

Topic 10 Part 1 [431 marks]

Let $f(n) = n^5 - n, n \in \mathbb{Z}^+$.

1a. Find the values of $f(3), f(4)$ and $f(5)$. [2 marks]

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1b. Use the Euclidean algorithm to find [4 marks]

- (i) $\gcd(f(3), f(4))$;
- (ii) $\gcd(f(4), f(5))$.

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1c. Explain why $f(n)$ is always exactly divisible by 5. [1 mark]

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- 1d. By factorizing $f(n)$ explain why it is always exactly divisible by 6. [4 marks]

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- 1e. Determine the values of n for which $f(n)$ is exactly divisible by 60. [3 marks]

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- 2a. Use the pigeon-hole principle to prove that for any simple graph that has two or more vertices and in which every vertex is connected to at least one other vertex, there must be at least two vertices with the same degree. [4 marks]

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2b.

[4 marks]

Each person shakes hands with at least one other person and no-one shakes hands with

the same person more than once. Use the result from part (a) to show that there must be at least two people who shake hands with the same number of people.

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2c.

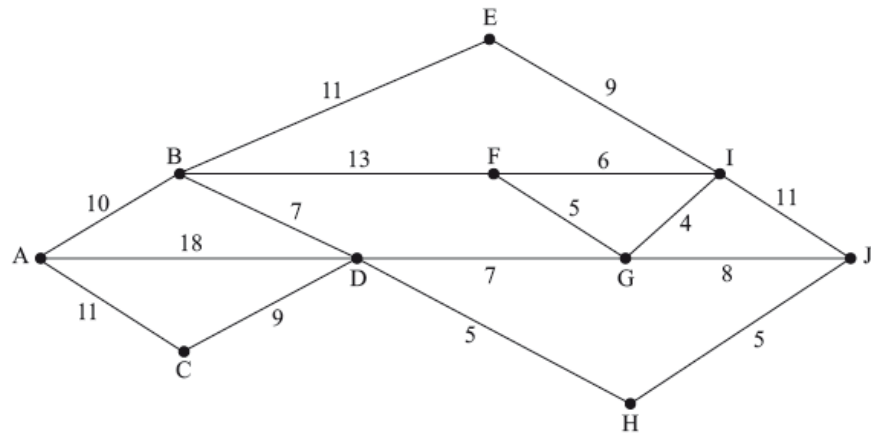
[2 marks]

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The following graph represents the cost in dollars of travelling by bus between 10 towns in a particular province.



- 3a. Use Dijkstra's algorithm to find the cheapest route between A and J , and state its cost. [7 marks]

- 3b. For the remainder of the question you may find the cheapest route between any two towns by inspection. [6 marks]

It is given that the total cost of travelling on all the roads without repeating any is \$139.

A tourist decides to go over all the roads at least once, starting and finishing at town A .

Find the lowest possible cost of his journey, stating clearly which roads need to be travelled more than once. You must fully justify your answer.

4a. Solve, by any method, the following system of linear congruences

[3 marks]

$$x \equiv 9 \pmod{11}$$

$$x \equiv 1 \pmod{5}$$

4b. Find the remainder when 41^{82} is divided by 11.

[4 marks]

4c. Using your answers to parts (a) and (b) find the remainder when 41^{82} is divided by 55.

[3 marks]

Andy and Roger are playing tennis with the rule that if one of them wins a game when serving then he carries on serving, and if he loses then the other player takes over the serve.

The probability Andy wins a game when serving is $\frac{1}{2}$ and the probability he wins a game when not serving is $\frac{1}{4}$. Andy serves in the first game. Let u_n denote the probability that Andy wins the n^{th} game.

5a. State the value of u_1 .

[1 mark]

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5b. Show that u_n satisfies the recurrence relation

[4 marks]

$$u_n = \frac{1}{4}u_{n-1} + \frac{1}{4}.$$

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5c. Solve this recurrence relation to find the probability that Andy wins the n^{th} game.

