Topic 1 Part 2 *[399 marks]*

1. Consider $a = \log_2 3 \times \log_3 4 \times \log_4 5 \times \ldots \times \log_{31} 32$. Given that $a \in \mathbb{Z}$, find the value of a. *[5 marks]*

2. A geometric sequence $\{u_n\}$, with complex terms, is defined by $u_{n+1} = (1 + \mathrm{i})u_n$ and $u_1 = 3$. *[17 marks]*

   (a) Find the fourth term of the sequence, giving your answer in the form $x + y\mathrm{i}$, $x,\ y \in \mathbb{R}$.

   (b) Find the sum of the first 20 terms of $\{u_n\}$, giving your answer in the form $a \times (1 + 2^m)$ where $a \in \mathbb{C}$ and $m \in \mathbb{Z}$ are to be determined.

   A second sequence $\{v_n\}$ is defined by $v_n = u_n u_{n+k}$, $k \in \mathbb{N}$.

   (c) (i) Show that $\{v_n\}$ is a geometric sequence.

   (ii) State the first term.

   (iii) Show that the common ratio is independent of k.

   A third sequence $\{w_n\}$ is defined by $w_n = |u_n - u_{n+1}|$.

   (d) (i) Show that $\{w_n\}$ is a geometric sequence.

   (ii) State the geometrical significance of this result with reference to points on the complex plane.

3. Solve the equation $8^{x-1} = 6^{3x}$. Express your answer in terms of $\ln 2$ and $\ln 3$. *[5 marks]*

4. (a) Show that the following system of equations has an infinite number of solutions. *[5 marks]*

   $$x + y + 2z = -2$$

   $$3x - y + 14z = 6$$

   $$x + 2y = -5$$

   The system of equations represents three planes in space.

   (b) Find the parametric equations of the line of intersection of the three planes.

5. Use mathematical induction to prove that $(2n)! \geq 2^n(n!)^2,\ n \in \mathbb{Z}^+$. *[7 marks]*

A set of positive integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is used to form a pack of nine cards.

Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

6a. Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7. *[3 marks]*

6b. Find the number of selections Grace could make if at least two of the four integers drawn are even. *[4 marks]*

7a. (i) Show that $(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}, \quad \cos\theta \neq 0.$ *[10 marks]*

(ii) Hence verify that $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation $(1+z)^4 + (1-z)^4 = 0, \quad z \in \mathbb{C}.$

(iii) State another root of the equation $(1+z)^4 + (1-z)^4 = 0, \quad z \in \mathbb{C}.$

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7b. (i) Use the double angle identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1.$ *[13 marks]*

(ii) Show that $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1.$

(iii) Hence find the value of $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\mathrm{d}x.$

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The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term
a and non-zero common difference
d.

8a. Show that $d = \frac{a}{2}$. *[3 marks]*

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8b. The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200. *[6 marks]*

Find the least value of n for which this occurs.

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Ava and Barry play a game with a bag containing one green marble and two red marbles. Each player in turn randomly selects a marble from the bag, notes its colour and replaces it. Ava wins the game if she selects a green marble. Barry wins the game if he selects a red marble. Ava starts the game.

9a. Find the probability that Ava wins on her first turn. *[1 mark]*

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9b. Find the probability that Barry wins on his first turn. *[2 marks]*

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9c. Find the probability that Ava wins in one of her first three turns. *[4 marks]*

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9d. Find the probability that Ava eventually wins. *[4 marks]*

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In this question you may assume that $\arctan x$ is continuous and differentiable for $x \in \mathbb{R}$.

10a. Consider the infinite geometric series *[1 mark]*

$$1 - x^2 + x^4 - x^6 + \dots \quad |x| < 1.$$

Show that the sum of the series is $\frac{1}{1+x^2}$.

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10b. Hence show that an expansion of $\arctan x$ is $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ *[4 marks]*

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10c. f is a continuous function defined on $[a,\ b]$ and differentiable on $]a,\ b[$ with $f'(x) > 0$ on $]a,\ b[$. *[4 marks]*

Use the mean value theorem to prove that for any $x,\ y \in [a,\ b]$, if $y > x$ then $f(y) > f(x)$.

