Topic 9 Part 1 *[597 marks]*

1a.

[2 marks]

Markscheme

any correct step before the given answer ***A1AG***

eg,

$f'(x) = \frac{(e^x)' + (e^{-x})'}{2} = \frac{e^x - e^{-x}}{2} = g(x)$

any correct step before the given answer ***A1AG***

eg,

$g'(x) = \frac{(e^x)' - (e^{-x})'}{2} = \frac{e^x + e^{-x}}{2} = f(x)$

[2 marks]

Examiners report

[N/A]

1b.

[5 marks]

Markscheme

METHOD 1

statement and attempted use of the general Maclaurin expansion formula ***(M1)***

$f(0) = 1;\ g(0) = 0$ (or equivalent in terms of derivative values) ***A1A1***

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$ or $f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ ***A1A1***

METHOD 2

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ ***A1***

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$ ***A1***

adding and dividing by 2 ***M1***

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$ or $f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ ***A1A1***

Notes: Accept 1, $\frac{x^2}{2}$ and $\frac{x^4}{24}$ or 1, $\frac{x^2}{2!}$ and $\frac{x^4}{4!}$. Award ***A1*** if two correct terms are seen.

[5 marks]

Examiners report

[N/A]

1c.

[3 marks]

Markscheme

METHOD 1

attempted use of the Maclaurin expansion from (b) ***M1***

$$\lim_{x\to 0}\frac{1-f(x)}{x^2}=\lim_{x\to 0}\frac{1-\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\ldots\right)}{x^2}$$ ***A1***

$$\lim_{x\to 0}\left(-\frac{1}{2}-\frac{x^2}{24}-\ldots\right)$$ ***A1***

$$-\frac{1}{2}$$ ***A1***

METHOD 2

attempted use of L'Hôpital and result from (a) ***M1***

$$\lim_{x\to 0}\frac{1-f(x)}{x^2}=\lim_{x\to 0}\frac{-g(x)}{2x}$$ ***A1***

$$\lim_{x\to 0}\frac{-f(x)}{2}$$ ***A1***

$$-\frac{1}{2}$$ ***A1***

[3 marks]

Examiners report

[N/A]

1d.

[6 marks]

Markscheme

METHOD 1

use of the substitution $u=f(x)$ and $(\mathrm{d}u=g(x)\mathrm{d}x)$ ***(M1)(A1)***

attempt to integrate $\int_1^\infty \frac{\mathrm{d}u}{u^2}$ ***(M1)***

obtain $\left[-\frac{1}{u}\right]_1^\infty$ or $\left[-\frac{1}{f(x)}\right]_0^\infty$ ***A1***

recognition of an improper integral by use of a limit or statement saying the integral converges ***R1***

obtain 1 ***A1*** ***N0***

METHOD 2

$$\int_0^\infty \frac{\frac{e^x-e^{-x}}{2}}{\left(\frac{e^x+e^{-x}}{2}\right)^2}\mathrm{d}x=\int_0^\infty \frac{2(e^x-e^{-x})}{(e^x+e^{-x})^2}\mathrm{d}x$$ ***(M1)***

use of the substitution $u=e^x+e^{-x}$ and $(\mathrm{d}u=e^x-e^{-x}\mathrm{d}x)$ ***(M1)***

attempt to integrate $\int_2^\infty \frac{2\mathrm{d}u}{u^2}$ ***(M1)***

obtain $\left[-\frac{2}{u}\right]_2^\infty$ ***A1***

recognition of an improper integral by use of a limit or statement saying the integral converges ***R1***

obtain 1 ***A1*** ***N0***

[6 marks]

Examiners report

[N/A]

2a. *[5 marks]*

Markscheme

(i) attempt at chain rule *(M1)*

$f'(x) = \frac{2\ln x}{x}$ ***A1***

(ii) attempt at chain rule *(M1)*

$g'(x) = \frac{2}{x\ln x}$ ***A1***

(iii) $g'(x)$ is positive on $]1,\ \infty[$ ***A1***

so $g(x)$ is increasing on $]1,\ \infty[$ ***AG***

[5 marks]

Examiners report

[N/A]

2b. *[12 marks]*

Markscheme

(i) rearrange in standard form:

$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2}{x\ln x}y=\frac{2x-1}{(\ln x)^2},\ x>1$ ***(A1)***

integrating factor:

$\mathrm{e}^{\int\frac{2}{x\ln x}\mathrm{d}x}$ ***(M1)***

$=\mathrm{e}^{\ln\left((\ln x)^2\right)}$ ***(A1)***

$=(\ln x)^2$

multiply by integrating factor ***(M1)***

$(\ln x)^2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2\ln x}{x}y=2x-1$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(y(\ln x)^2\right)=2x-1$ ***M1***

attempt to integrate: ***M1*** $\left(\text{or } y(\ln x)^2=\int 2x-1\mathrm{d}x\right)$

$(\ln x)^2y=x^2-x+c$ ***A1***

$y=\frac{x^2-x+c}{(\ln x)^2}$

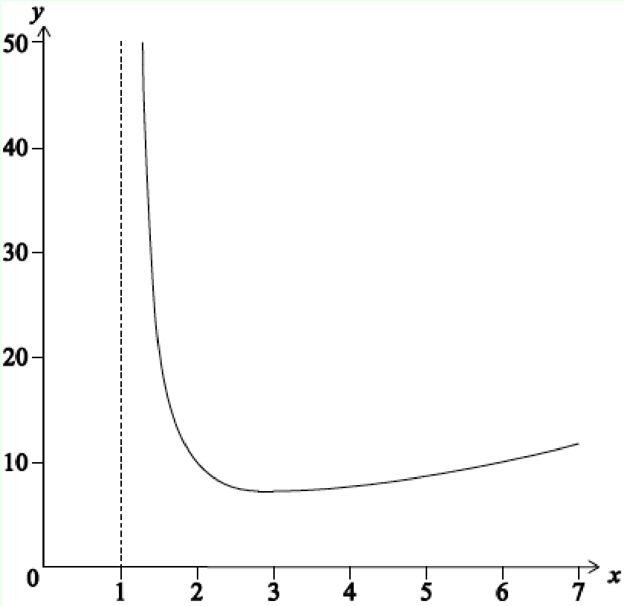
(ii) attempt to use the point $\left(\mathrm{e},\mathrm{e}^2\right)$ to determine c: ***M1***

eg, $(\ln\mathrm{e})^2\mathrm{e}^2=\mathrm{e}^2-\mathrm{e}+\mathrm{c}$ or $\mathrm{e}^2=\frac{\mathrm{e}^2-\mathrm{e}+\mathrm{c}}{(\ln\mathrm{e})^2}$ or $\mathrm{e}^2=\mathrm{e}^2-\mathrm{e}+\mathrm{c}$

$\mathrm{c}=\mathrm{e}$ ***A1***

$y=\frac{x^2-x+\mathrm{e}}{(\ln x)^2}$ ***AG***

(iii)



graph with correct shape ***A1***

minimum at $x=3.1$ (accept answers to a minimum of 2 s.f) ***A1***

asymptote shown at $x=1$ ***A1***

Note: y-coordinate of minimum not required for ***A1***;

Equation of asymptote not required for ***A1*** if VA appears on the sketch.

Award ***A0*** for asymptotes if more than one asymptote are shown

[12 marks]

Examiners report

[N/A]

3. *[12 marks]*

Markscheme

(a)

$b(n) = 3n + 1$ ***A1***

$c(n) = 3n + 2$ ***A1***

Note: $b(n)$ and $c(n)$ may be reversed.

[2 marks]

(b) consider the ratio of the $(n+1)^{\text{th}}$ and n^{th} terms: ***M1***

$\frac{3n+1}{3n+4} \times \frac{3n+2}{3n+5} \times \frac{x^{n+1}}{x^n}$ ***A1***

$\lim_{n \to \infty} \frac{3n+1}{3n+4} \times \frac{3n+2}{3n+5} \times \frac{x^{n+1}}{x^n} x$ ***A1***

radius of convergence: $R = 1$ ***A1***

[4 marks]

(c) any attempt to study the series for $x = -1$ or $x = 1$ ***(M1)***

converges for $x = 1$ by comparing with *p*-series $\sum \frac{1}{n^2}$ ***R1***

attempt to use the alternating series test for $x = -1$ ***(M1)***

Note: At least one of the conditions below needs to be attempted for ***M1*.**

$|\text{terms}| \approx \frac{1}{9n^2} \to 0$ and terms decrease monotonically in absolute value ***A1***

series converges for $x = -1$ ***R1***

interval of convergence: $[-1,\ 1]$ ***A1***

Note: Award the ***R1***s only if an attempt to corresponding correct test is made;
award the final ***A1*** only if at least one of the ***R1***s is awarded;
Accept study of absolute convergence at end points.

[6 marks]

Examiners report

[N/A]

4a. *[8 marks]*

Markscheme

$\lim_{x \to 1^-} \mathrm{e}^{-x^2}\left(-x^3+2x^2+x\right) = \lim_{x \to 1^+}(ax+b)$ ***M1***

$(= a + b)$ ***A1***

$2\mathrm{e}^{-1} = a + b$

differentiability: attempt to differentiate **both** expressions ***M1***

$f'(x) = -2x\mathrm{e}^{-x^2}\left(-x^3+2x^2+x\right) + \mathrm{e}^{-x^2}\left(-3x^2+4x+1\right)$ ***A1***

$(x < 1)$

(or $f'(x) = \mathrm{e}^{-x^2}\left(2x^4-4x^3-5x^2+4x+1\right)$)

$f'(x) = a$ ***A1***

$(x > 1)$

substitute $x = 1$ in **both** expressions and equate

$-2\mathrm{e}^{-1} = a$ ***A1***

substitute value of a and find

$b = 4\mathrm{e}^{-1}$ ***M1A1***

[8 marks]

Examiners report

[N/A]

4b.

[7 marks]

Markscheme

(i)

$f'(x) = \mathrm{e}^{-x^2}\left(2x^4 - 4x^3 - 5x^2 + 4x + 1\right)$ (for $x \leqslant 1$) ***M1***

$f(1) = f(-1)$ ***M1***

Rolle's theorem statement ***(A1)***

by Rolle's Theorem, $f'(x)$ has a zero in $]-1,1[$ ***R1***

hence quartic equation has a root in $]-1,1[$ ***AG***

(ii) let $g(x) = 2x^4 - 4x^3 - 5x^2 + 4x + 1$

$g(-1) = g(1) < 0$ and $g(0) > 0$ ***M1***

as g is a polynomial function it is continuous in $[-1, 0]$ and $[0, 1]$ ***R1***

(or g is a polynomial function continuous in any interval of real numbers)

then the graph of g must cross the *x*-axis at least once in $]-1, 0[$ and at least once in $]0, 1[$ ***R1***

[7 marks]

Examiners report

[N/A]

5.

[2 marks]

Markscheme

EITHER

$\sum_{n=1}^{\infty} \frac{2}{n^2+3n} < \sum_{n=1}^{\infty} \frac{2}{n^2}$ ***M1***

which is convergent ***A1***

the given series is therefore convergent using the comparison test ***AG***

OR

$\lim_{n\to\infty} \frac{\frac{2}{n^2+3n}}{\frac{1}{n^2}} = 2$ ***M1A1***

the given series is therefore convergent using the limit comparison test ***AG***

[2 marks]

Examiners report

Most candidates were able to answer part (a) and many gained a fully correct answer. A number of candidates ignored the factor 2 in the numerator and this led to candidates being penalised. In some cases candidates were not able to identify an appropriate series to compare with. Most candidates used the Comparison test rather than the Limit comparison test.

6.

[9 marks]

Markscheme

(a)

$a_n = \frac{e^n + 2^n}{2e^n} = \frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^n > \frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^{n+1} = a_{n+1}$ ***M1A1***

the sequence is decreasing (as terms are positive) ***A1***

Note: Accept reference to the sum of a constant and a decreasing geometric sequence.

Note: Accept use of derivative of $f(x) = \frac{e^x + 2^x}{2e^x}$ (and condone use of n) and graphical methods (graph of the sequence or graph of corresponding function f or graph of its derivative f'). Accept a list of consecutive terms of the sequence clearly decreasing (*eg* $0.8678\ldots,\ 0.77067\ldots,\ \ldots$).

[3 marks]

(b)

$L = \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^n = \frac{1}{2} + \frac{1}{2} \times 0 = \frac{1}{2}$ ***M1A1***

[2 marks]

(c)

EITHER

$\left|a_n - \frac{1}{2}\right| = \left|\frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^n - \frac{1}{2}\right| = \left|\frac{1}{2}\left(\frac{2}{e}\right)^n\right| < \frac{1}{1000}$ ***M1***

$\Rightarrow \left(\frac{e}{2}\right)^n > 500$ ***(A1)***

$\Rightarrow n > 20.25\ldots$ ***(A1)***

OR

$\Rightarrow \left(\frac{2}{e}\right)^n < 500$

$\Rightarrow n > 20.25\ldots$ ***(A1)(A1)***

Note: *A1* for correct inequality; *A1* for correct value.

THEN

therefore

$N = 21$ ***A1***

[4 marks]

Examiners report

Most candidates were successful in answering part (a) using a variety of methods. The majority of candidates scored some marks, if not full marks. Surprisingly, some candidates did not have the correct graph for the function the sequence represents. They obviously did not enter it correctly into their GDCs. Others used one of the two definitions for showing that a sequence is increasing/decreasing, but made mistakes with the algebraic manipulation of the expression, thereby arriving at an incorrect answer. Part (b) was less well answered with many candidates ignoring the command terms ‘show that’ and ‘find’ and just writing down the value of the limit. Some candidates attempted to use convergence tests for series with this sequence. Part (c) of this question was found challenging by the majority of candidates due to difficulties in solving inequalities involving absolute value.

