}$ is an equivalence class    ***A1***

*[10 marks]*

## Examiners report

Not a difficult question although using the relation definition to fully show transitivity was not well done. It was good to see some students use an operation binary matrix to show transitivity. This was a nice way given that the set was finite. The proof in (b) proved difficult.

**25b.**

## Markscheme

$a \sim a$ since $aa^{-1} = e \in H$, the identity must be in $H$ since it is a subgroup.    ***M1***

Hence reflexivity.    ***R1***

$a \sim b \Leftrightarrow ab^{-1} \in H$ but $H$ is a subgroup so it must contain

$(ab^{-1})^{-1} = ba^{-1}$    ***M1R1***

*i.e.*

$ba^{-1} \in H$ so $\sim$ is symmetric    ***A1***

$a \sim b$ and $b \sim c \Rightarrow ab^{-1} \in H$ and $bc^{-1} \in H$    ***M1***

But $H$ is closed, so

$(ab^{-1})(bc^{-1}) \in H$ or $a(b^{-1}b)c^{-1} \in H$    ***R1***

$ac^{-1} \in H \Rightarrow a \sim c$    ***A1***

Hence

$\sim$ is transitive and is thus an equivalence relation    ***R1AG***

*[9 marks]*

Not a difficult question although using the relation definition to fully show transitivity was not well done. It was good to see some students use an operation binary matrix to show transitivity. This was a nice way given that the set was finite. The proof in (b) proved difficult.

26. *[6 marks]*

## Markscheme

(a)    Each row and column contains all the elements of the set.    *A1A1*

*[2 marks]*

(b)    There are 5 elements therefore any subgroup must be of an order that is a factor of 5    *R2*

But there is a subgroup

$$\begin{array}{c c} & \begin{array}{c c} e & a \end{array} \\ \begin{array}{c} e \\ a \end{array} & \left(\begin{array}{c c} e & a \\ a & e \end{array}\right) \end{array}$$    of order 2 so the table is not a group table    *R2*

**Note:** Award *R0R2* for "*a* is an element of order 2 which does not divide the order of the group".

*[4 marks]*

*Total [6 marks]*

## Examiners report

Part (a) presented no problem but finding the order two subgroups (Lagrange's theorem was often quoted correctly) was beyond some candidates. Possibly presenting the set in non-alphabetical order was the problem.

27.                                                                                          [12 marks]

## Markscheme

(a)    by inspection, or otherwise,

$A = \{2, 3, 5, 7, 11, 13\}$    **A1**

$B = \{0, 2\}$    **A1**

$C = \{0, 1\}$    **A1**

$D = \{-1, 0, 1, 2, 3\}$    **A1**

*[4 marks]*


(b)    (i)    true    **A1**

$n(B) + n(B \cup C) = 2 + 3 = 5 = n(D)$    **R1**


(ii)    false    **A1**

$D \backslash B = \{-1, 1, 3\} \not\subset A$    **R1**


(iii)    false    **A1**

$B \cap A' = \{0\} \neq \emptyset$    **R1**


(iv)    true    **A1**

$n(B \Delta C) = n\{1, 2\} = 2$    **R1**

*[8 marks]*


*Total [12 marks]*

## Examiners report

It was surprising and disappointing that many candidates regarded 1 as a prime number. One of the consequences of this error was that it simplified some of the set-theoretic calculations in part(b), with a loss of follow-through marks. Generally speaking, it was clear that the majority of candidates were familiar with the set operations in part(b).

28.

# Markscheme

(a)   the Cayley table is

$$
\begin{array}{c}
\phantom{-1}\ \ -1\ \ \ \ 0\ \ \ \ 1 \\
\begin{array}{c} -1 \\ 0 \\ 1 \end{array}
\left(
\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & -1 \\
0 & 1 & 0
\end{array}
\right)
\end{array}
\quad \textbf{\textit{M1A2}}
$$

**Notes:** Award **M1** for setting up a Cayley table with labels.

Deduct **A1** for each error or omission.

*[3 marks]*

(b)   (i)   closed   **A1**

because all entries in table belong to $\{-1, 0, 1\}$   **R1**

(ii)   not commutative   **A1**

because the Cayley table is not symmetric, or counter-example given   **R1**

(iii)   not associative   **A1**

for example because   **M1**

$0 \odot (-1 \odot 0) = 0 \odot 1 = -1$

but

$(0 \odot -1) \odot 0 = -1 \odot 0 = 1$   **A1**

or alternative counter-example

*[7 marks]*

**Total [10 marks]**

# Examiners report

This question was generally well done, with the exception of part(b)(iii), showing that the operation is non-associative.