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10d. (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$. *[4 marks]*

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$.

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10e. Use the result from part (c) to prove that $\arctan x > x - \frac{x^3}{3}$, for $x > 0$. *[5 marks]*

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10f. Hence show that $\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$. *[4 marks]*

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11. Expand $(x + h)^3$. *[2 marks]*

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Let $y(x) = xe^{3x},\ x \in \mathbb{R}$.

12a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[2 marks]*

12b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. *[7 marks]*

12c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. *[5 marks]*

Justify whether any such point is a maximum or a minimum.

12d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. *[5 marks]*

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Let $\{u_n\},\ n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 0$. Let another sequence $\{v_n\},\ n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

13a. (i) Show that $\frac{v_{n+1}}{v_n}$ is a constant. *[4 marks]*

(ii) Write down the first term of the sequence $\{v_n\}$.

(iii) Write down a formula for v_n in terms of a, d and n.

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13b. Let S_n be the sum of the first n terms of the sequence $\{v_n\}$. *[8 marks]*

(i) Find S_n, in terms of a, d and n.

(ii) Find the values of d for which $\sum\limits_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum\limits_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

(iii) Write down S_∞ in terms of a and d.

(iv) Given that $S_\infty = 2^{a+1}$ find the value of d.

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13c. Let $\{w_n\},\ n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$. *[6 marks]*

Find $\sum_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q.

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14. Expand $(3 - x)^4$ in ascending powers of x and simplify your answer. *[4 marks]*

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15a. Find three distinct roots of the equation $8z^3 + 27 = 0,\ z \in \mathbb{C}$ giving your answers in modulus-argument form. *[6 marks]*

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15b. The roots are represented by the vertices of a triangle in an Argand diagram. *[3 marks]*

Show that the area of the triangle is $\frac{27\sqrt{3}}{16}$.

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The cubic equation $x^3 + px^2 + qx + c = 0$, has roots $\alpha,\ \beta,\ \gamma$. By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that

16a. (i) $p = -(\alpha + \beta + \gamma)$; *[3 marks]*

(ii) $q = \alpha\beta + \beta\gamma + \gamma\alpha$;

(iii) $c = -\alpha\beta\gamma$.

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16b. It is now given that $p = -6$ and $q = 18$ for parts (b) and (c) below. *[5 marks]*

(i) In the case that the three roots $\alpha,\ \beta,\ \gamma$ form an arithmetic sequence, show that one of the roots is 2.

(ii) Hence determine the value of c.

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16c. In another case the three roots $\alpha,\ \beta,\ \gamma$ form a geometric sequence. Determine the value of c. *[6 marks]*

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17a. Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0,\ n \in \mathbb{Z}$. *[2 marks]*

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17b. Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$. *[2 marks]*

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17c. Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2,\ n \in \mathbb{Z}$. *[9 marks]*

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18a. (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$. *[6 marks]*

(ii) Hence use De Moivre's theorem to prove

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

(iii) State a similar expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

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Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^5 - 1 = 0$ which has the smallest positive argument.

18b. Find the value of r and the value of α. *[4 marks]*

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18c. Using (a) (ii) and your answer from (b) show that $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$. *[4 marks]*

18d. Hence express *[5 marks]*

$\sin 72^\circ$ in the form $\frac{\sqrt{a+b\sqrt{c}}}{d}$ where $a,\ b,\ c,\ d \in \mathbb{Z}$.

From a group of five males and six females, four people are chosen.

19a. Determine how many possible groups can be chosen. *[2 marks]*

19b. Determine how many groups can be formed consisting of two males and two females. *[2 marks]*

19c. Determine how many groups can be formed consisting of at least one female. *[2 marks]*

Consider the following system of equations

$$2x + y + 6z = 0$$

$$4x + 3y + 14z = 4$$

$$2x - 2y + (\alpha - 2)z = \beta - 12.$$

20a. Find conditions on α and β for which *[6 marks]*

(i) the system has no solutions;

(ii) the system has only one solution;

(iii) the system has an infinite number of solutions.