7. *[19 marks]*

Markscheme

(a) let

$f(x,\ y) = \frac{y}{x + \sqrt{xy}}$ ***(M2)(A1)***

$y(1.2) = y(1) + 0.2f(1,\ 2)\ (= 2 + 0.1656\ldots)$ ***A1***

$= 2.1656\ldots$ ***(M1)***

$y(1.4) = 2.1656\ldots + 0.2f(1.2,\ 2.1256\ldots)\ (= 2.1656\ldots + 0.1540\ldots)$

Note: ***M1*** is for attempt to apply formula using point

$(1.2,\ y(1.2))$.

A1

$= 2.3197\ldots$

$y(1.6) = 2.3197\ldots + 0.2f(1.4,\ 2.3197\ldots)\ (= 2.3297\ldots + 0.1448\ldots)$

$= 2.46$ (3sf) ***A1 N3***

[7 marks]

(b)

$y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x + \sqrt{xy}} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx}{x + \sqrt{vx^2}}$ ***M1***

$\Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx}{x + x\sqrt{v}}$ (as $x > 0$) ***A1***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{1 + \sqrt{v}} - v$ ***AG***

[3 marks]

(c) (i)

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{1 + \sqrt{v}} - v$

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-v\sqrt{v}}{1 + \sqrt{v}} \Rightarrow \frac{1 + \sqrt{v}}{-v\sqrt{v}}\mathrm{d}v = \frac{1}{x}\mathrm{d}x$ ***M1***

$\int \frac{1 + \sqrt{v}}{-v\sqrt{v}}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***(M1)***

$\frac{2}{\sqrt{v}} - \ln v = \ln x + C$ ***A1A1***

Note: Do not penalize absence of $+C$ at this stage; ignore use of absolute values on \(v\)and \(x\) (which are positive anyway).

$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} = \ln x + C$ as $y = vx \Rightarrow v = \frac{y}{x}$ ***M1***

when $y = 2$, $x = 1 \Rightarrow \sqrt{2} - \ln 2 = 0 + C$ ***M1***

$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} = \ln x + \sqrt{2} - \ln 2$

$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} - \ln x - \sqrt{2} + \ln 2 = 0$ ***A1***

$\left(2\sqrt{\frac{x}{y}} - \ln y - \sqrt{2} + \ln 2 = 0\right)$

(ii)

$2\sqrt{\frac{1.6}{y}} - \ln\frac{y}{1.6} - \ln 1.6 - \sqrt{2} + \ln 2 = 0$ ***(M1)***

$y = 2.45$ ***A1***

[9 marks]

Examiners report

Part (a) was well answered by most candidates. In a few cases calculation errors and early rounding errors prevented candidates from achieving full marks, but most candidates scored at least a few marks here. In part (b) some candidates failed to convincingly show the given result. Part (c) proved to be a hard question for many candidates and a significant number of candidates had difficulty manipulating the algebraic expression, and either had the incorrect expression to integrate, or incorrectly integrated the correct expression. Many candidates reached as far as separating the variables correctly but integrating proved to be too difficult for many of them although most realised that the expression on v had to be split into two separate integrals. Most candidates made good attempts to evaluate the arbitrary constant and arrived at a correct or almost correct expression (sign errors were a common error) which allowed follow through for part b (ii). In some cases however the expression obtained was too simple or was omitted and it was not possible to grant follow through marks.

8a. *[4 marks]*

Markscheme

integrating factor $e^{\int -\frac{1}{t}\mathrm{d}t} = e^{-\ln t} \left(= \frac{1}{t}\right)$ ***M1A1***

$\frac{x}{t} = \int -\frac{2}{t^2}\mathrm{d}t = \frac{2}{t} + c$ ***A1A1***

Note: Award ***A1*** for $\frac{x}{t}$ and ***A1*** for $\frac{2}{t} + c$.

$x = 2 + ct$ ***AG***

[4 marks]

Examiners report

This was generally well done. Some candidates did not realize $e^{-\ln t}$ could be simplified to $\frac{1}{t}$.

8b. *[2 marks]*

Markscheme

given continuity at $x = 5$

$5c + 2 = 16 - \frac{35}{5} \Rightarrow c = \frac{7}{5}$ ***M1A1***

[2 marks]

Examiners report

This part was well done by the majority of candidates.

8c. *[2 marks]*

Markscheme

(i) 2 ***A1***

(ii) any value ≥ 16 ***A1***

Note: Accept values less than 16 if fully justified by reference to the maximum age for a dog.

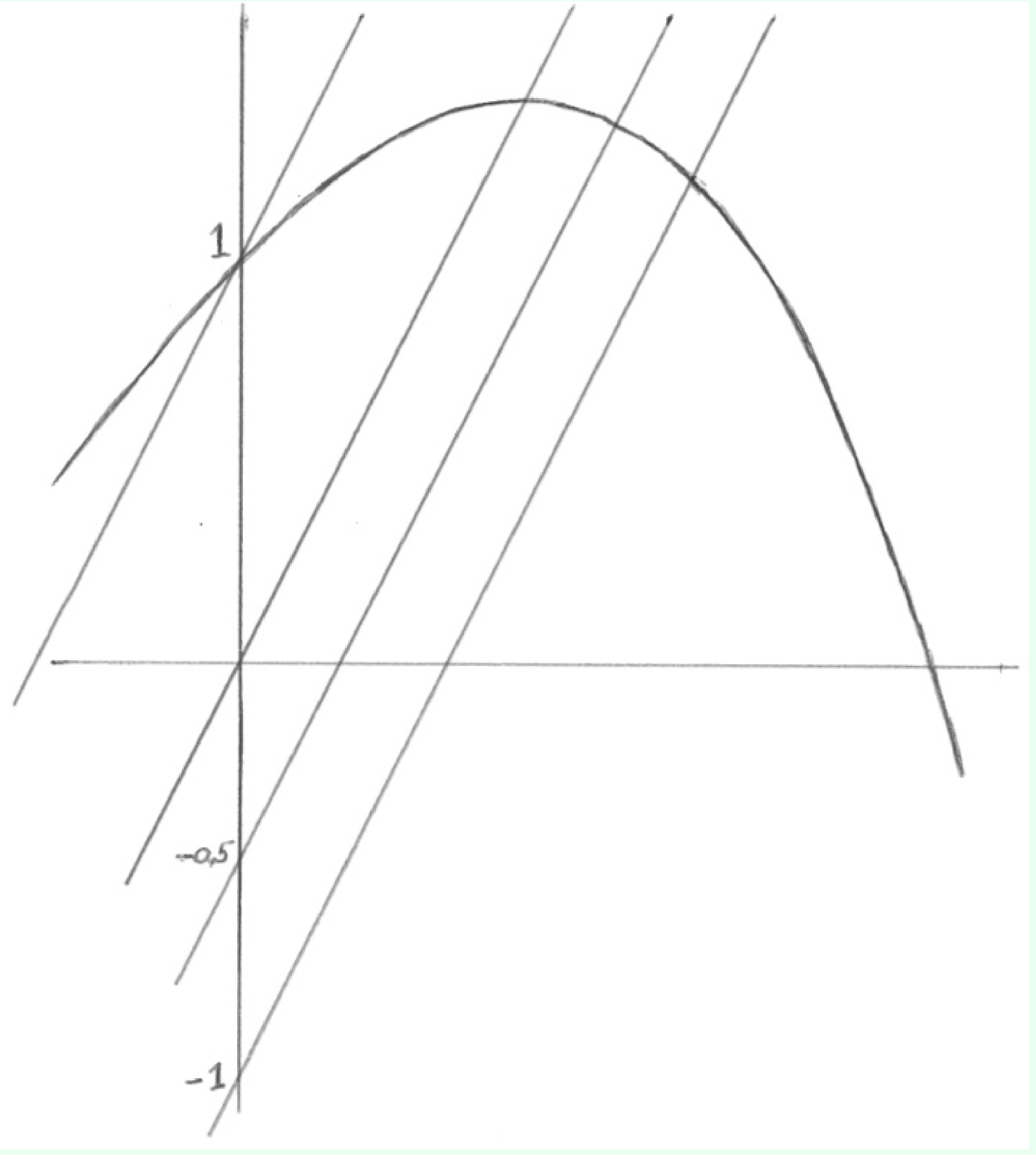
[2 marks]

Examiners report

This part was well done by the majority of candidates.

9a. *[2 marks]*

Markscheme



A1 for 4 parallel straight lines with a positive gradient ***A1***

A1 for correct y intercepts ***A1***

[2 marks]

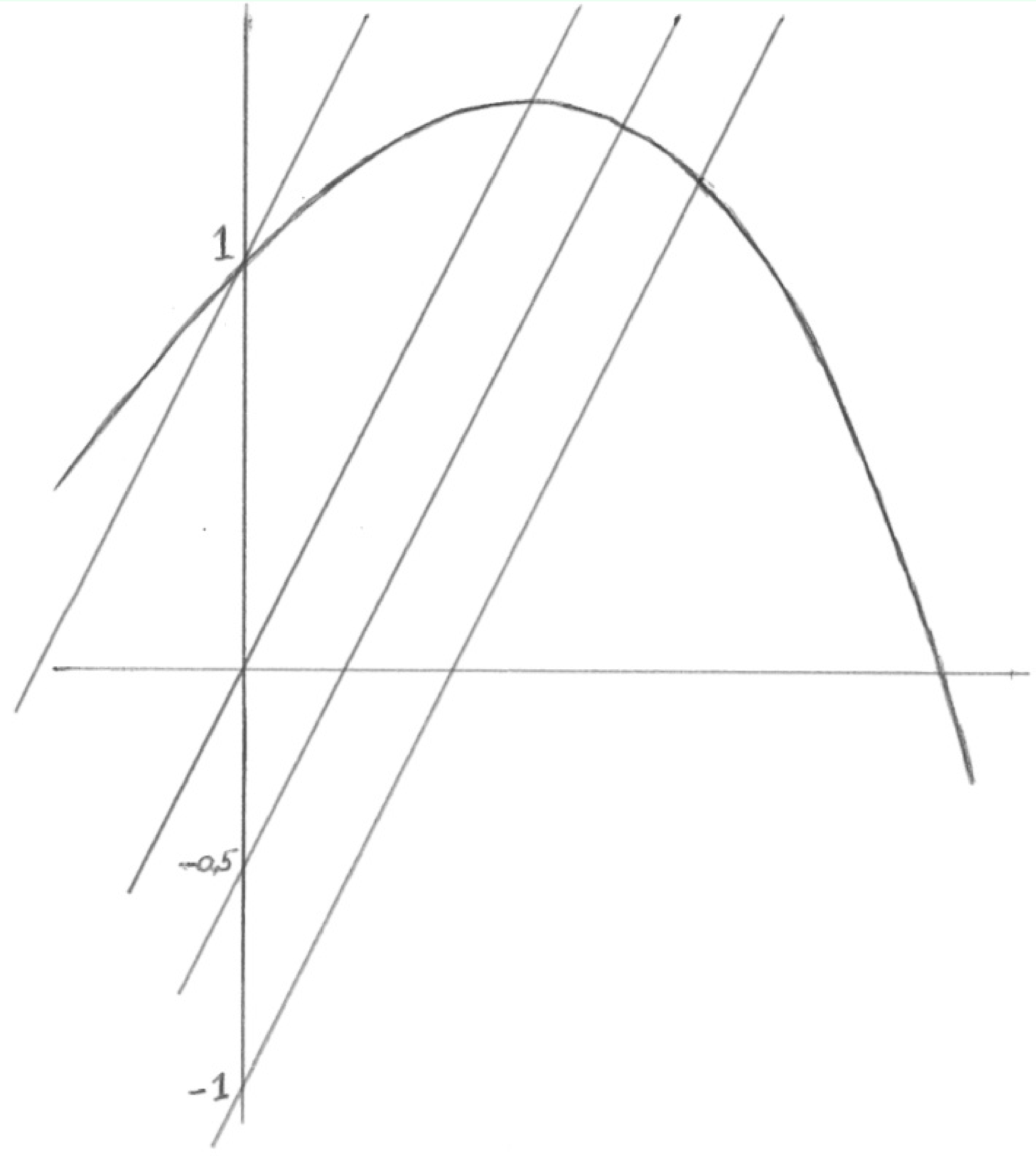
Examiners report

Some candidates ignored the instruction to prove from first principles and instead used standard differentiation. Some candidates also only found a derivative from one side.

9b.

[3 marks]

Markscheme



A1 for passing through $(0, 1)$ with positive gradient less than 2

A1 for stationary point on $y = 2x$

A1 for negative gradient on both of the other 2 isoclines ***A1A1A1***

[3 marks]

Examiners report

Parts (b) and (c) were attempted by very few candidates. Few recognized that the gradient of the curve had to equal the value of k on the isocline.

9c.

[1 mark]

Markscheme

The isocline is perpendicular to C ***R1***

[1 mark]

Examiners report

Parts (b) and (c) were attempted by very few candidates. Few recognized that the gradient of the curve had to equal the value of k on the isocline.

9d. *[4 marks]*

Markscheme

$y_{n+1} = y_n + 0.1(y_n - 2x_n) \quad (= 1.1y_n - 0.2x_n)$ ***(M1)(A1)***

Note: Also award ***M1A1*** if no formula seen but y_2 is correct.