29.

# Markscheme

(a)    the following two calculations show the required result

$F \circ F(x) = \frac{1}{\frac{1}{x}} = x$

$G \circ G(x) = 1 - (1 - x) = x$    *M1A1A1*

*[3 marks]*

(b)    part (a) shows that the identity function defined by $I(x) = x$ belongs to $S$    *A1*

the two compositions of $F$ and $G$ are:

$F \circ G(x) = \frac{1}{1-x};$    *(M1)A1*

$G \circ F(x) = 1 - \frac{1}{x} \left( = \frac{x-1}{x} \right)$    *(M1)A1*

the final element is

$G \circ F \circ G(x) = 1 - \frac{1}{1-x} \left( = \frac{x}{x-1} \right)$    *(M1)A1*

*[7 marks]*

*Total [10 marks]*

# Examiners report

This question was generally well done. In part(a), the quickest answer involved showing that squaring the function gave the identity.

Some candidates went through the more elaborate method of finding the inverse function in each case.

$F \circ F(x) = \frac{1}{\frac{1}{x}} = x$

$G \circ G(x) = 1 - (1 - x) = x$    *M1A1A1*

30.
# Markscheme

(a)    not a group    *A1*

**EITHER**

subtraction is not associative on

$\mathbb{Z}$ (or give counter-example)    *R1*

**OR**

there is a right-identity, 0, but it is not a left-identity    *R1*

*[2 marks]*

(b)    the set forms a group    *A1*

the closure is a consequence of the following relation (and the closure of

$\mathbb{C}$ itself):

$|z_1 z_2| = |z_1||z_2|$    *R1*

the set contains the identity 1    *R1*

that inverses exist follows from the relation

$|z^{-1}| = |z|^{-1}$

for non-zero complex numbers    *R1*

*[4 marks]*

(c)    not a group    *A1*

for example, only the identity element 1 has an inverse    *R1*

*[2 marks]*

(d)    the set forms a group    *A1*

$\frac{2m+1}{3n+1} \times \frac{3s+1}{3t+1} = \frac{9ms+3s+3m+1}{9nt+3n+3t+1} = \frac{3(3ms+s+m)+1}{3(3nt+n+t)+1}$    *M1R1*

shows closure

the identity 1 corresponds to $m = n = 0$    *R1*

an inverse corresponds to interchanging the parameters $m$ and $n$    *R1*

*[5 marks]*

*Total [13 marks]*

# Examiners report

There was a mixed response to this question. Some candidates were completely out of their depth. Stronger candidates provided satisfactory answers to parts (a) and (c). For the other parts there was a general lack of appreciation that, for example, closure and the existence of inverses, requires that products and inverses have to be shown to be members of the set.

31.

# Markscheme

(a)  (i)

$f_1 \circ g_1(k) = k + 4$  *M1*

Range

$(f_1 \circ g_1) = \mathbb{Z}$  *A1*

(ii)

$f_3 \circ g_2(k) = 0$  *M1*

Range

$(f_3 \circ g_2) = \{0\}$  *A1*

*[4 marks]*

(b)  the equation to solve is

$k - |k| + 4 = |2k|$  *M1A1*

the positive solution is $k = 2$  *A1*

the negative solution is $k = -1$  *A1*

*[4 marks]*

(c)  the equation factorizes:

$(m + n)(m - n) = p$  *(M1)*

for $p = 1$ , the possible factors over

$\mathbb{Z}$ are

$m + n = \pm 1, \ m - n = \pm 1$  *(M1)(A1)*

with solutions $(1, 0)$ and $(-1, 0)$  *A1*

for $p = 2$ , the possible factors over

$\mathbb{Z}$ are

$m + n = \pm 1, \ \pm 2; \ m - n = \pm 2, \ \pm 1$  *M1A1*

there are no solutions over

$\mathbb{Z} \times \mathbb{Z}$  *A1*

*[7 marks]*

*Total [15 marks]*

# Examiners report

The majority of candidates were able to compute the composite functions involved in parts (a) and (b). Part(c) was satisfactorily tackled by a minority of candidates. There were more GDC solutions than the more obvious approach of factorizing a difference of squares. Some candidates seemed to forget that $m$ and $n$ belonged to the set of integers.