[6 marks]

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- 5d. After they have played many games, Pat comes to watch. Use your answer from part (c) to estimate the probability that Andy will win the game they are playing when Pat arrives. [2 marks]

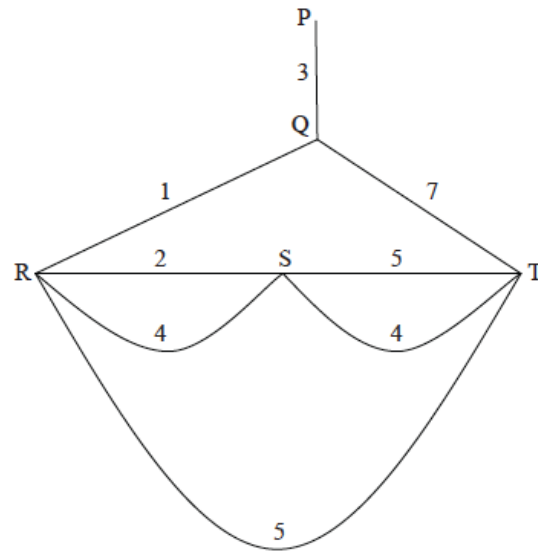
- 6a. The weights of the edges of a graph H are given in the following table. [8 marks]

	A	B	C	D	E	F	G
A	–	5	4	–	–	–	–
B	5	–	–	–	5	–	–
C	4	–	–	5	2	–	–
D	–	–	5	–	3	–	6
E	–	5	2	3	–	5	4
F	–	–	–	–	5	–	1
G	–	–	–	6	4	1	–

- (i) Draw the weighted graph H .
- (ii) Use Kruskal's algorithm to find the minimum spanning tree of H . Your solution should indicate the order in which the edges are added.
- (iii) State the weight of the minimum spanning tree.

6b. Consider the following weighted graph.

[3 marks]



(i) Write down a solution to the Chinese postman problem for this graph.

(ii) Calculate the total weight of the solution.

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6c. (i) State the travelling salesman problem.

[3 marks]

(ii) Explain why there is no solution to the travelling salesman problem for this graph.

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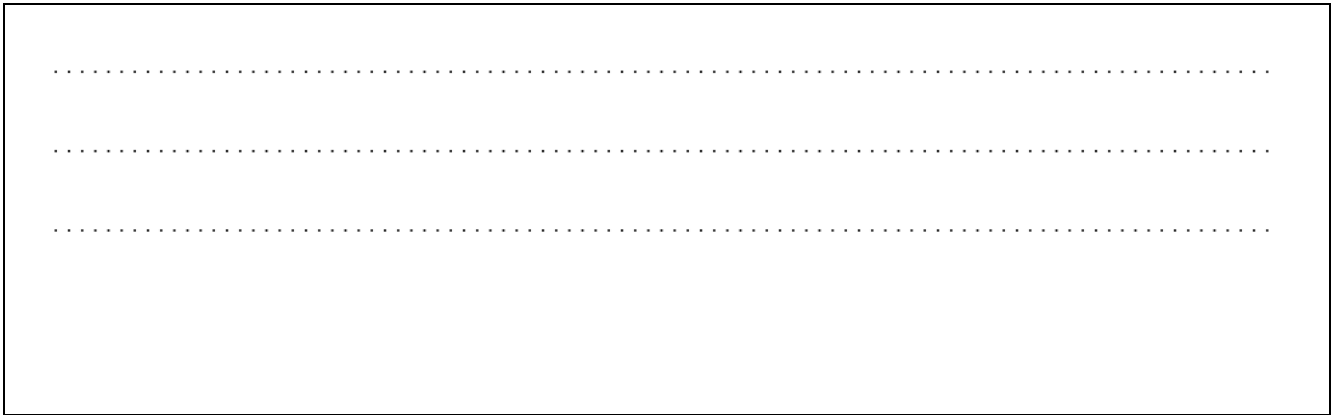
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The graph $K_{2,2}$ is the complete bipartite graph whose vertex set is the disjoint union of two subsets each of order two.