$y_0 = 1,\ y_1 = 1.1,\ y_2 = 1.19,\ y_3 = 1.269,\ y_4 = 1.3359$ ***(M1)***

$y_5 = 1.39 \text{ to } 3\text{sf}$ ***A1***

Note: ***M1*** is for repeated use of their formula, with steps of 0.1.

Note: Accept 1.39 or 1.4 only.

[4 marks]

Total [10 marks]

Examiners report

Those candidates who knew the method managed to score well on this part. On most calculators a short program can be written in the exam to make Euler's method very quick. Quite a few candidates were losing time by calculating and writing out many intermediate values, rather than just the x and y values.

10a. *[1 mark]*

Markscheme

$r = -x^2, \quad S = \frac{1}{1+x^2}$ ***A1AG***

[1 mark]

Examiners report

Most candidates picked up this mark for realizing the common ratio was $-x^2$.

10b. *[4 marks]*

Markscheme

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

EITHER

$\int \frac{1}{1+x^2}\mathrm{d}x = \int 1 - x^2 + x^4 - x^6 + \ldots \mathrm{d}x$ ***M1***

$\arctan x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***A1***

Note: Do not penalize the absence of c at this stage.

when $x = 0$ we have $\arctan 0 = 0$ hence $c = 0$ ***M1A1***

OR

$\int_0^x \frac{1}{1+t^2}\mathrm{d}t = \int_0^x 1 - t^2 + t^4 - t^6 + \ldots \mathrm{d}t$ ***M1A1A1***

Note: Allow x as the variable as well as the limit.

M1 for knowing to integrate, ***A1*** for each of the limits.

$[\arctan t]_0^x = \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right]_0^x$ ***A1***

hence $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***AG***

[4 marks]

Examiners report

Quite a few candidates did not recognize the importance of ‘hence’ in this question, losing a lot of time by trying to work out the terms from first principles.

Of those who integrated the formula from part (a) only a handful remembered to include the ‘$+c$’ term, and to verify that this must be equal to zero.

10c. *[4 marks]*

Markscheme

applying the MVT to the function f on the interval $[x,\ y]$ ***M1***

$\frac{f(y)-f(x)}{y-x} = f'(c) \quad (\text{for some } c \in]x,\ y[)$ ***A1***

$\frac{f(y)-f(x)}{y-x} > 0 \quad (\text{as } f'(c) > 0)$ ***R1***

$f(y) - f(x) > 0 \text{ as } y > x$ ***R1***

$\Rightarrow f(y) > f(x)$ ***AG***

Note: If they use x rather than c they should be awarded ***M1A0R0***, but could get the next ***R1***.

[4 marks]

Examiners report

Most candidates were able to achieve some marks on this question. The most commonly lost mark was through not stating that the inequality was unchanged when multiplying by $y - x \text{ as } y > x$.

10d.

[4 marks]

Markscheme

(i) $g(x) = x - \arctan x \Rightarrow g'(x) = 1 - \frac{1}{1+x^2}$ ***A1***

this is greater than zero because $\frac{1}{1+x^2} < 1$ ***R1***

so $g'(x) > 0$ ***AG***

(ii) (g is a continuous function defined on $[0,\ b]$ and differentiable on $]0,\ b[$ with $g'(x) > 0$ on $]0,\ b[$ for all $b \in \mathbb{R}$)

(If $x \in [0,\ b]$ then) from part (c) $g(x) > g(0)$ ***M1***

$x - \arctan x > 0 \Rightarrow \arctan x < x$ ***M1***

(as b can take any positive value it is true for all $x > 0$) ***AG***

[4 marks]

Examiners report

The first part of this question proved to be very straightforward for the majority of candidates.

In (ii) very few realized that they had to replace the lower variable in the formula from part (c) by zero.

10e.

[5 marks]

Markscheme

let $h(x) = \arctan x - \left(x - \frac{x^3}{3}\right)$ ***M1***

(h is a continuous function defined on $[0,\ b]$ and differentiable on $]0,\ b[$ with $h'(x) > 0$ on $]0,\ b[$)

$h'(x) = \frac{1}{1+x^2} - (1 - x^2)$ ***A1***

$= \frac{1-(1-x^2)(1+x^2)}{1+x^2} = \frac{x^4}{1+x^2}$ ***M1A1***

hence $h'(x) > 0$ (for $x \in [0,\ b]$) $h(x) > h(0)(= 0)$ ***R1***

$\Rightarrow \arctan x > x - \frac{x^3}{3}$ ***AG***

Note: Allow correct working with $h(x) = x - \frac{x^3}{3} - \arctan x$.

[5 marks]

Examiners report

Candidates found this part difficult, failing to spot which function was required.

10f.

[4 marks]

Markscheme

use of $x - \frac{x^3}{3} < \arctan x < x$ ***M1***

choice of $x = \frac{1}{\sqrt{3}}$ ***A1***

$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***M1***

$\frac{8}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***A1***

Note: Award final ***A1*** for a correct inequality with a single fraction on each side that leads to the final answer.

$\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$ ***AG***

[4 marks]

Total [22 marks]

Examiners report

Many candidates, even those who did not successfully complete (d) (ii) or (e), realized that these parts gave them the necessary inequality.

11.

[7 marks]

Markscheme

$f(0) = 0$ ***A1***

$f'(x) = -\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x + 1$ ***M1A1***

$f'(0) = 0$ ***(M1)***

$f''(x) = 2\mathrm{e}^{-x}\sin x$ ***A1***

$f''(0) = 0$

$f^{(3)}(x) = -2\mathrm{e}^{-x}\sin x + 2\mathrm{e}^{-x}\cos x$ ***A1***

$f^{(3)}(0) = 2$

the first non-zero term is $\frac{2x^3}{3!}\ \left(= \frac{x^3}{3}\right)$ ***A1***

Note: Award no marks for using known series.

[7 marks]

Examiners report

Most students had a good understanding of the techniques involved with this question. A surprising number forgot to show $f(0) = 0$. Some candidates did not simplify the second derivative which created extra work and increased the chance of errors being made.

12a. *[3 marks]*

Markscheme

METHOD 1

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}\int f(x)\mathrm{d}x + \frac{1}{x}f(x)$ ***M1M1A1***

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x),\ x > 0$ ***AG***

Note: ***M1*** for use of product rule, ***M1*** for use of the fundamental theorem of calculus, ***A1*** for all correct.

METHOD 2

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x)$

$\frac{\mathrm{d}(xy)}{\mathrm{d}x} = f(x)$ ***(M1)***

$xy = \int f(x)\mathrm{d}x$ ***M1A1***

$y = \frac{1}{x}\int f(x)\mathrm{d}x$ ***AG***

[3 marks]

Examiners report

This question allowed for several different approaches. The most common of these was the use of the integrating factor (even though that just took you in a circle). Other candidates substituted the solution into the differential equation and others multiplied the solution by x and then used the product rule to obtain the differential equation. All these were acceptable.

12b. *[5 marks]*

Markscheme

$y = \frac{1}{x}\left(2x^{\frac{1}{2}} + c\right)$ ***A1A1***

Note: ***A1*** for correct expression apart from the constant, ***A1*** for including the constant in the correct position.

attempt to use the boundary condition ***M1***

$c = 4$ ***A1***

$y = \frac{1}{x}\left(2x^{\frac{1}{2}} + 4\right)$ ***A1***

Note: Condone use of integrating factor.

[5 marks]

Total [8 marks]

Examiners report

This was a straightforward question. Some candidates failed to use the hint of ‘hence’, and worked from the beginning using the integrating factor. A surprising number made basic algebra errors such as putting the $+c$ term in the wrong place and so not dividing it by x.

13a. *[3 marks]*

Markscheme

METHOD 1

$(0 <)\frac{1}{n^2 \ln(n)} < \frac{1}{n^2}$, (for $n \geq 3$) ***A1***

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the comparison test ($\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges implies) $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$ converges ***R1***

Note: Mention of using the comparison test may have come earlier.

Only award ***R1*** if previous 2 ***A1***s have been awarded.

METHOD 2

$\lim_{n \to \infty}\left(\frac{\frac{1}{n^2 \ln n}}{\frac{1}{n^2}}\right) = \lim_{n \to \infty} \frac{1}{\ln n} = 0$ ***A1***

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the limit comparison test (if the limit is 0 and the series represented by the denominator converges, then so does the series represented by the numerator, hence) $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$ converges ***R1***

Note: Mention of using the limit comparison test may come earlier.

Do not award the ***R1*** if incorrect justifications are given, for example the series “converge or diverge together”.

Only award ***R1*** if previous 2 ***A1***s have been awarded.

[3 marks]

Examiners report

In this part the required test was not given in the question. This led to some students attempting inappropriate methods. When using the comparison or limit comparision test many candidates wrote the incorrect statement $\frac{1}{n^2}$ converges, (*p*-series) rather than the correct one with $\sum$. This perhaps indicates a lack of understanding of the concepts involved.

13b. *[8 marks]*

Markscheme

(i) consider $f(x) = \frac{1}{x \ln x}$ $(\text{for } x > 1)$ ***A1***

$f(x)$ is continuous and positive ***A1***

and is (monotonically) decreasing ***A1***

Note: If a candidate uses n rather than x, award as follows

$\frac{1}{n \ln n}$ is positive and decreasing ***A1A1***

$\frac{1}{n \ln n}$ is continuous for $n \in \mathbb{R},\ n > 1$ ***A1*** (only award this mark if the domain has been explicitly changed).

(ii) consider $\int_2^R \frac{1}{x \ln x}\mathrm{d}x$ ***M1***

$= [\ln(\ln x)]_2^R$ ***(M1)A1***

$\to \infty \text{ as } R \to \infty$ ***R1***

hence series diverges ***A1***

Note: Condone the use of ∞ in place of R.

Note: If the lower limit is not equal to 2, but the expression is integrated correctly award ***M0M1A1R0A0***.

[8 marks]

Total [17 marks]

Examiners report

(i) Candidates need to be aware of the necessary conditions for all the series tests.

(ii) The integration was well done by the candidates. Most also made the correct link between the integral being undefined and the series diverging. In this question it was not necessary to initially take a finite upper limit and the use of ∞ was acceptable. This was due to the command term being 'determine'. In q4b a finite upper limit was required, as the command term was 'show'. To ensure full marks are always awarded candidates should err on the side of caution and always use limit notation when working out indefinite integrals.

14a. *[7 marks]*

Markscheme

(i) $f'(x) = 3x^2 + 6x$ ***A1***

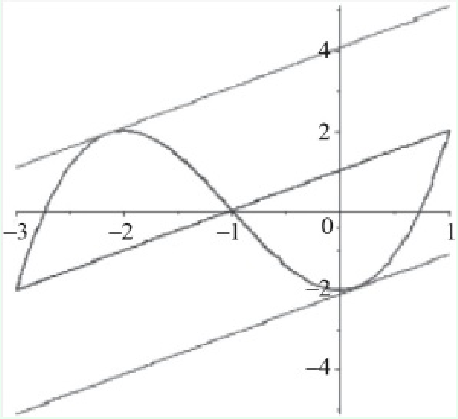
gradient of chord $= 1$ ***A1***

$3c^2 + 6c = 1$

$c = \frac{-3 \pm 2\sqrt{3}}{3}$ $(= -2.15,\ 0.155)$ ***A1A1***

Note: Accept any answers that round to the correct 2sf answers $(-2.2,\ 0.15)$.

(ii)



award ***A1*** for correct shape and clear indication of correct domain, ***A1*** for chord (from $x = -3$ to $x = 1$) and ***A1*** for two tangents drawn at their values of c ***A1A1A1***

[7 marks]

Examiners report

(i) This was well done by most candidates.

(ii) This was generally poorly done, with many candidates failing to draw the curve correctly as they did not appreciate the importance of the given domain. Another common error was to draw the graph of the derivative rather than the function.

14b. *[9 marks]*

Markscheme

(i) **METHOD 1**

(if a theorem is true for the interval $[a,\ b]$, it is also true for any interval $[x_1,\ x_2]$ which belongs to $[a,\ b]$)

suppose $x_1,\ x_2 \in [a,\ b]$ ***M1***

by the MVT, there exists c such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$ ***M1A1***

hence $f(x_1) = f(x_2)$ ***R1***

as $x_1,\ x_2$ are arbitrarily chosen, $f(x)$ is constant on $[a,\ b]$

Note: If the above is expressed in terms of a and b award ***M0M1A0R0***.

METHOD 2

(if a theorem is true for the interval $[a,\ b]$, it is also true for any interval $[x_1,\ x_2]$ which belongs to $[a,\ b]$)

suppose $x \in [a,\ b]$ ***M1***

by the MVT, there exists c such that $f'(c) = \frac{f(x)-f(a)}{x-a} = 0$ ***M1A1***

hence $f(x) = f(a) =$ constant ***R1***

(ii) attempt to differentiate $(x) = 2\arccos x + \arccos(1-2x^2)$ ***M1***

$-2\frac{1}{\sqrt{1-x^2}} - \frac{-4x}{\sqrt{1-(1-2x^2)^2}}$ ***A1A1***

$= -2\frac{1}{\sqrt{1-x^2}} + \frac{4x}{\sqrt{4x^2-4x^4}} = 0$ ***A1***

Note: Only award ***A1*** for 0 if a correct attempt to simplify the denominator is also seen.

$f(x) = f(0) = 2 \times \frac{\pi}{2} + 0 = \pi$ ***A1AG***

Note: This ***A1*** is not dependent on previous marks.

Note: Allow any value of $x \in [0,\ 1]$.