32.

# Markscheme

(a)

|   | 1 | −1 | i | −i |
|---|---|---|---|---|
| 1 | 1 | −1 | i | −i |
| −1 | −1 | 1 | −i | i |
| i | i | −i | −1 | 1 |
| −i | −i | i | 1 | −1 |

see the Cayley table, (since there are no new elements) the set is closed   *A1*

1 is the identity element   *A1*

1 and −1 are self inverses and i and -i form an inverse pair, hence every element has an inverse   *A1*

multiplication is associative   *A1*

hence {1, −1, i, −i} form a group $G$ under the operation of multiplication   *AG*

*[4 marks]*


(b)   (i)   $aba = aab$

$= eb$   *A1*

$= b$   *AG*


(ii)   $abab = aabb$

$= ee$   *A1*

$= e$   *AG*

*[2 marks]*


(c)   (i)

| * | e | a | b | ab |
|---|---|---|---|---|
| e | e | a | b | ab |
| a | a | e | ab | b |
| b | b | ab | e | a |
| ab | ab | b | a | e |

*A2*

**Note:** Award *A1* for 1 or 2 errors, *A0* for more than 2.


(ii)   see the Cayley table, (since there are no new elements) the set is closed   *A1*

$H$ has an identity element $e$   *A1*

all elements are self inverses, hence every element has an inverse   *A1*

the operation is associative as stated in the question

hence {$e$, $a$, $b$, $ab$} forms a group $G$ under the operation

*   *AG*


(iii)   since there is symmetry across the leading diagonal of the group table, the group is Abelian   *A1*

*[6 marks]*


(d)   consider the element i from the group $G$   *(M1)*

$i^2 = −1$

$i^3 = −i$

$i^4 = 1$

thus i is a generator for $G$ and hence $G$ is a cyclic group   *A1*

−i is the other generator for $G$   *A1*

for the group $H$ there is no generator as all the elements are self inverses   *R1*

*[4 marks]*

(e)    since one group is cyclic and the other group is not, they are not isomorphic    **R1**

*[1 mark]*

***Total [17 marks]***

## Examiners report

Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. A number of candidates did not understand the term "Abelian". Many candidates understood the conditions for a group to be cyclic. Many candidates did not realise that the answer to part (e) was actually found in part (d), hence the reason for this part only being worth 1 mark. Overall, a number of fully correct solutions to this question were seen.

33a.                                                                                                  *[5 marks]*

## Markscheme

(i)    if

$*$ is commutative

$a * b = b * a$

since

$a + b + 1 = b + a + 1$ ,

$*$ is commutative    **R1**

(ii)    let $e$ be the identity element

$a * e = a + e + 1 = a$    **M1**

$\Rightarrow e = -1$    **A1**

(iii)    let $a$ have an inverse,

$a^{-1}$

$a * a^{-1} = a + a^{-1} + 1 = -1$    **M1**

$\Rightarrow a^{-1} = -2 - a$    **A1**

*[5 marks]*

## Examiners report

Part (a) of this question was the most accessible on the paper and was completed correctly by the majority of candidates.

# Markscheme

$(x_1, y_1) \odot ((x_2, y_2) \odot (x_3, y_3)) = (x_1, y_1) \odot (x_2 + x_3 + 1, 3y_2 y_3)$   *M1*

$= (x_1 + x_2 + x_3 + 2, 9y_1 y_2 y_3)$   *A1A1*

$((x_1, y_1) \odot (x_2, y_2)) \odot (x_3, y_3) = (x_1 + x_2 + 1, 3y_1 y_2) \odot (x_3, y_3)$   *M1*

$= (x_1 + x_2 + x_3 + 2, 9y_1 y_2 y_3)$   *A1*

hence

$\odot$ is associative   *R1*

*[6 marks]*

# Examiners report

Part (b) was completed by many candidates, but a significant number either did not understand what was meant by associative, confused associative with commutative, or were unable to complete the algebra.