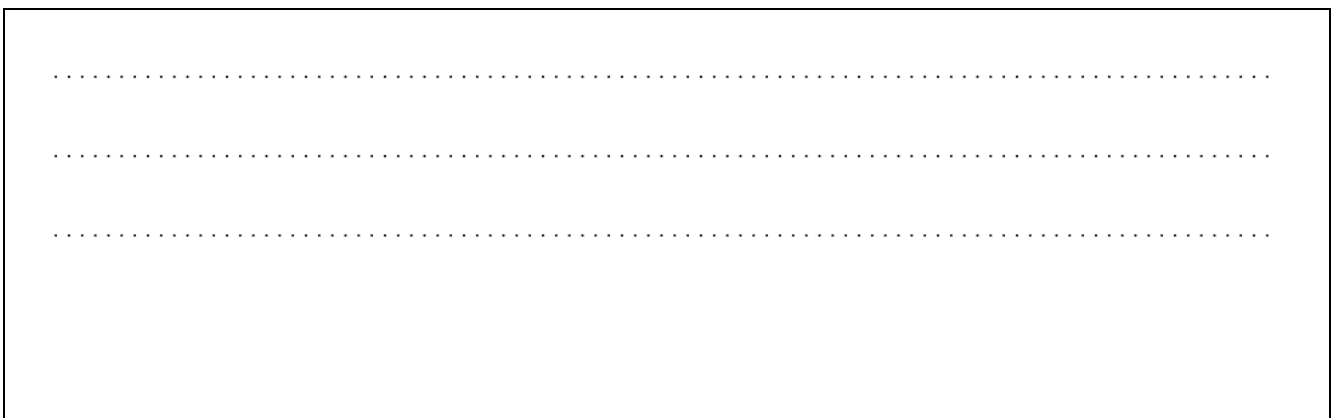
7a. Draw $K_{2,2}$ as a planar graph.

[2 marks]



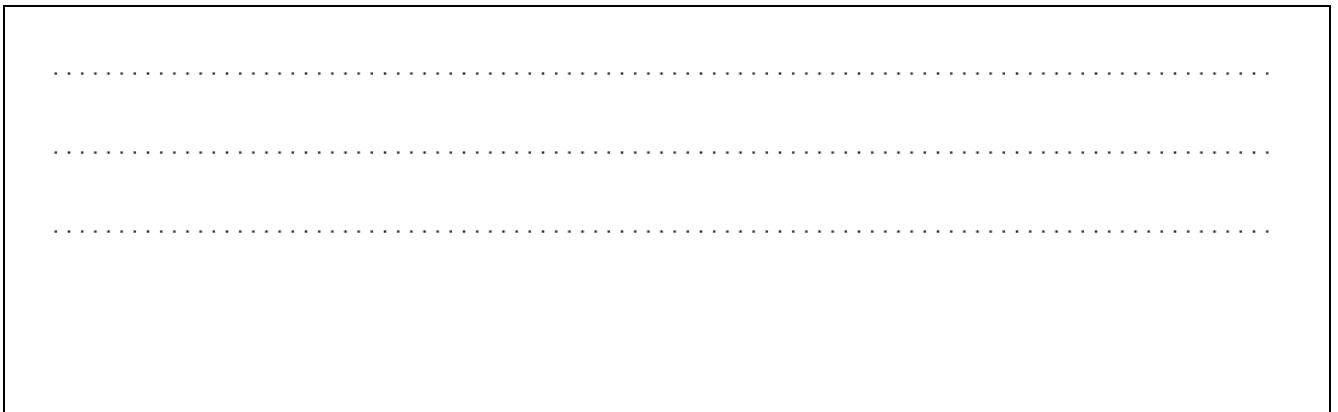
7b. Draw a spanning tree for $K_{2,2}$.

[1 mark]



7c. Draw the graph of the complement of $K_{2,2}$.

[1 mark]



- 7d. Show that the complement of any complete bipartite graph does not possess a spanning tree. [3 marks]

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- 8a. The sequence $\{u_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $u_{n+1} = 7u_n - 6$. [5 marks]

Given that $u_0 = 5$, find an expression for u_n in terms of n .

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- 8b. The sequence $\{v_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $v_{n+2} = 10v_{n+1} + 11v_n$. [7 marks]

Given that $v_0 = 4$ and $v_1 = 44$, find an expression for v_n in terms of n .

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- 8c. The sequence $\{v_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $v_{n+2} = 10v_{n+1} + 11v_n$. [4 marks]
 Show that $v_n - u_n \equiv 15 \pmod{16}$, $n \in \mathbb{N}$.

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A simple connected planar graph, has e edges, v vertices and f faces.

- 9a. (i) Show that $2e \geq 3f$ if $v > 2$. [6 marks]
 (ii) Hence show that K_5 , the complete graph on five vertices, is not planar.