[9 marks]

Total [16 marks]

Examiners report

(i) This was very poorly done. A lot of the arguments seemed to be stating what was being required to be proved, eg ‘because the derivative is equal to 0 the line is flat’. Most candidates did not realise the importance of testing a point inside the interval, so the most common solutions seen involved the Mean Value Theorem applied to the end points. In addition there was some confusion between the Mean Value Theorem and Rolle’s Theorem.

(ii) It was pleasing that so many candidates spotted the link with the previous part of the question. The most common error after this point was to differentiate incorrectly. Candidates should be aware this is a ‘prove’ question, and so it was not sufficient simply to state, for example, $f(0) = \pi$.

15a. *[3 marks]*

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} \Rightarrow \int \frac{1}{y}\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x$ ***M1A1***

$\Rightarrow \ln y = \ln x + c$

$\Rightarrow \ln y = \ln x + \ln k = \ln kx$ ***A1***

$\Rightarrow y = kx$

[3 marks]

Examiners report

This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from to

$\ln y = \ln x + c$

$y = x + c$

15b. *[4 marks]*

Markscheme

$y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

so

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v$ ***M1***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 0 \ \ (\text{as } x \neq 0)$ ***R1***

$\Rightarrow v = k$

$\Rightarrow \frac{y}{x} = k \ \ (\Rightarrow y = kx)$ ***A1***

[4 marks]

Examiners report

This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from to

$\ln y = \ln x + c$

$y = x + c$

15c. *[5 marks]*

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{-1}{x}\right)y = 0$ ***(M1)***

$\mathrm{IF} = \mathrm{e}^{\int \frac{-1}{x}\mathrm{d}x} = \mathrm{e}^{-\ln x} = \frac{1}{x}$ ***M1A1***

$x^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} - x^{-2}y = 0$

$\Rightarrow \frac{\mathrm{d}[x^{-1}y]}{\mathrm{d}x} = 0$ ***(M1)***

$\Rightarrow x^{-1}y = k \ \ (\Rightarrow y = kx)$ ***A1***

[5 marks]

Examiners report

This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from to

$\ln y = \ln x + c$

$y = x + c$

15d. *[1 mark]*

Markscheme

$20 = 2k \Rightarrow k = 10$ so $y(5) = 10 \times 5 = 50$ ***A1***

[1 mark]

Examiners report

This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from to

$\ln y = \ln x + c$

$y = x + c$

16a. *[6 marks]*

Markscheme

using

$x_0 = 1,\ y_0 = 1$

$x_n = 1 + 0.1n,\ y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$ ***(M1)(M1)(A1)***

Note: If they have not written down formulae but have $x_1 = 1.1$ and $y_1 = 1.14142\dots$ award ***M1M1A1***.

gives by GDC

$x_{10} = 2,\ y_{10} = 2.770114792\dots$ ***(M1)(A1)***

so

$a \simeq 2.7701$ (4dp) ***A1 N6***

Note: Do not penalize over-accuracy.

[6 marks]

Examiners report

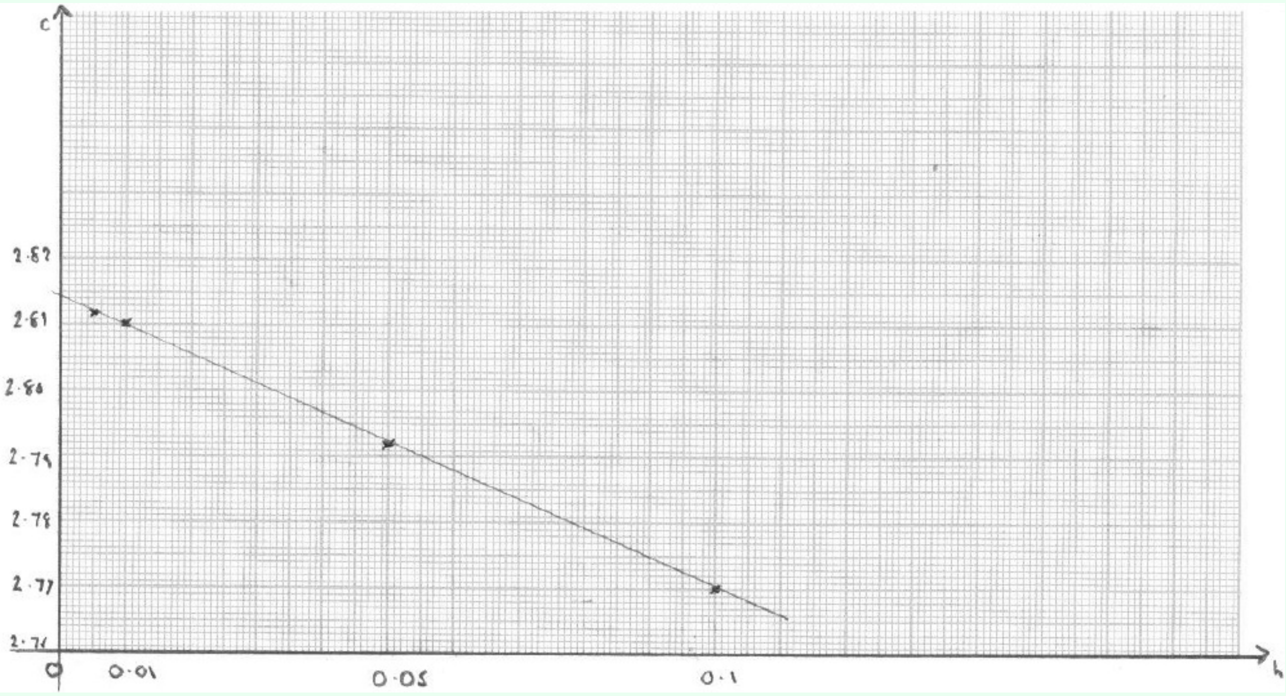
Part (a) was done well. We would recommend that candidates write down the equation they are using, in this case, $y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$, to ensure they get all the method marks. Beyond this the answer is all that is needed (or if a student wishes to show working, simply each of the values of x_n and y_n). Many candidates wasted a lot of time by writing out values of each part of the function, perhaps indicating they did not how to do it more quickly using their calculators.

Part (b) Surprisingly when drawing the graph a lot of candidates had (0.01, 2.8099) closer to 2.80 than 2.81

Most realised that the best possible estimate was given by the *y*-intercept of the line they had drawn.

16b. *[3 marks]*

Markscheme



points drawn on graph above ***A1A1A1***

Note: Award ***A1*** for scales, ***A1*** for 2 points correctly plotted, ***A1*** for other 2 points correctly plotted (second and third ***A1*** dependent on the first being correct).

[3 marks]

Examiners report

Part (a) was done well. We would recommend that candidates write down the equation they are using, in this case, $y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$, to ensure they get all the method marks. Beyond this the answer is all that is needed (or if a student wishes to show working, simply each of the values of x_n and y_n). Many candidates wasted a lot of time by writing out values of each part of the function, perhaps indicating they did not how to do it more quickly using their calculators.

Part (b) Surprisingly when drawing the graph a lot of candidates had (0.01, 2.8099) closer to 2.80 than 2.81

Most realised that the best possible estimate was given by the *y*-intercept of the line they had drawn.

16c. *[1 mark]*

Markscheme

suitable line of best fit placed on graph ***A1***

[1 mark]

Examiners report

Part (a) was done well. We would recommend that candidates write down the equation they are using, in this case, $y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$, to ensure they get all the method marks. Beyond this the answer is all that is needed (or if a student wishes to show working, simply each of the values of x_n and y_n). Many candidates wasted a lot of time by writing out values of each part of the function, perhaps indicating they did not how to do it more quickly using their calculators.

Part (b) Surprisingly when drawing the graph a lot of candidates had (0.01, 2.8099) closer to 2.80 than 2.81

Most realised that the best possible estimate was given by the *y*-intercept of the line they had drawn.

16d. *[2 marks]*

Markscheme

letting $h \to 0$ we approach the *y* intercept on the graph so ***(R1)***

$c \simeq 2.814$ (3dp) ***A1***

Note: Accept 2.815.

[2 marks]

Examiners report

Part (a) was done well. We would recommend that candidates write down the equation they are using, in this case, $y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$, to ensure they get all the method marks. Beyond this the answer is all that is needed (or if a student wishes to show working, simply each of the values of x_n and y_n). Many candidates wasted a lot of time by writing out values of each part of the function, perhaps indicating they did not how to do it more quickly using their calculators.

Part (b) Surprisingly when drawing the graph a lot of candidates had (0.01, 2.8099) closer to 2.80 than 2.81

Most realised that the best possible estimate was given by the *y*-intercept of the line they had drawn.

17a. *[3 marks]*

Markscheme

$\lim\limits_{H\to\infty}\int_a^H \frac{1}{x^2}\mathrm{d}x = \lim\limits_{H\to\infty}\left[\frac{-1}{x}\right]_a^H$ ***A1***

$\lim\limits_{H\to\infty}\left(\frac{-1}{H} + \frac{1}{a}\right)$ ***A1***

$= \frac{1}{a}$ ***A1***

[3 marks]

Examiners report

Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

17b. *[3 marks]*

Markscheme

as $\left\{ \frac{1}{n^2} \right\}$ is a positive decreasing sequence we consider the function $\frac{1}{x^2}$

we look at

$\int_1^\infty \frac{1}{x^2} \mathrm{d}x$ ***M1***

$\int_1^\infty \frac{1}{x^2} \mathrm{d}x = 1$ ***A1***

since this is finite (allow "limit exists" or equivalent statement) ***R1***

$\sum_{n=1}^\infty \frac{1}{n^2}$ converges ***AG***

[3 marks]

Examiners report

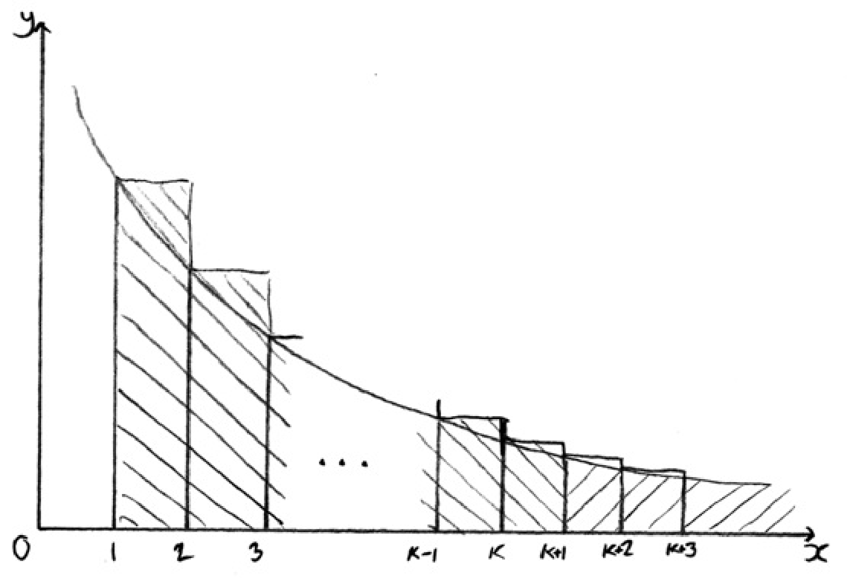
Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

In part (b) the calculation of the integral as equal to 1 only scored 2 of the 3 marks. The final mark was for stating that 'because the value of the integral is finite (or 'the limit exists' or an equivalent statement) then the series converges. Quite a few candidates left out this phrase.

17c. *[6 marks]*

Markscheme

(i)



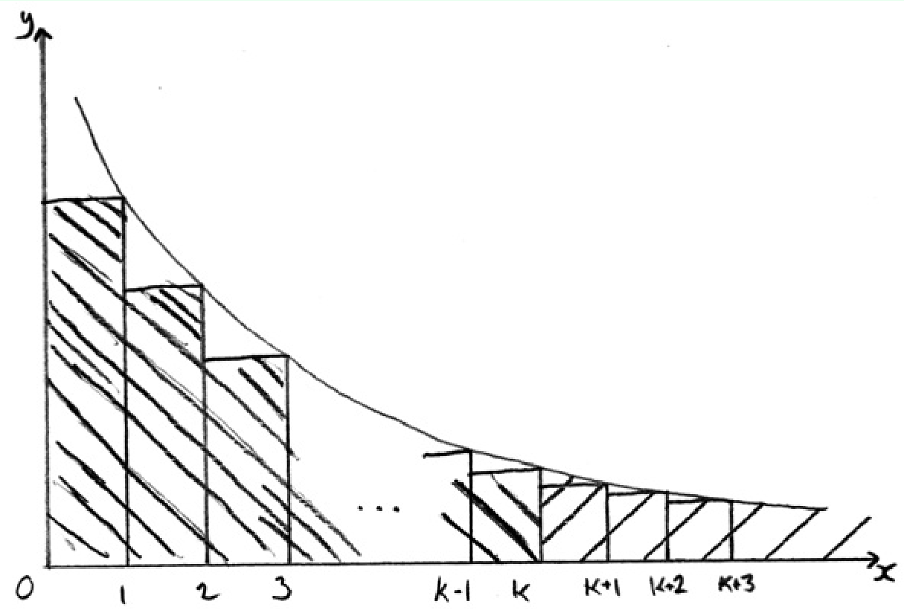
attempt to shade rectangles ***M1***

correct start and finish points for rectangles ***A1***

since the area shaded is less that the area of the required staircase we have ***R1***

$\sum_{n=1}^{k} \frac{1}{n^2} + \int_{k+1}^{\infty} \frac{1}{x^2} \mathrm{d}x < L$ ***AG***

(ii)



attempt to shade rectangles ***M1***

correct start and finish points for rectangles ***A1***

since the area shaded is greater that the area of the required staircase we have ***R1***

$L < \sum_{n=1}^{k} \frac{1}{n^2} + \int_{k}^{\infty} \frac{1}{x^2} \mathrm{d}x$ ***AG***

Note: Alternative shading and rearranging of the inequality is acceptable.