20b. In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form. *[3 marks]*

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21a. If $w = 2 + 2\mathrm{i}$, find the modulus and argument of w. *[2 marks]*

21b. Given *[4 marks]*

$z = \cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)$, find in its simplest form

$w^4 z^6$.

22. The first terms of an arithmetic sequence are *[6 marks]*

$\frac{1}{\log_2 x}, \frac{1}{\log_8 x}, \frac{1}{\log_{32} x}, \frac{1}{\log_{128} x}, \ldots$

Find x if the sum of the first 20 terms of the sequence is equal to 100.

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is

$\frac{2}{3}$.

23a. Show that the probability that Alfred wins exactly 4 of the games is *[3 marks]*

$\frac{80}{243}$.

23b. (i) Explain why the total number of possible outcomes for the results of the 6 games is 64. *[4 marks]*

(ii) By expanding

$(1 + x)^6$ and choosing a suitable value for x, prove

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$

(iii) State the meaning of this equality in the context of the 6 games played.

23c. The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still *[9 marks]*

$\frac{2}{3}$.

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2 \left(\frac{2}{3}\right)^s \left(\frac{1}{3}\right)^t$ where the values of *r*, *s* and *t* are to be found.

(ii) Using your answer to (c) (i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that

$\binom{12}{6} = \binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2$.

23d. Alfred and Beatrice play *n* games. Let *A* denote the number of games Alfred wins. The expected value of *A* can be written as *[6 marks]*

$\mathrm{E}(A) = \sum_{r=0}^{n} r \binom{n}{r} \frac{a^r}{b^n}$.

(i) Find the values of *a* and *b*.

(ii) By differentiating the expansion of

$(1+x)^n$, prove that the expected number of games Alfred wins is

$\frac{2n}{3}$.

24. Expand *[4 marks]*

$(2-3x)^5$ in ascending powers of *x*, simplifying coefficients.

25. A geometric sequence has first term *a*, common ratio *r* and sum to infinity 76. A second geometric sequence has first term *a*, common ratio *[7 marks]*

r^3 and sum to infinity 36.

Find *r*.

Given the complex numbers

$z_1 = 1 + 3\mathrm{i}$ and

$z_2 = -1 - \mathrm{i}$.

26a. Write down the exact values of *[2 marks]*

$|z_1|$ and

$\arg(z_2)$.

26b. Find the minimum value of *[5 marks]*

$|z_1 + \alpha z_2|$, where

$\alpha \in \mathbb{R}$.

27. One root of the equation $x^2 + ax + b = 0$ is $2 + 3\mathrm{i}$ where $a,\ b \in \mathbb{R}$. Find the value of a and the value of b. *[4 marks]*

28. Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each. *[4 marks]*

29. A system of equations is given below. *[6 marks]*

$x + 2y - z = 2$

$2x + y + z = 1$

$-x + 4y + az = 4$

(a) Find the value of a so that the system does not have a unique solution.

(b) Show that the system has a solution for any value of a.

30. Prove, by mathematical induction, that $7^{8n+3} + 2,\ n \in \mathbb{N}$, is divisible by 5. *[8 marks]*

31a. Find the term in x^5 in the expansion of $(3x + A)(2x + B)^6$. *[4 marks]*

31b. Mina and Norbert each have a fair cubical die with faces labelled 1, 2, 3, 4, 5 and 6; they throw it to decide if they are going to eat a cookie. *[4 marks]*

Mina throws her die just once and she eats a cookie if she throws a four, a five or a six.

Norbert throws his die six times and each time eats a cookie if he throws a five or a six.

Calculate the probability that five cookies are eaten.