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- 9b. (i) State the handshaking lemma. [4 marks]
 (ii) Determine the value of f , if each vertex has degree 2.

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9c. Draw an example of a simple connected planar graph on 6 vertices each of degree 3. [2 marks]

10a. State the Fundamental theorem of arithmetic for positive whole numbers greater than 1. [2 marks]

10b. Use the Fundamental theorem of arithmetic, applied to 5577 and 99 099, to calculate $\gcd(5577, 99\,099)$ and $\text{lcm}(5577, 99\,099)$, expressing each of your answers as a product of prime numbers. [3 marks]

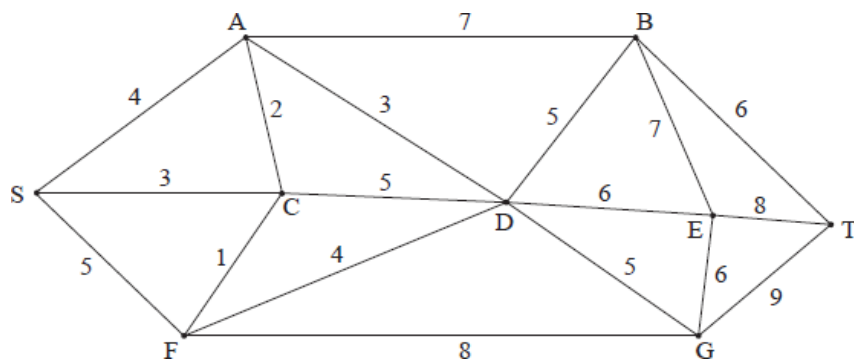
10c. Prove that $\gcd(n, m) \times \text{lcm}(n, m) = n \times m$ for all $n, m \in \mathbb{Z}^+$.

[6 marks]

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In the graph given above, the numbers shown represent the distance along that edge.

11a. Using Dijkstra's algorithm, find the length of the shortest path from vertex S to vertex T . Write down this shortest path.

[6 marks]

11b. (i) Does this graph have an Eulerian circuit? Justify your answer.
(ii) Does this graph have an Eulerian trail? Justify your answer.

[4 marks]

11c. The graph above is now to be considered with the edges representing roads in a town and with the distances being the length of that road in kilometres. Huan is a postman and he has to travel along every road in the town to deliver letters to all the houses in that road. He has to start at the sorting office at S and also finish his route at S . Find the shortest total distance of such a route. Fully explain the reasoning behind your answer.

[4 marks]

12a. Using the Euclidean algorithm, show that

[4 marks]

$$\gcd(99, 332) = 1.$$

12b. (i) Find the general solution to the diophantine equation

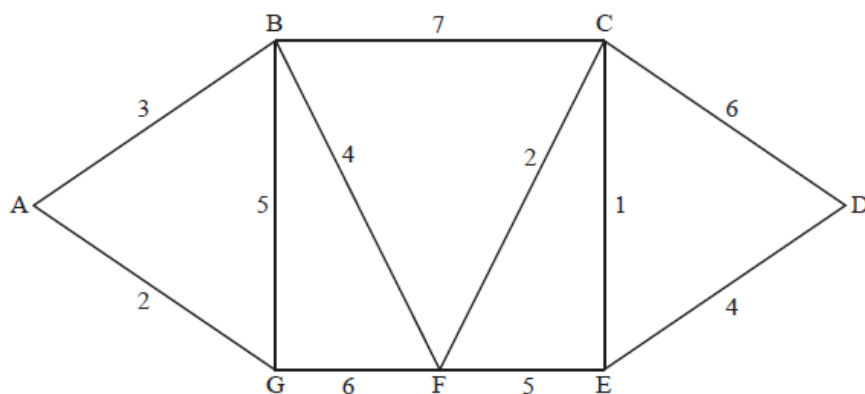
[11 marks]

$$332x - 99y = 1.$$

(ii) Hence, or otherwise, find the smallest positive integer satisfying the congruence

$$17z \equiv 1 \pmod{57}.$$

The diagram shows a weighted graph with vertices A, B, C, D, E, F, G. The weight of each edge is marked on the diagram.



13a. (i) Explain briefly why the graph contains an Eulerian trail but not an Eulerian circuit. [4 marks]

(ii) Write down an Eulerian trail.