[6 marks]

Examiners report

Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

Candidates found part (c) difficult. Very few drew the correct series of rectangles and some clearly had no idea of what was expected of them.

17d. *[2 marks]*

Markscheme

$\int_{k+1}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{k+1}$, $\int_{k}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{k}$ ***A1A1***

$\sum\limits_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k+1} < L < \sum\limits_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k}$ ***AG***

[2 marks]

Examiners report

Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

17e. *[3 marks]*

Markscheme

$\frac{205}{144} + \frac{1}{6} < \frac{\pi^2}{6} < \frac{205}{144} + \frac{1}{4}\ \left(1.6236... < \frac{\pi^2}{6} < 1.6736...\right)$ ***(M1)A1***

$\sqrt{6\left(\frac{205}{144} + \frac{1}{6}\right)} < \pi < \sqrt{6\left(\frac{205}{144} + \frac{1}{4}\right)}$

$3.12 < \pi < 3.17$ ***A1 N2***

[3 marks]

Examiners report

Though part (e) could be done without doing any of the previous parts of the question many students were probably put off by the notation because only a minority attempted it.

18a. *[5 marks]*

Markscheme

apply the limit comparison test with $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ ***M1***

$\lim\limits_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{n^2}{n(n+1)} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$ ***M1A1***

(since the limit is finite and $\neq 0$) both series do the same ***R1***

we know that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges and hence $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ also converges ***R1AG***

[5 marks]

Examiners report

Candidates and teachers need to be aware that the Limit comparison test is distinct from the comparison test. Quite a number of candidates lost most of the marks for this part by doing the wrong test.

Some candidates failed to state that because the result was finite and not equal to zero then the two series converge or diverge together. Others forgot to state, with a reason, that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

18b. *[3 marks]*

Markscheme

A1

$$(1+x)\ln(1+x) = (1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\ldots\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\ldots\right) + \left(x^2 - \frac{x3}{2} + \frac{x^4}{3} - \frac{x^5}{4}\ldots\right)$$

EITHER

A1

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n}$$

M1

$$= x + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n+1} \left(\frac{-1}{n+1} + \frac{1}{n}\right)$$

OR

A1

$$x + \left(1 - \frac{1}{2}\right)x^2 - \left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{1}{3} - \frac{1}{4}\right)x^4 - \ldots$$

M1

$$= x + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n+1} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

AG

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)}$$

[3 marks]

Examiners report

Candidates and teachers need to be aware that the Limit comparison test is distinct from the comparison test. Quite a number of candidates lost most of the marks for this part by doing the wrong test.

Some candidates failed to state that because the result was finite and not equal to zero then the two series converge or diverge together. Others forgot to state, with a reason, that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In part (b) finding the partial fractions was well done. The second part involving the use of telescoping series was less well done, and students were clearly not as familiar with this technique as with some others.

Part (c) was the least well done of all the questions. It was expected that students would use explicitly the result from the first part of 4(b) or show it once again in order to give a complete answer to this question, rather than just assuming that a pattern spotted in the first few terms would continue.

Candidates need to be informed that unless specifically told otherwise they may use without proof any of the Maclaurin expansions given in the Information Booklet. There were many candidates who lost time and gained no marks by trying to derive the expansion for $\ln(1+x)$.

19a.

[6 marks]

Markscheme

let

$f(x) = \sqrt{x}, \ f(1) = 1$ ***(A1)***

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}, \ f'(1) = \frac{1}{2}$ ***(A1)***

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}, \ f''(1) = -\frac{1}{4}$ ***(A1)***

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}, \ f'''(1) = \frac{3}{8}$ ***(A1)***

$a_1 = \frac{1}{2} \cdot \frac{1}{1!}, \ a_2 = -\frac{1}{4} \cdot \frac{1}{2!}, \ a_3 = \frac{3}{8} \cdot \frac{1}{3!}$ ***(M1)***

$a_0 = 1, \ a_1 = \frac{1}{2}, \ a_2 = -\frac{1}{8}, \ a_3 = \frac{1}{16}$ ***A1***

Note: Accept

$y = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + \ldots$

[6 marks]

Examiners report

Many candidates achieved full marks on this question but there were still a large minority of candidates who did not seem familiar with the application of Taylor series. Whilst all candidates who responded to this question were aware of the need to use derivatives many did not correctly use factorials to find the required coefficients. It should be noted that the formula for Taylor series appears in the Information Booklet.

19b.

[3 marks]

Markscheme

METHOD 1

$\lim\limits_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim\limits_{x \to 1} \frac{\frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \ldots}{x-1}$ ***M1***

$= \lim\limits_{x \to 1} \left(\frac{1}{2} - \frac{1}{8}(x-1) + \ldots\right)$ ***A1***

$= \frac{1}{2}$ ***A1***

METHOD 2

using l’Hôpital’s rule, ***M1***

$\lim\limits_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim\limits_{x \to 1} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1}$ ***A1***

$= \frac{1}{2}$ ***A1***

METHOD 3

$\frac{\sqrt{x}-1}{x+1} = \frac{1}{\sqrt{x}+1}$ ***M1A1***

$\lim\limits_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$ ***A1***

[3 marks]

Examiners report

Many candidates achieved full marks on this question but there were still a large minority of candidates who did not seem familiar with the application of Taylor series. Whilst all candidates who responded to this question were aware of the need to use derivatives many did not correctly use factorials to find the required coefficients. It should be noted that the formula for Taylor series appears in the Information Booklet.

20a. *[5 marks]*

Markscheme

use of

$y \to y + \frac{h\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

x	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{h\mathrm{d}y}{\mathrm{d}x}$
0	2	1	0.1
0.1	2.1	0.7793304775	0.07793304775
0.2	2.17793304775	0.5190416116	0.05190416116
0.3	2.229837209		

Note: Award ***A1*** for $y(0.1)$ and ***A1*** for $y(0.2)$

$y(0.3) = 2.23$ ***A2***

[5 marks]

Examiners report

Most candidates knew Euler's method and were able to apply it to the differential equation to answer part (a). Some candidates who knew Euler's method completed one iteration too many to arrive at an incorrect answer. Surprisingly few candidates were able to efficiently use their GDCs to answer this question and this led to many final answers that were incorrect due to rounding errors.

20b. *[10 marks]*

Markscheme

(i)

$\text{IF} = \mathrm{e}^{\left(\int \tan x \mathrm{d}x\right)}$ ***(M1)***

$\text{IF} = \mathrm{e}^{\left(\int \frac{\sin x}{\cos x} \mathrm{d}x\right)}$ ***(M1)***

Note: Only one of the two ***(M1)*** above may be implied.

$= \mathrm{e}^{(-\ln \cos x)}$ (or $\mathrm{e}^{(\ln \sec x)}$) ***A1***

$= \sec x$ ***AG***

(ii) multiplying by the IF ***(M1)***

$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sec x \tan x = \cos x$ ***(A1)***

$\frac{\mathrm{d}}{\mathrm{d}x}(y \sec x) = \cos x$ ***(A1)***

$y \sec x = \sin x + c$ ***A1A1***

putting

$x = 0,\ y = 2 \Rightarrow c = 2$

$y = \cos x(\sin x + 2)$ ***A1***

[10 marks]

Examiners report

Most candidates were able to correctly derive the Integration Factor in part (b) but some lost marks due to not showing all the steps that would be expected in a "show that" question. The differential equation was solved correctly by a significant number of candidates but there were errors when candidates multiplied by $\sec x$ before the inclusion of the arbitrary constant.

21a. *[4 marks]*

Markscheme

$\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n \to \infty} \frac{\frac{(n+1)^2 x^{n+1}}{2^{n+1}}}{\frac{n^2 x^n}{2^n}}$ ***M1***

$= \lim\limits_{n \to \infty} \frac{(n+1)^2}{n^2} \times \frac{x}{2}$ ***A1***

$= \frac{x}{2}$ (since $\lim \to$) ***A1*** as $n \to \infty$

the radius of convergence R is found by equating this limit to 1, giving $R = 2$ ***A1***

[4 marks]

Examiners report

It was pleasing that most candidates were aware of the Radius of Convergence and Interval of Convergence required by parts (a) and (b) of this problem. Many candidates correctly handled the use of the Ratio Test for convergence and there was also the use of Cauchy's n$^{\text{th}}$ root test by a small number of candidates to solve part (a). Candidates need to take care to justify correctly the divergence or convergence of series when finding the Interval of Convergence.

21b. *[3 marks]*

Markscheme

when $x = 2$, the series is

$\sum n^2$ which is divergent because the terms do not converge to 0 ***R1***

when $x = -2$, the series is

$\sum (-1)^n n^2$ which is divergent because the terms do not converge to 0 ***R1***

the interval of convergence is

$]-2,\ 2[$ ***A1***

[3 marks]

Examiners report

It was pleasing that most candidates were aware of the Radius of Convergence and Interval of Convergence required by parts (a) and (b) of this problem. Many candidates correctly handled the use of the Ratio Test for convergence and there was also the use of Cauchy's n$^{\text{th}}$ root test by a small number of candidates to solve part (a). Candidates need to take care to justify correctly the divergence or convergence of series when finding the Interval of Convergence.

21c. *[4 marks]*

Markscheme

putting $x = -0.1$, ***(M1)***

for any correct partial sum ***(A1)***

– 0.05

– 0.04

– 0.041125

– 0.041025

– 0.0410328 ***(A1)***

the sum is – 0.0410 correct to 3 significant figures ***A1***

[4 marks]

Examiners report

The summation of the series in part (c) was poorly handled by a significant number of candidates, which was surprising on what was expected to be quite a straightforward problem. Again efficient use of the GDC seemed to be a problem. A number of candidates found the correct sum but not to the required accuracy.

22a. *[9 marks]*

Markscheme

(i) the area under the curve between $a - 1$ and $a + 1$

M1
$= \int_{a-1}^{a+1} \frac{\mathrm{d}x}{x}$
A1
$= [\ln x]_{a-1}^{a+1}$
A1
$= \ln\left(\frac{a+1}{a-1}\right)$
lower sum

M1A1
$= \frac{1}{a} + \frac{1}{a+1}$
AG
$= \frac{2a+1}{a(a+1)}$
upper sum

A1
$= \frac{1}{a-1} + \frac{1}{a}$
AG
$= \frac{2a-1}{a(a-1)}$
it follows that

$\frac{2a+1}{a(a+1)} < \ln\left(\frac{a+1}{a-1}\right) < \frac{2a-1}{a(a-1)}$
because the area of the region under the curve lies between the areas of the regions defined by the lower and upper sums ***R1***

(ii) putting

A1
$\left(\frac{a+1}{a-1} = 1.2\right) \Rightarrow a = 11$
therefore,

A1
$\text{UB} = \frac{21}{110}(= 0.191),\ \text{LB} = \frac{23}{132}(= 0.174)$
[9 marks]

Examiners report

Many candidates made progress with this problem. This was pleasing since whilst being relatively straightforward it was not a standard problem. There were still some candidates who did not use the definite integral correctly to find the area under the curve in part (a) and part (b). Also candidates should take care to show all the required working in a “show that” question, even when demonstrating familiar results. The ability to find upper and lower bounds was often well done in parts (a) (ii) and (b) (ii).

22b. *[5 marks]*

Markscheme

(i) the area under the curve between a – 1 and a

A1

$= \int_{a-1}^{a} \frac{\mathrm{d}x}{x}$

$= [\ln x]_{a-1}^{a} = \ln\left(\frac{a}{a-1}\right)$

attempt to find area of trapezium ***M1***

area of trapezoidal "upper sum"

or equivalent ***A1***

$= \frac{1}{2}\left(\frac{1}{a-1} + \frac{1}{a}\right)$

$= \frac{2a-1}{2a(a-1)}$

it follows that

AG

$\ln\left(\frac{a}{a-1}\right) < \frac{2a-1}{2a(a-1)}$

(ii) putting

A1

$\left(\frac{a}{a-1} = 1.2\right) \Rightarrow a = 6$

therefore,

A1

$\mathrm{UB} = \frac{11}{60} (= 0.183)$

[5 marks]

Examiners report

Many candidates made progress with this problem. This was pleasing since whilst being relatively straightforward it was not a standard problem. There were still some candidates who did not use the definite integral correctly to find the area under the curve in part (a) and part (b). Also candidates should take care to show all the required working in a "show that" question, even when demonstrating familiar results. The ability to find upper and lower bounds was often well done in parts (a) (ii) and (b) (ii).

23. *[6 marks]*

Markscheme

METHOD 1

$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{40}(60 - v)$ ***(M1)***

attempting to separate variables

$\int \frac{\mathrm{d}v}{60 - v} = \int \frac{\mathrm{d}t}{40}$ ***M1***

$-\ln(60 - v) = \frac{t}{40} + c$ ***A1***

(or equivalent) ***A1***

$c = -\ln 60$

attempting to solve for v when $t = 30$ ***(M1)***

$v = 60 - 60e^{-\frac{3}{4}}$

$v = 31.7\ (\mathrm{ms}^{-1})$ ***A1***

METHOD 2

$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{40}(60 - v)$ ***(M1)***

(or equivalent) ***M1***

$\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{40}{60 - v}$

where

$\int_0^{v_f} \frac{40}{60 - v}\mathrm{d}v = 30$

is the velocity of the car after 30 seconds. ***A1A1***

v_f

attempting to solve

for

$\int_0^{v_f} \frac{40}{60 - v}\mathrm{d}v = 30$ ***(M1)***

v_f

$v = 31.7\ (\mathrm{ms}^{-1})$ ***A1***

[6 marks]

Examiners report

Most candidates experienced difficulties with this question. A large number of candidates did not attempt to separate the variables and instead either attempted to integrate with respect to v or employed constant acceleration formulae. Candidates that did separate the variables and attempted to integrate both sides either made a sign error, omitted the constant of integration or found an incorrect value for this constant. Almost all candidates were not aware that this question could be solved readily on a GDC.