$(x_1, y_1) \odot ((x_2, y_2) \odot (x_3, y_3)) = (x_1, y_1) \odot (x_2 + x_3 + 1, 3y_2 y_3)$   *M1*

$= (x_1 + x_2 + x_3 + 2, 9y_1 y_2 y_3)$   *A1A1*

$((x_1, y_1) \odot (x_2, y_2)) \odot (x_3, y_3) = (x_1 + x_2 + 1, 3y_1 y_2) \odot (x_3, y_3)$   *M1*

$= (x_1 + x_2 + x_3 + 2, 9y_1 y_2 y_3)$   *A1*

hence

$\odot$ is associative   *R1*

34.                                                                                  [14 marks]

## Markscheme

(a)    consider

$(x, y)R(x, y)$

since $x - x = 0$ and $y - y = 0$ , $R$ is reflexive    ***A1***

assume

$(x, y)R(a, b)$

$\Rightarrow x - a = 3M$ and

$y - b = 2N$    ***M1***

$\Rightarrow a - x = -3M$ and

$b - y = -2N$    ***A1***

$\Rightarrow (a, b)R(x, y)$

hence $R$ is symmetric

assume

$(x, y)R(a, b)$

$\Rightarrow x - a = 3M$ and

$y - b = 2N$

assume

$(a, b)R(c, d)$

$\Rightarrow a - c = 3P$ and

$b - d = 2Q$    ***M1***

$\Rightarrow x - c = 3(M + P)$ and

$y - d = 2(N + Q)$    ***A1***

hence

$(x, y)R(c, d)$    ***A1***

hence $R$ is transitive

therefore $R$ is an equivalence relation    ***AG***

***[7 marks]***


(b)

$\{(x, y) : x = 3m + 2, y = 2n + 1, m, n \in \mathbb{Z}\}$    ***A1A1***

***[2 marks]***


(c)

$\{3m, 2n\} \{ 3m + 1, 2n\} \{3m + 2, 2n\}$

$\{3m, 2n + 1\} \{ 3m + 1, 2n + 1\} m, n \in \mathbb{Z}$    ***A1A1A1A1A1***

***[5 marks]***

***Total [14 marks]***

35.

## Markscheme

(a)    we need to show that the function is both injective and surjective to be a bijection    *R1*

suppose

$f(x,\ y) = f(u,\ v)$    *M1*

$(2x + y,\ x - y) = (2u + v,\ u - v)$

forming a pair of simultaneous equations    *M1*

$2x + y = 2u + v$    (i)

$x - y = u - v$    (ii)

$(i) + (ii) \Rightarrow 3x = 3u \Rightarrow x = u$    *A1*

$(i) - 2(ii) \Rightarrow 3y = 3v \Rightarrow y = v$    *A1*

hence function is injective    *R1*

let

$2x + y = s$ and

$x - y = t$    *M1*

$\Rightarrow 3x = s + t$

$\Rightarrow x = \frac{s+t}{3}$    *A1*

also

$3y = s - 2t$

$\Rightarrow y = \frac{s-2t}{3}$    *A1*

for any

$(s,\ t) \in \mathbb{R} \times \mathbb{R}$ there exists

$(x,\ y) \in \mathbb{R} \times \mathbb{R}$ and the function is surjective    *R1*

*[10 marks]*

(b)    the inverse is

$f^{-1}(x,\ y) = \left( \frac{x+y}{3},\ \frac{x-2y}{3} \right)$    *A1*

*[1 mark]*

*Total [11 marks]*

36.

## Markscheme

we are trying to prove

$(A \backslash B) \backslash C \neq A \backslash (B \backslash C)$    *M1(A1)*

$\text{LHS} = (A \cap B') \backslash C$    *(A1)*

$= (A \cap B') \cap C'$    *A1*

$\text{RHS} = A \backslash (B \cap C')$

$= A \cap (B \cap C')'$    *(A1)*

$= A \cap (B' \cup C)$    *A1*

as LHS does not contain any element of *C* and RHS does,

$\text{LHS} \neq \text{RHS}$    *R1*

hence set difference is not associative    *AG*

**Note:** Accept answers which use a proof containing a counter example.

*Total [7 marks]*

## Examiners report

This question was found difficult by a large number of candidates, but a number of correct solutions were seen. A number of candidates who understood what was required failed to gain the final reasoning mark. Many candidates seemed to be ill-prepared to deal with this style of question.

37.