13b. (i) Use Dijkstra's algorithm to find the path of minimum total weight joining A to D. [8 marks]

(ii) State the minimum total weight.

When numbers are written in base n ,

$$33^2 = 1331.$$

14a. By writing down an appropriate polynomial equation, determine the value of n . [4 marks]

14b. Rewrite the above equation with numbers in base 7. [6 marks]

The positive integer p is an odd prime number.

15a. Show that [4 marks]

$$\sum_{k=1}^p k^p \equiv 0 \pmod{p}.$$

15b. Given that [4 marks]

$$\sum_{k=1}^p k^{p-1} \equiv n \pmod{p} \text{ where}$$

$0 \leq n \leq p-1$, find the value of n .

The weighted graph K , representing the travelling costs between five customers, has the following adjacency table.

	A	B	C	D	E
A	0	1	6	7	4
B	1	0	9	8	10
C	6	9	0	11	3
D	7	8	11	0	12
E	4	10	3	12	0

16a. Draw the graph K . [2 marks]

16b. Starting from customer D, use the nearest-neighbour algorithm, to determine an upper bound to the travelling salesman problem for K . [4 marks]

16c. By removing customer A, use the method of vertex deletion, to determine a lower bound to the travelling salesman problem for K . [4 marks]

17a. Consider the integers [7 marks]

$$a = 871 \text{ and}$$

$$b = 1157, \text{ given in base}$$

10.

(i) Express

a and

b in base

13.

(ii) Hence show that

$$\gcd(a, b) = 13.$$

17b. A list [4 marks]

L contains

$n + 1$ distinct positive integers. Prove that at least two members of

L leave the same remainder on division by

n .

17c. Consider the simultaneous equations [6 marks]

$$4x + y + 5z = a$$

$$2x + z = b$$

$$3x + 2y + 4z = c$$

where

$$x, y, z \in \mathbb{Z}.$$

(i) Show that 7 divides

$$2a + b - c.$$

(ii) Given that $a = 3$, $b = 0$ and $c = -1$, find the solution to the system of equations modulo 2.

17d. Consider the set

[6 marks]

P of numbers of the form

$$n^2 - n + 41, n \in \mathbb{N}.$$

- Prove that all elements of P are odd.
- List the first six elements of P for $n = 0, 1, 2, 3, 4, 5$.
- Show that not all elements of P are prime.

18. (a) Draw a spanning tree for

[10 marks]

(i) the complete graph,

K_4 ;

(ii) the complete bipartite graph,

$K_{4,4}$.

(b) Prove that a simple connected graph with n vertices, where $n > 1$, must have two vertices of the same degree.

(c) Prove that every simple connected graph has at least one spanning tree.

19. (a) (i) Write down the general solution of the recurrence relation

[17 marks]

$$u_n + 2u_{n-1} = 0, n \geq 1.$$

(ii) Find a particular solution of the recurrence relation

$$u_n + 2u_{n-1} = 3n - 2, n \geq 1, \text{ in the}$$

form

$$u_n = An + B, \text{ where}$$

$$A, B \in \mathbb{Z}.$$

(iii) Hence, find the solution to

$$u_n + 2u_{n-1} = 3n - 2, n \geq 1 \text{ where}$$

$$u_1 = 7.$$

(b) Find the solution of the recurrence relation

$$u_n = 2u_{n-1} - 2u_{n-2}, n \geq 2, \text{ where}$$

$$u_0 = 2, u_1 = 2. \text{ Express your solution in the form}$$

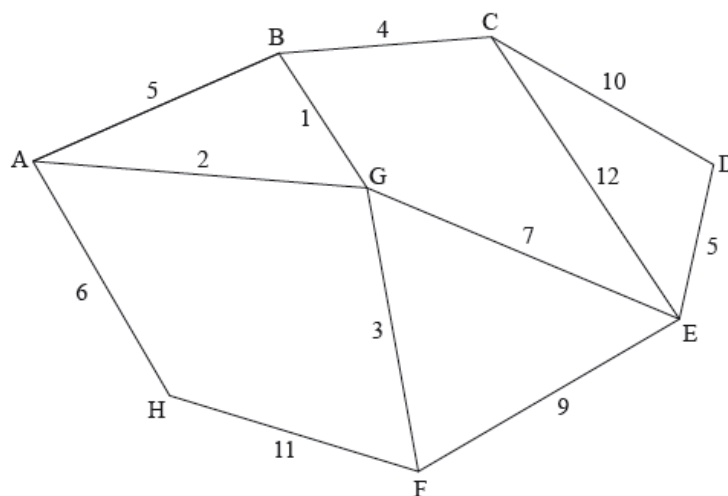
$$2^{f(n)} \cos(g(n)\pi), \text{ where the functions } f \text{ and } g \text{ map}$$

\mathbb{N} to

\mathbb{R} .