24a. *[10 marks]*

Markscheme

(i) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x + \cos x$ ***A1***

$y\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x + \sin x)(-\sin x + \cos x)$ ***M1***

$= \cos^2 x - \sin^2 x$ ***A1***

$= \cos 2x$ ***AG***

METHOD 2

$y^2 = (\sin x + \cos x)^2$ ***A1***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2(\cos x + \sin x)(\cos x - \sin x)$ ***M1***

$y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 x - \sin^2 x$ ***A1***

$= \cos 2x$ ***AG***

(ii) attempting to separate variables

$\int y \, \mathrm{d}y = \int \cos 2x \, \mathrm{d}x$ ***M1***

$\frac{1}{2}y^2 = \frac{1}{2}\sin 2x + C$ ***A1A1***

Note: Award ***A1*** for a correct LHS and ***A1*** for a correct RHS.

$y = \pm(\sin 2x + A)^{\frac{1}{2}}$ ***A1***

(iii)

$\sin 2x + A \equiv (\cos x + \sin x)^2$ ***(M1)***

$(\cos x + \sin x)^2 = \cos^2 x + 2\sin x \cos x + \sin^2 x$

use of $\sin 2x \equiv 2\sin x \cos x$. ***(M1)***

$A = 1$ ***A1***

[10 marks]

Examiners report

Part (a) was not well done and was often difficult to mark. In part (a) (i), a large number of candidates did not know how to verify a solution, $y(x)$, to the given differential equation. Instead, many candidates attempted to solve the differential equation. In part (a) (ii), a large number of candidates began solving the differential equation by correctly separating the variables but then either neglected to add a constant of integration or added one as an afterthought. Many simple algebraic and basic integral calculus errors were seen. In part (a) (iii), many candidates did not realize that the solution given in part (a) (i) and the general solution found in part (a) (ii) were to be equated. Those that did know to equate these two solutions, were able to square both solution forms and correctly use the trigonometric identity $\sin 2x = 2\sin x \cos x$. Many of these candidates however started with incorrect solution(s).

24b. *[12 marks]*

Markscheme

(i) substituting $x = \frac{\pi}{4}$ and $y = 2$ into $y = (\sin 2x + A)^{\frac{1}{2}}$ ***M1***

so $g(x) = (\sin 2x + 3)^{\frac{1}{2}}$. ***A1***

range g is $\left[\sqrt{2},\, 2\right]$ ***A1A1A1***

Note: Accept [1.41, 2]. Award ***A1*** for each correct endpoint and ***A1*** for the correct closed interval.

(ii)

$\int_0^{\frac{\pi}{2}} (\sin 2x + 3)^{\frac{1}{2}}\, dx$ ***(M1)(A1)***

= 2.99 ***A1***

(iii)

$\pi \int_0^{\frac{\pi}{2}} (\sin 2x + 3) dx - \pi(1)\left(\frac{\pi}{2}\right)$ (or equivalent) ***(M1)(A1)(A1)***

Note: Award ***(M1)(A1)(A1)*** for $\pi \int_0^{\frac{\pi}{2}} (\sin 2x + 2) dx$

$= 17.946 - 4.935 \left(= \frac{\pi}{2}(3\pi + 2) - \pi\left(\frac{\pi}{2}\right)\right)$ ***A1***

Note: Award ***A1*** for $\pi(\pi + 1)$.

[12 marks]

Examiners report

In part (b), a large number of candidates knew how to find a required area and a required volume of solid of revolution using integral calculus. Many candidates, however, used incorrect expressions obtained in part (a). In part (b) (ii), a number of candidates either neglected to state 'π' or attempted to calculate the volume of a solid of revolution of 'radius' $f(x) - g(x)$.

25a. *[4 marks]*

Markscheme

METHOD 1

$\lim\limits_{x \to 0} \frac{\sin 4x^2 - \sin 9x^2}{4x^2}$ ***M1***

$= \lim\limits_{x \to 0} \frac{\sin 4x^2}{4x^2} - \frac{9}{4} \lim\limits_{x \to 0} \frac{\sin 9x^2}{9x^2}$ ***A1A1***

$= 1 - \frac{9}{4} \times 1 = -\frac{5}{4}$ ***A1***

METHOD 2

$\lim\limits_{x \to 0} \frac{\sin 4x^2 - \sin 9x^2}{4x^2}$ ***M1***

$= \lim\limits_{x \to 0} \frac{8x \cos 4x^2 - 18x \cos 9x^2}{8x}$ ***M1A1***

$= \frac{8 - 18}{8} = -\frac{10}{8} = -\frac{5}{4}$ ***A1***

[4 marks]

Examiners report

Part (a) of this question was accessible to the vast majority of candidates, who recognised that L'Hôpital's rule could be used. Most candidates were successful in finding the limit, with some making calculation errors. Candidates that attempted to use $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ or a combination of this result and L'Hôpital's rule were less successful.

25b. *[2 marks]*

Markscheme

since

$\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$ $\left(\text{or } \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

$\sin x^2 = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{(2n+1)!}$ ***(M1)***

$\left(\text{or } \sin x = \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots\right)$ ***A1***

$g(x) = \sin x^2 = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ ***AG***

[2 marks]

Examiners report

In part (b) most candidates showed to be familiar with the substitution given and were successful in showing the result.

25c. *[7 marks]*

Markscheme

let

$I = \int_0^1 \sin x^2 \, dx$

M1

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^1 x^{4n+2} \, dx \quad \left(\int_0^1 \frac{x^2}{1!} dx - \int_0^1 \frac{x^6}{3!} dx + \int_0^1 \frac{x^{10}}{5!} dx - \ldots \right)$ ***M1***

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{\left[x^{4n+3}\right]_0^1}{(4n+3)} \quad \left(\left[\frac{x^3}{3 \times 1!}\right]_0^1 - \left[\frac{x^7}{7 \times 3!}\right]_0^1 + \left[\frac{x^{11}}{11 \times 5!}\right]_0^1 - \ldots \right)$ ***A1***

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!(4n+3)} \quad \left(\frac{1}{3 \times 1!} - \frac{1}{7 \times 3!} + \frac{1}{11 \times 5!} - \ldots \right)$

$= \sum_{n=0}^{\infty} (-1)^n a_n$

where

$a_n = \frac{1}{(4n+3)(2n+1)!} > 0$

for all

$n \in \mathbb{N}$

as

$\{a_n\}$

is decreasing the sum of the alternating series

$\sum_{n=0}^{\infty} (-1)^n a_n$

lies between

$\sum_{n=0}^{N} (-1)^n a_n$

and

$\sum_{n=0}^{N} (-1)^n a_n \pm a_{N+1}$ ***R1***

hence for four decimal place accuracy, we need

$|a_{N+1}| < 0.00005$ ***M1***

N	$\|a_{N+1}\|$
1	$\frac{1}{11(5!)} = 0.0000757576$
2	$\frac{1}{15(7!)} = 0.0000132275$

since

$a_{2+1} < 0.00005$ ***R1***

so

$N = 2$ (or 3 terms) ***A1***

[7 marks]

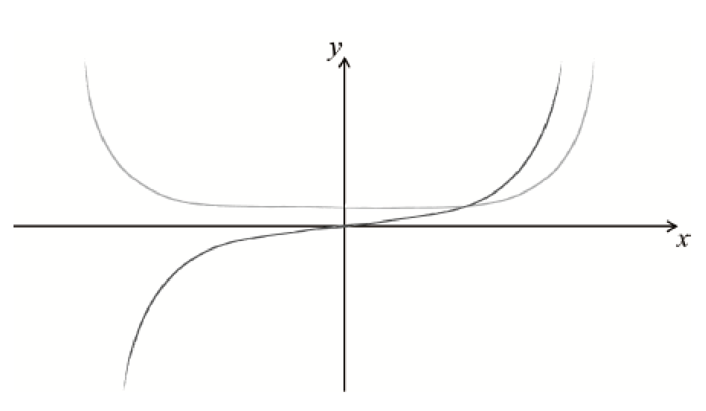
Examiners report

Very few candidates were able to do part (c) successfully. Most used trial and error to arrive at the answer.

26. *[9 marks]*

Markscheme

(a)



A1 for shape, ***A1*** for passing through origin ***A1A1***

Note: Asymptotes not required.

[2 marks]

(b)

$$p(x) = \underbrace{f(0)}_{a} + \underbrace{f'(0)}_{b}x + \underbrace{\frac{f''(0)}{2!}}_{c}x^2 + \underbrace{\frac{f^{(3)}(0)}{3!}}_{d}x^3 + \ldots$$

(i) because the y-intercept of f is positive ***R1***

(ii)

$b = 0$ ***A1***

$c \geqslant 0$ ***A1A1***

Note: ***A1*** for $>$ and *A1* for $=$.

$d = 0$ ***A1***

[5 marks]

(c) as the graph has vertical asymptotes $x = \pm k,\ k > 0$, ***R1***

the radius of convergence has an upper bound of k ***A1***

Note: Accept $r < k$.

[2 marks]

Examiners report

Overall candidates made good attempts to parts (a) and most candidates realized that the graph contained the origin; however many candidates had difficulty rendering the correct shape of the graph of f'. Part b(i) was also well answered although some candidates where not very clear and digressed a lot. Part (ii) was less successful with most candidates scoring just part of the marks. A small number of candidates answered part (c) correctly with a valid reason.

27.

[11 marks]

Markscheme

put $y = vx$ so that

M1
$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$
substituting, ***M1***

(A1)
$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2x^2+3vx^2+2x^2}{x^2}\ (= v^2 + 3v + 2)$
A1
$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 + 2v + 2$
M1
$\int \frac{\mathrm{d}v}{v^2+2v+2} = \int \frac{\mathrm{d}x}{x}$
(A1)
$\int \frac{\mathrm{d}v}{(v+1)^2+1} = \int \frac{\mathrm{d}x}{x}$
A1
$\arctan(v+1) = \ln x + c$
Note: Condone absence of c at **this** stage.

M1
$\arctan(\frac{y}{x} + 1) = \ln x + c$
When $x = 1$, $y = -1$ ***M1***

$c = 0$ ***A1***

$\frac{y}{x} + 1 = \tan \ln x$
A1
$y = x(\tan \ln x - 1)$
[11 marks]

Examiners report

Most candidates recognised this differential equation as one in which the substitution $y = vx$ would be helpful and many reached the stage of separating the variables. However, the integration of

$\frac{1}{x^2+2v+2}$ proved beyond many candidates who failed to realise that completing the square would lead to an arctan integral. This highlights the importance of students having a full understanding of the core calculus if they are studying this option.

28a.

[4 marks]

Markscheme

using a ratio test,

M1A1

$\left|\frac{T_{n+1}}{T_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\right| \times \left|\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$
Note: Condone omission of modulus signs.

$\to 0$ as $n \to \infty$ for all values of x ***R1***
the series is therefore convergent for

A1
$x \in \mathbb{R}$
[4 marks]

Examiners report

Solutions to (a) were generally good although some candidates failed to reach the correct conclusion from correct application of the ratio test. Solutions to (b) and (c), however, were generally disappointing with many candidates unable to make use of the signposting in the question. Candidates who were unable to solve (b) and (c) often picked up marks in (d).

28b. *[6 marks]*

Markscheme

(i)

$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{2\times 3} + \ldots$ ***M1***

$< x + \frac{x^2}{2} + \frac{x^3}{2\times 2} + \ldots \quad (\text{for } x > 0)$ ***A1***

$= \frac{x}{1-\frac{x}{2}} \quad (\text{for } x < 2)$ ***A1***

$= \frac{2x}{2-x} \quad (\text{for } 0 < x < 2)$ ***AG***

(ii)

$e^x < 1 + \frac{2x}{2-x} = \frac{2+x}{2-x}$ ***A1***

$e < \left(\frac{2+x}{2-x}\right)^{\frac{1}{x}}$ ***A1***

replacing x by $\frac{1}{n}$ (and noting that the result is true for $n > \frac{1}{2}$ and therefore $\mathbb{Z}^+$) ***M1***

$e < \left(\frac{2n+1}{2n-1}\right)^n$ ***AG***

[6 marks]

Examiners report

Solutions to (a) were generally good although some candidates failed to reach the correct conclusion from correct application of the ratio test. Solutions to (b) and (c), however, were generally disappointing with many candidates unable to make use of the signposting in the question. Candidates who were unable to solve (b) and (c) often picked up marks in (d).

28c. *[4 marks]*

Markscheme

(i)

$1 - \mathrm{e}^{-x} = x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ ***A1***

for $0 < x < 2$, the series is alternating with decreasing terms so that the sum is greater than the sum of an even number of terms ***R1***

therefore

$1 - \mathrm{e}^{-x} > x - \frac{x^2}{2}$ ***AG***

(ii)

$\mathrm{e}^{-x} < 1 - x + \frac{x^2}{2}$

$\mathrm{e}^{x} > \frac{1}{\left(1 - x + \frac{x^2}{2}\right)}$ ***M1***

$\mathrm{e} > \left(\frac{2}{2 - 2x + x^2}\right)^{\frac{1}{x}}$ ***A1***

replacing x by $\frac{1}{n}$ (and noting that the result is true for $n > \frac{1}{2}$ and therefore $\mathbb{Z}^+$)

$\mathrm{e} > \left(\frac{2n^2}{2n^2 - 2n + 1}\right)^n$ ***AG***

[4 marks]

Examiners report

Solutions to (a) were generally good although some candidates failed to reach the correct conclusion from correct application of the ratio test. Solutions to (b) and (c), however, were generally disappointing with many candidates unable to make use of the signposting in the question. Candidates who were unable to solve (b) and (c) often picked up marks in (d).