# Markscheme

(a)    (i)

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 0 | 2 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 0 | 1 | 2 |

*A3*

**Note:** Award *A3* for no errors, *A2* for one error, *A1* for two errors and *A0* for three or more errors.

(ii)    it is not a Latin square because some rows/columns contain the same digit more than once    *A1*

*[4 marks]*

(b)    (i)    **EITHER**

it is not commutative because the table is not symmetric about the leading diagonal    *R2*

**OR**

it is not commutative because

$a + 2b + ab \neq 2a + b + ab$ in general    *R2*

**Note:** Accept a counter example *e.g.*

$1 * 2 = 3$ whereas

$2 * 1 = 2$ .

(ii)    **EITHER**

for example

$(0 * 1) * 1 = 2 * 1 = 2$    *M1*

and

$0 * (1 * 1) = 0 * 0 = 0$    *A1*

so

$*$ is not associative    *A1*

**OR**

associative if and only if

$a * (b * c) = (a * b) * c$    *M1*

which gives

$a + 2b + 4c + 2bc + ab + 2ac + abc = a + 2b + ab + 2c + ac + 2bc + abc$    *A1*

so

$*$ is not associative as

$2ac \neq 2c + ac$ , in general    *A1*

*[5 marks]*

(c)    $x = 0$ is a solution    *A2*

$x = 2$ is a solution    *A2*

## Examiners report

This question was generally well answered.

38.

## Markscheme

(a)

$f'(x) = 2e^x - e^{-x}$   *A1*

*[1 mark]*

(b)   *f* is an injection because

$f'(x) > 0$ for

$x \in [0, \infty[$   *R2*

(accept GDC solution backed up by a correct graph)

since

$f(0) = 0$ and

$f(x) \to \infty$ as

$x \to \infty$ , (and *f* is continuous) it is a surjection   *R1*

hence it is a bijection   *AG*

*[3 marks]*

(c)   let

$y = 2e^x + e^{-x} - 3$   *M1*

so

$2e^{2x} - (y+3)e^x + 1 = 0$   *A1*

$e^x = \dfrac{y+3\pm\sqrt{(y+3)^2-8}}{4}$   *A1*

$x = \ln\left(\dfrac{y+3\pm\sqrt{(y+3)^2-8}}{4}\right)$   *A1*

since

$x \geqslant 0$ we must take the positive square root   *(R1)*

$f^{-1}(x) = \ln\left(\dfrac{x+3+\sqrt{(x+3)^2-8}}{4}\right)$   *A1*

*[6 marks]*

Total [10 marks]

## Examiners report

In many cases the attempts at showing that $f$ is a bijection were unconvincing. The candidates were guided towards showing that $f$ is an injection by noting that

$f'(x) > 0$ for all $x$, but some candidates attempted to show that

$f(x) = f(y) \Rightarrow x = y$ which is much more difficult. Solutions to (c) were often disappointing, with the algebra defeating many candidates.

# Markscheme

(a)    (i)    *R* is reflexive, *i.e. PRP* because the sum of the zeroes of *P* is equal to the sum of the zeros of *P*    **R1**

*R* is symmetric, *i.e.*

$P_1 R P_2 \Rightarrow P_2 R P_1$ because the sums of the zeros of

$P_1$ and

$P_2$ are equal implies that the sums of the zeros of

$P_2$ and

$P_1$ are equal    **R1**

suppose that

$P_1 R P_2$ and

$P_2 R P_3$    **M1**

it follows that

$P_1 R P_3$ so *R* is transitive, because the sum of the zeros of

$P_1$ is equal to the sum of the zeros of

$P_2$ which in turn is equal to the sum of the zeros of

$P_3$ , which implies that the sum of the zeros of

$P_1$ is equal to the sum of the zeros of

$P_3$    **R1**

the three requirements for an equivalence relation are therefore satisfied    **AG**

(ii)    the zeros of

$z^2 - 4z + 5$ are

$2 \pm i$ , for which the sum is 4    **M1A1**

$z^2 + az + b$ has zeros of

$\frac{-a \pm \sqrt{a^2 - 4b}}{2}$ , so the sum is $-a$    **(M1)**

**Note:** Accept use of the result (although not in the syllabus) that the sum of roots is minus the coefficient of *z*.

hence $-a = 4$ and so $a = -4$    **A1**

the equivalence class is

$z^2 - 4z + k$ , $(k \in \mathbb{R})$    **A1**

*[9 marks]*

(b)    for example,

$(z-1)(z-2)S(z-1)(z-3)$ and

$(z-1)(z-3)S(z-3)(z-4)$ but

$(z-1)(z-2)S(z-3)(z-4)$ is not true    **M1A1**

so *S* is not transitive    **A1**

*[3 marks]*

*Total [12 marks]*

# Examiners report

Most candidates were able to show, in (a), that *R* is an equivalence relation although few were able to identify the required

equivalence class. In (b), the explanation that *S* is not transitive was often unconvincing.