20. The following diagram shows a weighted graph.

[7 marks]

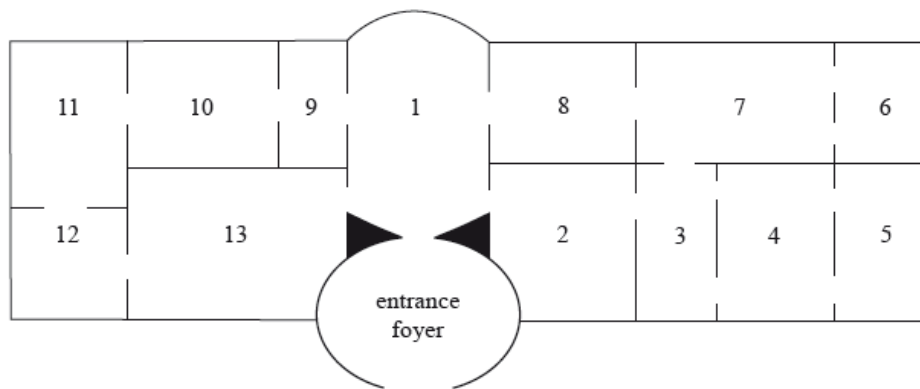


(a) Use Kruskal's algorithm to find a minimum spanning tree, clearly showing the order in which the edges are added.

(b) Sketch the minimum spanning tree found, and write down its weight.

21. The following figure shows the floor plan of a museum.

[11 marks]



- (a) (i) Draw a graph G that represents the plan of the museum where each exhibition room is represented by a vertex labelled with the exhibition room number and each door between exhibition rooms is represented by an edge. Do not consider the entrance foyer as a museum exhibition room.
- (ii) Write down the degrees of the vertices that represent each exhibition room.
- (iii) Virginia enters the museum through the entrance foyer. Use your answers to (i) and (ii) to justify why it is possible for her to visit the thirteen exhibition rooms going through each internal doorway exactly once.
- (b) Let G and H be two graphs whose adjacency matrices are represented below.

G

	A	B	C	D	E	F
A	0	2	0	2	0	0
B	2	0	1	1	0	1
C	0	1	0	1	2	1
D	2	1	1	0	2	0
E	0	0	2	2	0	2
F	0	1	1	0	2	0

H

	P	Q	R	S	T	U
P	0	1	3	0	1	2
Q	1	0	1	3	2	0
R	3	1	0	2	1	3
S	0	3	2	0	2	0
T	1	2	1	2	0	1
U	2	0	3	0	1	0

Using the adjacency matrices,

- (i) find the number of edges of each graph;
- (ii) show that exactly one of the graphs has a Eulerian trail;
- (iii) show that neither of the graphs has a Eulerian circuit.

22. Consider an integer

[15 marks]

a with

$(n + 1)$ digits written as

$a = 10^n a_n + 10^{n-1} a_{n-1} + \dots + 10a_1 + a_0$, where

$0 \leq a_i \leq 9$ for

$0 \leq i \leq n$, and

$a_n \neq 0$.

(a) Show that

$a \equiv (a_n + a_{n-1} + \dots + a_0) \pmod{9}$.

(b) Hence find all pairs of values of the single digits

x and

y such that the number

$a = 476x212y$ is a multiple of

45.

(c) (i) If

$b = 34390$ in base 10, show that

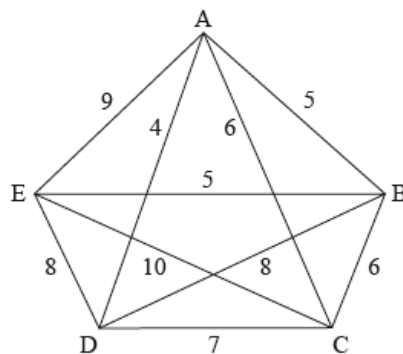
b is

52151 written in base 9.