28d. *[2 marks]*

Markscheme

from (b) and (c),

$\mathrm{e} < 2.718282\ldots$ and $\mathrm{e} > 2.718281\ldots$ ***A1***

we conclude that e = 2.71828 correct to 5 decimal places ***A1***

[2 marks]

Examiners report

Solutions to (a) were generally good although some candidates failed to reach the correct conclusion from correct application of the ratio test. Solutions to (b) and (c), however, were generally disappointing with many candidates unable to make use of the signposting in the question. Candidates who were unable to solve (b) and (c) often picked up marks in (d).

29a.

[4 marks]

Markscheme

M1A1A1

$\lim\limits_{x \to 0} \frac{\tan x}{x+x^2} = \lim\limits_{x \to 0} \frac{\sec^2 x}{1+2x}$

A1

$\lim\limits_{x \to 0} \frac{\tan x}{x+x^2} = \frac{1}{1} = 1$

[4 marks]

Examiners report

This question was accessible to the vast majority of candidates, who recognised that L'Hopital's rule was required. A few of the weaker candidates did not realise that it needed to be applied twice in part (b). Many fully correct solutions were seen.

29b.

[7 marks]

Markscheme

M1A1A1

$\lim\limits_{x \to 1} \frac{1-x^2+2x^2 \ln x}{1-\sin \frac{\pi x}{2}} = \lim\limits_{x \to 1} \frac{-2x+2x+4x \ln x}{-\frac{\pi}{2} \cos \frac{\pi x}{2}}$

M1A1A1

$= \lim\limits_{x \to 1} \frac{4+4 \ln x}{\frac{\pi^2}{4} \sin \frac{\pi x}{2}}$

A1

$\lim\limits_{x \to 1} \frac{1-x^2+2x^2 \ln x}{1-\sin \frac{\pi x}{2}} = \frac{4}{\frac{\pi^2}{4}} = \frac{16}{\pi^2}$

[7 marks]

Examiners report

This question was accessible to the vast majority of candidates, who recognised that L'Hopital's rule was required. A few of the weaker candidates did not realise that it needed to be applied twice in part (b). Many fully correct solutions were seen.

30. *[17 marks]*

Markscheme

(a) from

$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x + \cos x$

A1

$f'(0) = 1$

now

M1A1A1A1

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = y\sec^2 x + \frac{\mathrm{d}y}{\mathrm{d}x}\tan x - \sin x$

Note: Award ***A1*** for each term on RHS.

A1

$\Rightarrow f''(0) = -\frac{\pi}{2}$

A1

$\Rightarrow y = -\frac{\pi}{2} + x - \frac{\pi x^2}{4}$

[7 marks]

(b) recognition of integrating factor ***(M1)***

integrating factor is

$\mathrm{e}^{\int -\tan x\mathrm{d}x}$

(A1)

$= \mathrm{e}^{\ln\cos x}$

(A1)

$= \cos x$

M1

$\Rightarrow y\cos x = \int \cos^2 x\mathrm{d}x$

A1

$\Rightarrow y\cos x = \frac{1}{2}\int (1 + \cos 2x)\mathrm{d}x$

A1

$\Rightarrow y\cos x = \frac{x}{2} + \frac{\sin 2x}{4} + k$

when

M1A1

$x = \pi,\ y = 0 \Rightarrow k = -\frac{\pi}{2}$

(A1)

$\Rightarrow y\cos x = \frac{x}{2} + \frac{\sin 2x}{4} - \frac{\pi}{2}$

A1

$\Rightarrow y = \sec x\left(\frac{x}{2} + \frac{\sin 2x}{4} - \frac{\pi}{2}\right)$

[10 marks]

Total [17 marks]

Examiners report

Part (a) of the question was set up in an unusual way, which caused a problem for a number of candidates as they tried to do part (b) first and then find the Maclaurin series by a standard method. Few were successful as they were usually weaker candidates and made errors in finding the solution $y = f(x)$. The majority of candidates knew how to start part (b) and recognised the need to use an integrating factor, but a number failed because they missed out the negative sign on the integrating factor, did not realise that $\mathrm{e}^{\ln\cos x} = \cos x$ or were unable to integrate $\cos^2 x$. Having said this, a number of candidates succeeded in gaining full marks on this question.

31a. *[5 marks]*

Markscheme

comparing with the series

$\sum\limits_{n=1}^{\infty} \frac{1}{n}$ ***A1***

using the limit comparison test ***(M1)***

since

$\lim\limits_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \left(= \lim\limits_{x \to 0} \frac{\sin x}{x} \right) = 1$ ***M1A1***

$\sum \frac{1}{n}$ diverges,

$\sum \sin \frac{1}{n}$ diverges ***A1***

[5 marks]

Examiners report

This question was found to be the hardest on the paper, with only the best candidates gaining full marks on it. Part (a) was very poorly done with a significant number of candidates unable to start the question. More students recognised part (b) as an integral test, but often could not progress beyond this. In many cases, students appeared to be guessing at what might constitute a valid test.

31b. *[7 marks]*

Markscheme

using integral test ***(M1)***

let

$u = \ln x$ ***(M1)***

$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{1}{x(\ln x)^2} \mathrm{d}x = \int \frac{1}{u^2} \mathrm{d}u = -\frac{1}{u} \left(= -\frac{1}{\ln x} \right)$ ***A1***

$\Rightarrow \int_2^{\infty} \frac{1}{x(\ln x)^2} \mathrm{d}x = \lim\limits_{a \to \infty} \left[-\frac{1}{\ln x} \right]_2^a$

$= \lim\limits_{a \to \infty} \left(-\frac{1}{\ln a} + \frac{1}{\ln 2} \right)$ ***(M1)(A1)***

as

$a \to \infty, \ -\frac{1}{\ln a} \to 0$ ***A1***

$\Rightarrow \int_2^{\infty} \frac{1}{x(\ln x)^2} \mathrm{d}x = \frac{1}{\ln 2}$

hence the series is convergent ***AG***

[7 marks]

Examiners report

This question was found to be the hardest on the paper, with only the best candidates gaining full marks on it. Part (a) was very poorly done with a significant number of candidates unable to start the question. More students recognised part (b) as an integral test, but often could not progress beyond this. In many cases, students appeared to be guessing at what might constitute a valid test.

32a. *[9 marks]*

Markscheme

(a)

,
$|z| = z$
A1A1
$\arg(z) = 0$
so

AG N0
$L(z) = \ln z$
[2 marks]

(b) (i)

A1A1 N2
$L(-1) = \ln 1 + \mathrm{i}\pi = \mathrm{i}\pi$
(ii)

A1A1 N2
$L(1 - \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{7\pi}{4}$
(iii)

A1 N1
$L(-1 + \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{3\pi}{4}$
[5 marks]

(c) for comparing the product of two of the above results with the third ***M1***

for stating the result

and
$-1 + \mathrm{i} = -1 \times (1 - \mathrm{i})$
R1
$L(-1 + \mathrm{i}) \neq L(-1) + L(1 - \mathrm{i})$
hence, the property

$L(z_1 z_2) = L(z_1) + L(z_2)$
does not hold for all values of

and
z_1
AG N0
z_2
[2 marks]

Total [9 marks]

Examiners report

Part A was answered well by a fair amount of candidates, with some making mistakes in calculating the arguments of complex numbers, as well as careless mistakes in finding the products of complex numbers.

32b.

[14 marks]

Markscheme

(a) from

$f(x+y) = f(x)f(y)$
for $x = y = 0$ ***M1***

we have

A1
$f(0+0) = f(0)f(0) \Leftrightarrow f(0) = (f(0))^2$
as

, this implies that
$f(0) \neq 0$
R1AG ***N0***
$f(0) = 1$
[3 marks]

(b) **METHOD 1**

from

$f(x+y) = f(x)f(y)$
for $y = -x$, we have

M1A1
$f(x-x) = f(x)f(-x) \Leftrightarrow f(0) = f(x)f(-x)$
as

this implies that
$f(0) \neq 0$
R1AG ***N0***
$f(x) \neq 0$
METHOD 2

suppose that, for a value of x,

M1
$f(x) = 0$
from

$f(x+y) = f(x)f(y)$
for

, we have
$y = -x$
A1
$f(x-x) = f(x)f(-x) \Leftrightarrow f(0) = f(x)f(-x)$
substituting

by 0 gives
$f(x)$
which contradicts part (a) ***R1***
$f(0) = 0$
therefore

for all x. ***AG*** ***N0***
$f(x) \neq 0$
[3 marks]

(c) by the definition of derivative

(M1)
$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$
A1(A1)
$= \lim_{h \to 0}\left(\frac{f(x)f(h)-f(x)f(0)}{h}\right)$
A1

$= \lim_{h \to 0} \left(\frac{f(h) - f(0)}{h} \right) f(x)$
AG ***N0***
$= f'(0) f(x) \quad (= k\, f(x))$
[4 marks]

(d)

M1A1
$\int \frac{f'(x)}{f(x)} \mathrm{d}x = \int k \mathrm{d}x \Rightarrow \ln f(x) = kx + C$
A1
$\ln f(0) = C \Rightarrow C = 0$
A1 ***N1***
$f(x) = \mathrm{e}^{kx}$
Note: Award ***M1A0A0A0*** if no arbitrary constant C .

[4 marks]

Total [14 marks]

Examiners report

Part B proved demanding for most candidates, particularly parts (c) and (d). A surprising number of candidates did not seem to know what was meant by the 'definition of derivative' in part (c) as they attempted to use quotient rule rather than first principles.

33.

[7 marks]

Markscheme

(a)

$\frac{\mathrm{d}y}{\mathrm{d}t} = ky \cos(kt)$
(M1)
$\frac{\mathrm{d}y}{y} = k \cos(kt) \mathrm{d}t$
M1
$\int \frac{\mathrm{d}y}{y} = \int k \cos(kt) \mathrm{d}t$
A1
$\ln y = \sin(kt) + c$

$y = A\mathrm{e}^{\sin(kt)}$
(M1)
$t = 0 \Rightarrow y_0 = A$
A1
$\Rightarrow y = y_0 \mathrm{e}^{\sin kt}$

(b)

(M1)
$-1 \leqslant \sin kt \leqslant 1$

$y_0 \mathrm{e}^{-1} \leqslant y \leqslant y_0 \mathrm{e}^{1}$
so the ratio is

A1
$\frac{1}{\mathrm{e}} : \mathrm{e}$ or $1 : \mathrm{e}^2$
[7 marks]

Examiners report

Part (a) was done successfully by many candidates. However, very few attempted part (b).

34.

[13 marks]

Markscheme

put $y = vx$ so that

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1A1***

the equation becomes

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + v^2$ ***(A1)***

leading to

$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2$ ***A1***

separating variables,

$\int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}v}{v^2}$ ***M1A1***

hence

$\ln x = -v^{-1} + C$ ***A1A1***

substituting for v,

$\ln x = \frac{-x}{y} + C$ ***M1***

Note: Do not penalise absence of C at the above stages.

substituting the boundary conditions,

$0 = -\frac{1}{2} + C$ ***M1***

$C = \frac{1}{2}$ ***A1***

the solution is

$\ln x = \frac{-x}{y} + \frac{1}{2}$ ***(A1)***

leading to

$y = \frac{2x}{1-2\ln x}$ (or equivalent form) ***A1***

Note: Candidates are not required to note that

$x \neq \sqrt{\mathrm{e}}$.

[13 marks]

Examiners report

Many candidates were able to make a reasonable attempt at this question with many perfect solutions seen.

35. *[10 marks]*

Markscheme

(a)

A1
$\mathrm{e}^x - 1 = x + \frac{x^2}{2} + \frac{x^2}{6} + \ldots$
M1A1
$\mathrm{e}^{\mathrm{e}^x - 1} = 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right)^2}{2} + \frac{\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right)^3}{6} + \ldots$
M1A1
$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^3}{6} + \ldots$
AG
$= 1 + x + x^2 + \frac{5}{6}x^3 + \ldots$
[5 marks]

(b) **EITHER**

A1
$f'(x) = 1 + 2x + \frac{5x^2}{2} + \ldots$
M1A1
$\frac{f(x)-1}{f'(x)-1} = \frac{x+x^2+5x^3/6+\ldots}{2x+5x^2/2+\ldots}$
A1
$= \frac{1+x+\ldots}{2+5x/2+\ldots}$
A1
$\to \frac{1}{2}$ as $x \to 0$
[5 marks]

OR

using l'Hopital's rule, ***M1***

M1A1
$\lim\limits_{x\to 0} \frac{\mathrm{e}^{(\mathrm{e}^x-1)} - 1}{\mathrm{e}^{(\mathrm{e}^x-1)} - 1' - 1} = \lim\limits_{x\to 0} \frac{\mathrm{e}^{(\mathrm{e}^x-1)} - 1}{\mathrm{e}^{(\mathrm{e}^x+x-1)} - 1}$
A1
$= \lim\limits_{x\to 0} \frac{\mathrm{e}^{(\mathrm{e}^x+x-1)}}{\mathrm{e}^{(\mathrm{e}^x+x-1)} \times (\mathrm{e}^x+1)}$
A1
$= \frac{1}{2}$
[5 marks]

Total [10 marks]

Examiners report

Many candidates obtained the required series by finding the values of successive derivatives at $x = 0$, failing to realise that the intention was to start with the exponential series and replace x by the series for

. Candidates who did this were given partial credit for using this method. Part (b) was reasonably well answered using a variety of
$\mathrm{e}^x - 1$
methods.