40.
# Markscheme

(a)    if

$h \in H$ then

$h \in G$   **R1**

hence, (by Lagrange) the order of $h$ exactly divides $n$

and so the order of $h$ is smaller than or equal to $n$    **R2**

*[3 marks]*

(b)    the associativity in $G$ ensures associativity in $H$    **R1**

(closure within $H$ is given)

as $H$ is non-empty there exists an

$h \in H$, let the order of $h$ be $m$ then

$h^m = e$ and as $H$ is closed

$e \in H$   **R2**

it follows from the earlier result that

$h * h^{m-1} = h^{m-1} * h = e$    **R1**

thus, the inverse of $h$ is

$h^{m-1}$ which

$\in H$   **R1**

the four axioms are satisfied showing that

$\{H, *\}$ is a subgroup    **R1**

*[6 marks]*

*Total [9 marks]*

# Examiners report

Solutions to this question were extremely disappointing. This property of subgroups is mentioned specifically in the Guide and yet

most candidates were unable to make much progress in (b) and even solutions to (a) were often unconvincing.

41.

# Markscheme

(a)  (i)

| * | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 7 |
| 3 | 3 | 1 | 7 | 5 |
| 5 | 5 | 7 | 1 | 3 |
| 7 | 7 | 5 | 3 | 1 |

*A3*

**Note:** Award *A2* for 15 correct, *A1* for 14 correct and *A0* otherwise.

(ii)  it is a group because:

the table shows closure    *A1*

multiplication is associative    *A1*

it possesses an identity 1    *A1*

justifying that every element has an inverse *e.g.* all self-inverse    *A1*

(iii)  (since

$*$ is commutative,

$5 * x = y$)

so solutions are $(1, 5), (3, 7), (5, 1), (7, 3)$    *A2*

**Notes:** Award *A1* for 3 correct and *A0* otherwise.

Do not penalize extra incorrect solutions.

*[9 marks]*

(b)

| ⊗ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 7 | 9 |
| 3 | 3 | 9 | 5 | 1 | 7 |
| 5 | 5 | 5 | 5 | 5 | 5 |
| 7 | 7 | 1 | 5 | 9 | 3 |
| 9 | 9 | 7 | 5 | 3 | 1 |

**Note:** It is not necessary to see the Cayley table.

a valid reason    *R2*

*e.g.* from the Cayley table the 5 row does not give a Latin square, or 5 does not have an inverse, so it cannot be a group

*[2 marks]*

(c)  (i)  remove the 5    *A1*

(ii)  they are not isomorphic because all elements in *A* are self-inverse this is not the case in *C*, (e.g.

$3 \otimes 3 = 9 \neq 1$)    *R2*

**Note:** Accept any valid reason.

*[3 marks]*

*Total [14 marks]*

# Examiners report

Candidates are generally confident when dealing with a specific group and that was the situation again this year. Some candidates lost marks in (a)(ii) by not giving an adequate explanation for the truth of some of the group axioms, eg some wrote 'every element has an inverse'. Since the question told the candidates that
$\{A, *)$ was a group, this had to be the case and the candidates were expected to justify their statement by noting that every element was self-inverse. Solutions to (c)(ii) were reasonably good in general, certainly better than solutions to questions involving isomorphisms set in previous years.

42.

# Markscheme

(a) (i) the inverse is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \quad \textit{A1}$$

(ii) **EITHER**

$1 \to 2 \to 4 \to 3 \to 1$ (is a cycle of length 4) **R3**

so

$p_1$ is of order 4 **A1 N2**

**OR**

consider

$$p_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \quad \textit{M1A1}$$

it is now clear that

$$p_1^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \textit{A1}$$

so

$p_1$ is of order 4 **A1 N2**

*[5 marks]*

(b) (i) consider

$$p_1 p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \textit{M1A1}$$
$$p_2 p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \textit{A1}$$

composition is not commutative **A1**

**Note:** In this part do not penalize candidates who incorrectly reverse the order both times.

(ii) **EITHER**

pre and postmultiply by

$p_1^{-1},$

$p_2^{-1}$ to give

$p_3 = p_1^{-1} p_2^{-1}$ **(M1)(A1)**

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad \textit{A1}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \textit{A1}$$