(ii) Hence find

b^2 in base 9, by performing all the calculations without changing base.

The following diagram shows a weighted graph G with vertices A, B, C, D and E.



23a. Show that graph

[3 marks]

G is Hamiltonian. Find the total number of Hamiltonian cycles in

G , giving reasons for your answer.

23b. State an upper bound for the travelling salesman problem for graph

[1 mark]

G .

23c. Hence find a lower bound for the travelling salesman problem for

[2 marks]

G .

23d. Show that the lower bound found in (d) cannot be the length of an optimal solution for the travelling salesman problem for the graph G . [3 marks]

24a. Show that

[5 marks]

30 is a factor of

$n^5 - n$ for all

$n \in \mathbb{N}$.

- 24b. (i) Show that [8 marks]
 $3^{3^m} \equiv 3 \pmod{4}$ for all
 $m \in \mathbb{N}$.
 (ii) Hence show that there is exactly one pair
 (m, n) where
 $m, n \in \mathbb{N}$, satisfying the equation
 $3^{3^m} = 2^{2^n} + 5^2$.

- 25a. Use the Euclidean algorithm to express $\gcd(123, 2347)$ in the form $123p + 2347q$, where [8 marks]
 $p, q \in \mathbb{Z}$.

- 25b. Find the least positive solution of [3 marks]
 $123x \equiv 1 \pmod{2347}$.

- 25c. Find the general solution of [3 marks]
 $123z \equiv 5 \pmod{2347}$.

- 25d. State the solution set of [1 mark]
 $123y \equiv 1 \pmod{2346}$.

The graph G has adjacency matrix M given below.

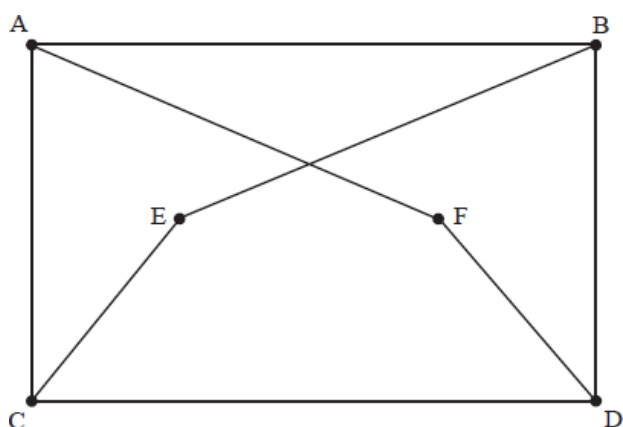
$$\begin{array}{c} \text{A} \text{ B} \text{ C} \text{ D} \text{ E} \text{ F} \\ \text{A} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \text{B} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \\ \text{C} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \\ \text{D} \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \\ \text{E} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \\ \text{F} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

- 26a. Draw the graph G . [2 marks]

- 26b. What information about G is contained in the diagonal elements of M [1 mark]
 2?

- 26c. List the trails of length 4 starting at A and ending at C. [3 marks]

- 27a. Draw the complement of the following graph as a planar graph. [3 marks]



- 27b. A simple graph G has v vertices and e edges. The complement G' of G has e' edges. [14 marks]
- Prove that $e \leq \frac{1}{2}v(v-1)$.
 - Find an expression for $e + e'$ in terms of v .
 - Given that G' is isomorphic to G , prove that v is of the form $4n$ or $4n + 1$ for $n \in \mathbb{Z}^+$.
 - Prove that there is a unique simple graph with 4 vertices which is isomorphic to its complement.
 - Prove that if $v \geq 11$, then G and G' cannot both be planar.
- 28a. Use the result $2003 = 6 \times 333 + 5$ and Fermat's little theorem to show that $2^{2003} \equiv 4 \pmod{7}$. [3 marks]
- 28b. Find $2^{2003} \pmod{11}$ and $2^{2003} \pmod{13}$. [3 marks]
- 28c. Use the Chinese remainder theorem, or otherwise, to evaluate $2^{2003} \pmod{1001}$, noting that $1001 = 7 \times 11 \times 13$. [7 marks]
- Let the greatest common divisor of 861 and 957 be h .
- 29a. Using the Euclidean algorithm, find h . [4 marks]
- 29b. Hence find integers A and B such that $861A + 957B = h$. [3 marks]
- 29c. Using part (b), solve $287w \equiv 2 \pmod{319}$, where $w \in \mathbb{N}$, $w < 319$. [5 marks]
- 29d. Find the general solution to the diophantine equation $861x + 957y = 6$. [6 marks]
30. State Fermat's little theorem. [2 marks]
- 31a. Use the Euclidean algorithm to find the greatest common divisor of the numbers 56 and 315. [4 marks]
- 31b. (i) Find the general solution to the diophantine equation $56x + 315y = 21$. [9 marks]
- (ii) Hence or otherwise find the smallest positive solution to the congruence $315x \equiv 21 \pmod{56}$.