32. Consider the complex numbers $u = 2 + 3\mathrm{i}$ and $v = 3 + 2\mathrm{i}$. *[7 marks]*

(a) Given that $\frac{1}{u} + \frac{1}{v} = \frac{10}{w}$, express w in the form $a + b\mathrm{i},\ a,\ b \in \mathbb{R}$.

(b) Find w^* and express it in the form $re^{\mathrm{i}\theta}$.

33. The first three terms of a geometric sequence are *[7 marks]*

$\sin x,\ \sin 2x$ and

$4\sin x\cos^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) Find the common ratio r.

(b) Find the set of values of x for which the geometric series

$\sin x + \sin 2x + 4\sin x\cos^2 x + \ldots$ converges.

Consider

$x = \arccos\left(\frac{1}{4}\right),\ x > 0$.

(c) Show that the sum to infinity of this series is

$\frac{\sqrt{15}}{2}$.

34. (a) (i) Find the sum of all integers, between 10 and 200, which are divisible by 7. *[6 marks]*

(ii) Express the above sum using sigma notation.

An arithmetic sequence has first term 1000 and common difference of −6 . The sum of the first n terms of this sequence is negative.

(b) Find the least value of n.

35. Find the coefficient of *[6 marks]*

x^{-2} in the expansion of

$(x-1)^3\left(\frac{1}{x}+2x\right)^6$.

36a. Consider *[7 marks]*

$z = r(\cos\theta + \mathrm{i}\sin\theta),\ z \in \mathbb{C}$.

Use mathematical induction to prove that

$z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta),\ n \in \mathbb{Z}^+$.

36b. Given *[4 marks]*

$u = 1 + \sqrt{3}\mathrm{i}$ and

$v = 1 - \mathrm{i}$,

(i) express

u and

v in modulus-argument form;

(ii) hence find

u^3v^4.

The complex numbers

u and

v are represented by point A and point B respectively on an Argand diagram.

36c. Plot point A and point B on the Argand diagram. *[1 mark]*

Point A is rotated through

$\frac{\pi}{2}$ in the anticlockwise direction about the origin O to become point

A'. Point B is rotated through

$\frac{\pi}{2}$ in the clockwise direction about O to become point

B'.

36d. Find the area of triangle O

$\mathrm{A}'\mathrm{B}'$. *[3 marks]*

36e. Given that u and v are roots of the equation $z^4 + bz^3 + cz^2 + dz + e = 0$, where $b,\ c,\ d,\ e \in \mathbb{R}$, find the values of $b,\ c,\ d$ and e. *[5 marks]*

37. Prove by mathematical induction that $n^3 + 11n$ is divisible by 3 for all $n \in \mathbb{Z}^+$. *[7 marks]*

38. The sum of the first two terms of a geometric series is 10 and the sum of the first four terms is 30. *[7 marks]*

(a) Show that the common ratio r satisfies $r^2 = 2$.

(b) Given $r = \sqrt{2}$

(i) find the first term;

(ii) find the sum of the first ten terms.

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

39a. Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta,\ n \in \mathbb{Z}^+$. *[2 marks]*

39b. Expand $\left(z + z^{-1}\right)^4$. *[1 mark]*

39c. Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where $p,\ q$ and r are constants to be determined. *[4 marks]*

39d. Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. *[3 marks]*

39e. Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \mathrm{d}\theta$. *[3 marks]*

The region S is bounded by the curve
$y = \sin x\cos^2 x$ and the x-axis between
$x = 0$ and
$x = \frac{\pi}{2}$.

39f. S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4 marks]*

39g. (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$. *[3 marks]*

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta$ in terms of k.

40. The fourth term in an arithmetic sequence is 34 and the tenth term is 76. *[7 marks]*

(a) Find the first term and the common difference.

(b) The sum of the first n terms exceeds 5000. Find the least possible value of n.

41. A complex number z is given by *[6 marks]*
$z = \frac{a+\mathrm{i}}{a-\mathrm{i}}$, $a \in \mathbb{R}$.

(a) Determine the set of values of a such that

(i) z is real;

(ii) z is purely imaginary.

(b) Show that $|z|$ is constant for all values of a.