**OR**

starting from

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad \textit{M1}$$

successively deducing each missing number, to get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad \textit{A3}$$

*[8 marks]*

## Examiners report

Many candidates scored well on this question although some gave the impression of not having studied this topic. The most common error in (b) was to believe incorrectly that

$p_1 p_2$ means

$p_1$ followed by

$p_2$. This was condoned in (i) but penalised in (ii). The Guide makes it quite clear that this is the notation to be used.

43.

# Markscheme

(a) let $a$ be a generator and consider the (general) elements

$b = a^m$, $c = a^n$ **M1**

then

$bc = a^m a^n$ **A1**

$= a^n a^m$ (using associativity) **R1**

$= cb$ **A1**

therefore $G$ is Abelian **AG**

**[4 marks]**

(b) let $G$ be of order $p$ and let

$m \in \{1, \ldots\ldots, p\}$, let $a$ be a generator

consider

$aa^{-1} = e \Rightarrow a^m (a^{-1})^m = e$ **M1R1**

this shows that

$(a^{-1})^m$ is the inverse of

$a^m$ **R1**

as $m$ increases from 1 to $p$,

$a^m$ takes $p$ different values and it generates $G$ **R1**

it follows from the uniqueness of the inverse that

$(a^{-1})^m$ takes $p$ different values and is a generator **R1**

**[5 marks]**

(c) **EITHER**

by Lagrange, the order of any element divides the order of the group, *i.e.* 5 **R1**

the only numbers dividing 5 are 1 and 5 **R1**

the identity element is the only element of order 1 **R1**

all the other elements must be of order 5 **R1**

so they all generate $G$ **AG**

**OR**

let $a$ be a generator.

successive powers of $a$ and therefore the elements of $G$ are

$a$, $a^2$, $a^3$, $a^4$ and $a^5 = e$ **A1**

successive powers of

$a^2$ are

$a^2$, $a^4$, $a$, $a^3$, $a^5 = e$ **A1**

successive powers of

$a^3$ are

$a^3$, $a$, $a^4$, $a^2$, $a^5 = e$ **A1**

successive powers of

$a^4$ are

$a^4$, $a^3$, $a^2$, $a$, $a^5 = e$ **A1**

this shows that

$a^2$, $a^3$, $a^4$ are also generators in addition to $a$ **AG**

**[4 marks]**

*Total [13 marks]*

44a.                                                                                                    *[5 marks]*

## Markscheme

reflexive: if $a$ is odd,

$a \times a$ is odd so $R$ is not reflexive     *R1*

symmetric: if $ab$ is even then $ba$ is even so $R$ is symmetric     *R1*

transitive: let $aRb$ and $bRc$; it is necessary to determine whether or not $aRc$     *(M1)*

for example $5R2$ and $2R3$     *A1*

since

$5 \times 3$ is not even, 5 is not related to 3 and $R$ is not transitive     *R1*

*[5 marks]*

## Markscheme

(i)    reflexive:

$a^2 \equiv a^2 (\bmod 6)$ so $S$ is reflexive    **R1**

symmetric:

$a^2 \equiv b^2 (\bmod 6) \Rightarrow 6|(a^2 - b^2) \Rightarrow 6|(b^2 - a^2) \Rightarrow b^2 \equiv a^2 (\bmod 6)$   **R1**

so $S$ is symmetric

transitive: let $aSb$ and $bSc$ so that

$a^2 = b^2 + 6M$ and

$b^2 = c^2 + 6N$    **M1**

it follows that

$a^2 = c^2 + 6(M + N)$ so $aSc$ and $S$ is transitive    **R1**

$S$ is an equivalence relation because it satisfies the three conditions    **AG**


(ii)    by considering the squares of integers (mod 6), the equivalence    **(M1)**

classes are

$\{1, 5, 7, 11,$

$\ldots\}$    **A1**

$\{2, 4, 8, 10,$

$\ldots\}$    **A1**

$\{3, 9, 15, 21,$

$\ldots\}$    **A1**

$\{6, 12, 18, 24,$

$\ldots\}$    **A1**

*[9 marks]*

## Examiners report

[N/A]