32. The vertices of a graph L are A, B, C, D, E, F, G and H. The weights of the edges in L are given in the following table. [2 marks]

	A	B	C	D	E	F	G	H
A	–	4	–	–	1	4	1	5
B	4	–	1	–	–	1	–	13
C	–	1	–	11	2	–	–	–
D	–	–	11	–	8	3	–	7
E	1	–	2	8	–	10	–	–
F	4	1	–	3	10	–	6	–
G	1	–	–	–	–	6	–	7
H	5	13	–	7	–	–	7	–

Draw the graph L .

- 33a. Use the Euclidean algorithm to find [4 marks]

$\gcd(752, 352)$.

- 33b. A farmer spends £8128 buying cows of two different breeds, A and B, for her farm. A cow of breed A costs £752 and a [10 marks]

cow of breed B costs £352.

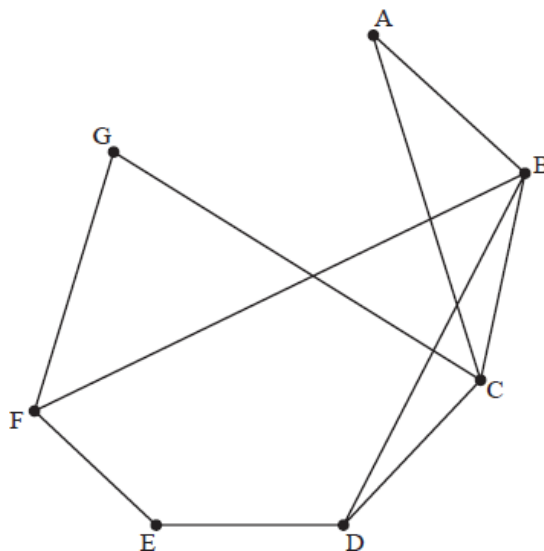
- (i) Set up a diophantine equation to show this.
- (ii) Using your working from part (a), find the general solution to this equation.
- (iii) **Hence** find the number of cows of each breed bought by the farmer.

- 34a. In any graph, show that [5 marks]

- (i) the sum of the degrees of all the vertices is even;
- (ii) there is an even number of vertices of odd degree.

34b. Consider the following graph, M .

[12 marks]



- (i) Show that M is planar.
- (ii) Explain why M is not Eulerian.
- (iii) By adding one edge to M it is possible to make it Eulerian. State which edge must be added.

This new graph is called N .

- (iv) Starting at A , write down a possible Eulerian circuit for N .
- (v) Define a Hamiltonian cycle. If possible, write down a Hamiltonian cycle for N , and if not possible, give a reason.
- (vi) Write down the adjacency matrix for N .
- (vii) Which pair of distinct vertices has exactly 30 walks of length 4 between them?

35. Anna is playing with some cars and divides them into three sets of equal size. However, when she tries to divide them into five sets of equal size, there are four left over. Given that she has fewer than 50 cars, what are the possible numbers of cars she can have? [7 marks]

A version of Fermat's little theorem states that when p is prime, a is a positive integer and a and p are relatively prime, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

- 36a. Use the above result to show that if p is prime then [4 marks]

$$a^p \equiv a \pmod{p} \text{ where } a \text{ is any positive integer.}$$

- 36b. Show that [7 marks]

$$2^{341} \equiv 2 \pmod{341}.$$

- 36c. (i) State the converse of the result in part (a). [2 marks]
 (ii) Show that this converse is not true.