## Markscheme

$a \odot b = \sqrt{ab} = \sqrt{ba} = b \odot a$    **A1**

since

$a \odot b = b \odot a$ it follows that

$\odot$ is commutative    **R1**

*[2 marks]*

## Examiners report

[N/A]

## Markscheme

$a * (b * c) = a * b^2 c^2 = a^2 b^4 c^4$   **M1A1**

$(a * b) * c = a^2 b^2 * c = a^4 b^4 c^2$   **A1**

these are different, therefore

$*$ is not associative   **R1**

**Note:** Accept numerical counter-example.

*[4 marks]*

## Examiners report

[N/A]

---

## Markscheme

$a * (b \odot c) = a * \sqrt{bc} = a^2 bc$   **M1A1**

$(a * b) \odot (a * c) = a^2 b^2 \odot a^2 c^2 = a^2 bc$   **A1**

these are equal so

$*$ is distributive over

$\odot$   **R1**

*[4 marks]*

## Examiners report

[N/A]

---

## Markscheme

the identity e would have to satisfy

$a \odot e = a$ for all $a$   **M1**

now

$a \odot e = \sqrt{ae} = a \Rightarrow e = a$   **A1**

therefore there is no identity element   **A1**

*[3 marks]*

## Examiners report

[N/A]

# Markscheme

(i) the Cayley table is

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 |

*A3*

**Note:** Deduct 1 mark for each error up to a maximum of 3.

(ii) by considering powers of elements, *(M1)*

it follows that 3 (or 5) is of order 6 *A1*

so the group is cyclic *A1*

(iii) we see that 2 and 4 are of order 3 so the subgroup of order 3 is {1, 2, 4} *M1A1*

(iv) the element of order 2 is 6 *A1*

the coset is {3, 5, 6} *A1*

*[10 marks]*

# Examiners report

[N/A]

# Markscheme

(i) consider for example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$ *M1A1*

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$ *M1A1*

**Note:** Award *M1A1M1A0* if both compositions are done in the wrong order.

**Note:** Award *M1A1M0A0* if the two compositions give the same result, if no further attempt is made to find two permutations which are not commutative.

these are different so the group is not Abelian *R1AG*

(ii) they are not isomorphic because

$\{G, \times_7\}$ is Abelian and

$\{K, \circ\}$ is not *R1*

*[6 marks]*

47.

# Markscheme

consider the function $f$ given by

$f(E) = e$

$f(A) = e$

$f(B) = a$   *M1A1*

$f(C) = a$

then, it has to be shown that

$f(X * Y) = f(X) \odot f(Y)$ for all $X, Y \in G$   *(M1)*

consider

$f((E \text{ or } A) * (E \text{ or } A)) = f(E \text{ or } A) = e;\ f(E \text{ or } A) \odot f(E \text{ or } A) = e \odot e = e$   *M1A1*

$f((E \text{ or } A) * (B \text{ or } C)) = f(B \text{ or } C) = a;\ f(E \text{ or } A) \odot f(B \text{ or } C) = e \odot a = a$   *A1*

$f((B \text{ or } C) * (B \text{ or } C)) = f(E \text{ or } A) = e;\ f(B \text{ or } C) \odot f(B \text{ or } C) = a \odot a = e$   *A1*

since the groups are Abelian, there is no need to consider

$f((B \text{ or } C) * (E \text{ or } A))$   *R1*

the required property is satisfied in all cases so the homomorphism exists

**Note:** A comprehensive proof using tables is acceptable.

the kernel is

$\{E, A\}$   *A1*

*[9 marks]*

48.

## Markscheme

the associativity property carries over from $G$     *R1*

closure is given     *R1*

let

$h \in H$ and let $n$ denote the order of $h$, (this is finite because $G$ is finite)     *M1*

it follows that

$h^n = e$, the identity element     *R1*

and since $H$ is closed,

$e \in H$     *R1*

since

$h * h^{n-1} = e$     *M1*

it follows that

$h^{n-1}$ is the inverse,

$h^{-1}$, of $h$     *R1*

and since $H$ is closed,

$h^{-1} \in H$ so each element of $H$ has an inverse element     *R1*

the four requirements for $H$ to be a group are therefore satisfied     *AG*

*[8 marks]*

## Examiners report

[N/A]