36a. *[7 marks]*

Markscheme

the nth term is

$u_n = \frac{1\times 3\times 5\ldots(2n-1)}{2\times 5\times 8\ldots(3n-1)}x^n$ ***M1A1***

(using the ratio test to test for absolute convergence)

$\frac{|u_{n+1}|}{|u_n|} = \frac{(2n+1)}{(3n+2)}|x|$ ***M1A1***

$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \frac{2|x|}{3}$ ***A1***

let R denote the radius of convergence

then

$\frac{2R}{3} = 1$ so

$r = \frac{3}{2}$ ***M1A1***

Note: Do not penalise the absence of absolute value signs.

[7 marks]

Examiners report

Solutions to this question were generally disappointing. In (a), many candidates were unable even to find an expression for the nth term so that they could not apply the ratio test.

36b. *[8 marks]*

Markscheme

using the compound angle formula or a graphical method the series can be written in the form ***(M1)***

$\sum_{n=1}^{\infty} u_n$ where

$u_n = (-1)^n \sin\left(\frac{1}{n}\right)$ ***A2***

since

$\frac{1}{n} < \frac{\pi}{2}$ *i.e.* an angle in the first quadrant, ***R1***

it is an alternating series ***R1***

$u_n \to 0$ as $n \to \infty$ ***R1***

and

$|u_{n+1}| < |u_n|$ ***R1***

it follows that the series is convergent ***R1***

[8 marks]

Examiners report

Solutions to this question were generally disappointing. In (b), few candidates were able to rewrite the nth term in the form $\sum(-1)^n \sin\left(\frac{1}{n}\right)$ so that most candidates failed to realise that the series was alternating.

37.

[8 marks]

Markscheme

$\frac{dy}{dx} = e^x + 2y^2$ *(A1)*

x	y	dy/dx	δy	
0	1	3	0.3	*M1A1*
0.1	1.3	4.485170918	0.4485170918	*A1*
0.2	1.7485170918	7.336026799	0.7336026799	*A1*
0.3	2.482119772	13.67169593	1.367169593	*A1*
0.4	3.849289365			*A1*

required approximation = 3.85 *A1*

[8 marks]

Examiners report

Most candidates seemed familiar with Euler's method. The most common way of losing marks was either to round intermediate answers to insufficient accuracy despite the advice in the question or simply to make an arithmetic error. Many candidates were given an accuracy penalty for not rounding their answer to three significant figures.

38. *[11 marks]*

Markscheme

(a)

M1A1

$\int_0^\infty \mathrm{e}^{-x}\cos x\mathrm{d}x = [\mathrm{e}^{-x}\sin x]_0^\infty - \int_0^\infty \sin x\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x})\mathrm{d}x$

since

as

$\mathrm{e}^{-x} \to 0$

and

$x \to \infty$

is bounded

$\sin x$

as

$\mathrm{e}^{-x}\sin x \to 0$

R1

$x \to \infty$

(or alternative convincing argument)

R1

$\mathrm{e}^{-x}\sin x = 0 \text{ when } x = 0$

the second term

A1

$\int_0^\infty \mathrm{e}^{-x}\sin x\mathrm{d}x$

so

AG

$\int_0^\infty \mathrm{e}^{-x}\cos x\mathrm{d}x = \int_0^\infty \mathrm{e}^{-x}\sin x\mathrm{d}x$

[5 marks]

(b) continuing the process

M1A1

$\int_0^\infty \mathrm{e}^{-x}\cos x\mathrm{d}x = -[\mathrm{e}^{-x}\cos x]_0^\infty + \int_0^\infty \cos x\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x})\mathrm{d}x$

the value of the first term is 1 ***A1***

the second term

A1

$= -\int_0^\infty \mathrm{e}^{-x}\cos x\mathrm{d}x$

so

A1

$2\int_0^\infty \mathrm{e}^{-x}\cos x\mathrm{d}x = 1$

the common value of the integrals is

A1

$\frac{1}{2}$

[6 marks]

Total [11 marks]

Examiners report

Although this question is based on core material, many candidates were unable to perform the double integration by parts successfully. The difficulty in the method often lies in the choice of u and v and wrong choices were often made. Many candidates failed to consider adequately what happens at the upper limit (infinity). The question was structured so that the solution to (a) led to the solution for (b) but in many cases, the solutions to (a) and (b) were mixed up often to the candidates' disadvantage. In this case, candidates who obtained the required results, in whatever order, were of course given full credit.

39. *[9 marks]*

Markscheme

put

so that
$y = vx$

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

the equation becomes

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 + v + 4$ ***A1***

$\int \frac{\mathrm{d}v}{v^2+4} = \int \frac{\mathrm{d}x}{x}$ ***A1***

$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = \ln x + C$ ***A1A1***

substituting

$(x,\ v) = (1,\ 2)$

$C = \frac{\pi}{8}$ ***M1A1***

the solution is

$\arctan\left(\frac{y}{2x}\right) = 2\ln x + \frac{\pi}{4}$ ***A1***

$y = 2x\tan\left(2\ln x + \frac{\pi}{4}\right)$ ***A1***

[9 marks]

Examiners report

Most candidates recognised this differential equation as one in which the substitution $y = vx$ would be helpful and many carried the method through to a successful conclusion. The most common error seen was an incorrect integration of $\frac{1}{4+v^2}$ with partial fractions and/or a logarithmic evaluation seen. Some candidates failed to include an arbitrary constant which led to a loss of marks later on.

40. *[17 marks]*

Markscheme

(a) using or obtaining

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$ ***(M1)***

$(1-n^2)^{-\frac{1}{2}} = 1 + (-x^2) \times \left(-\frac{1}{2}\right) + \frac{(-x^2)^2}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) + \ldots$ ***(A1)***

$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \ldots$ ***A1***

[3 marks]

(b) integrating, and changing sign

$\arccos x = -x - \frac{1}{6}x^3 - \frac{3}{40}x^5 + C + \ldots$ ***M1A1***

put $x = 0$,

$\frac{\pi}{2} = C$ ***M1***

$\left(\arccos x \approx \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5\right)$ ***AG***

[3 marks]

(c) **EITHER**

using

$\arccos x^2 \approx \frac{\pi}{2} - x^2 - \frac{1}{6}x^6 - \frac{3}{40}x^{10}$ ***M1A1***

$\lim\limits_{x\to 0} \frac{\frac{\pi}{2} - \arccos x^2 - x^2}{x^6} = \lim\limits_{x\to 0} \frac{\frac{x^6}{6} + \text{higher powers}}{x^6}$ ***M1A1***

$= \frac{1}{6}$ ***A1***

OR

using l'Hôpital's Rule ***M1***

$\text{limit} = \lim\limits_{x\to 0} \frac{\frac{1}{\sqrt{1-x^4}} \times 2x - 2x}{6x^5}$ ***M1***

$= \lim\limits_{x\to 0} \frac{\frac{1}{\sqrt{1-x^4}} - 1}{3x^4}$ ***A1***

$= \lim\limits_{x\to 0} \frac{-\frac{1}{2} \times \frac{1}{(1-x^4)^{3/2}} \times -4x^3}{12x^3}$ ***M1***

$= \frac{1}{6}$ ***A1***

[5 marks]

(d)

$\int_0^{0.2} \arccos\sqrt{x}\,\mathrm{d}x \approx \int_0^{0.2} \left(\frac{\pi}{2} - x^{\frac{1}{2}} - \frac{1}{6}x^{\frac{3}{2}} - \frac{3}{40}x^{\frac{5}{2}}\right) \mathrm{d}x$ ***M1***

$= \left[\frac{\pi}{2}x - \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{15}x^{\frac{5}{2}} - \frac{3}{140}x^{\frac{7}{2}}\right]_0^{0.2}$ ***(A1)***

$= \frac{\pi}{2} \times 0.2 - \frac{2}{3} \times 0.2^{\frac{3}{2}} - \frac{1}{15} \times 0.2^{\frac{5}{2}} - \frac{3}{140} \times 0.2^{\frac{7}{2}}$ ***(A1)***

= 0.25326 (to 5 decimal places) ***A1***

Note: Accept integration of the series approximation using a GDC.

using a GDC, the actual value is 0.25325 ***A1***

so the approximation is not correct to 5 decimal places ***R1***

[6 marks]

Total [17 marks]

Examiners report

Many candidates ignored the instruction in the question to use the series for $(1+x)^n$ to deduce the series for $(1-x^2)^{-1/2}$ and attempted instead to obtain it by successive differentiation. It was decided at the standardisation meeting to award full credit for this method although in the event the algebra proved to be too difficult for many. Many candidates used l'Hopital's Rule in (c) – this was much more difficult algebraically than using the series and it usually ended unsuccessfully. Candidates should realise that if a question on evaluating an indeterminate limit follows the determination of a Maclaurin series then it is likely that the series will be helpful in evaluating the limit. Part (d) caused problems for many candidates with algebraic errors being common. Many candidates failed to realise that the best way to find the exact value of the integral was to use the calculator.

41a. *[10 marks]*

Markscheme

(i) consider

$\frac{T_{n+1}}{T_n} = \frac{\left|\frac{(n-1)x^{n+1}}{2^{n+1}}\right|}{\left|\frac{nx^n}{2^n}\right|}$ ***M1***

$= \frac{(n+1)|x|}{2n}$ ***A1***

$\to \frac{|x|}{2}$ as $n \to \infty$ ***A1***

the radius of convergence satisfies $\frac{R}{2} = 1$, *i.e.* $R = 2$ ***A1***

(ii) the series converges for $-2 < x < 2$, we need to consider $x = \pm 2$ ***(R1)***

when $x = 2$, the series is $1 + 2 + 3 + \dots$ ***A1***

this is divergent for any one of several reasons *e.g.* finding an expression for or a comparison test with the harmonic series or noting that $\lim\limits_{n\to\infty} u_n \neq 0$ etc. ***R1***

when $x = -2$, the series is $-1 + 2 - 3 + 4 \dots$ ***A1***

this is divergent for any one of several reasons

e.g. partial sums are $-1,\ 1,\ -2,\ 2,\ -3,\ 3 \dots$ or noting that $\lim\limits_{n\to\infty} u_n \neq 0$ *etc.* ***R1***

the interval of convergence is $-2 < x < 2$ ***A1***

[10 marks]

Examiners report

Most candidates found the radius of convergence correctly but examining the situation when $x = \pm 2$ often ended in loss of marks through inadequate explanations. In (b)(i) many candidates were able to justify the convergence of the given series. In (b)(ii), however, many candidates seemed unaware of the fact the sum to infinity lies between any pair of successive partial sums.

41b. *[5 marks]*

Markscheme

(i) this alternating series is convergent because the moduli of successive terms are monotonic decreasing ***R1***

and the n^{th} term tends to zero as $n \to \infty$ ***R1***

(ii) consider the partial sums

0.333, 0.111, 0.269, 0.148, 0.246 ***M1A1***

since the sum to infinity lies between any pair of successive partial sums, it follows that the sum to infinity lies between 0.148 and 0.246 so that it is less than 0.25 ***R1***

Note: Accept a solution which looks only at 0.333, 0.269, 0.246 and states that these are successive upper bounds.

[5 marks]

Examiners report

Most candidates found the radius of convergence correctly but examining the situation when $x = \pm 2$ often ended in loss of marks through inadequate explanations. In (b)(i) many candidates were able to justify the convergence of the given series. In (b)(ii), however, many candidates seemed unaware of the fact the sum to infinity lies between any pair of successive partial sums.

42. *[7 marks]*

Markscheme

M1

$(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^2\tan y \Rightarrow \int \frac{\mathrm{d}y}{\tan y} = \int \frac{2x^2}{1+x^3}\mathrm{d}x$

(A1)(A1)

$\int \frac{\cos y}{\sin y}\mathrm{d}y = \frac{2}{3}\int \frac{3x^2}{1+x^3}\mathrm{d}x$

A1A1

$\ln|\sin y| = \frac{2}{3}\ln|1+x^3| + C$

Notes: Do not penalize omission of modulus signs.

Do not penalize omission of constant at this stage.

EITHER

M1

$\ln\left|\sin\frac{\pi}{2}\right| = \frac{2}{3}\ln|1| + C \Rightarrow C = 0$

OR

$|\sin y| = A|1+x^3|^{\frac{2}{3}}$, $A = \mathrm{e}^C$

M1

$\left|\sin\frac{\pi}{2}\right| = A|1+0^3|^{\frac{2}{3}} \Rightarrow A = 1$

THEN

A1

$y = \arcsin\left((1+x^3)^{\frac{2}{3}}\right)$

Note: Award ***M0A0*** if constant omitted earlier.

[7 marks]

Examiners report

Many candidates separated the variables correctly but were then unable to perform the integrations.

43. *[7 marks]*

Markscheme

METHOD 1

$f(0) = \frac{0}{0}$, hence using l'Hôpital's Rule, ***(M1)***

$g(x) = 1 - \cos(x^6),\ h(x) = x^{12};\ \frac{g'(x)}{h'(x)} = \frac{6x^5 \sin(x^6)}{12x^{11}} = \frac{\sin(x^6)}{2x^6}$ ***A1A1***

EITHER

$\frac{g'(0)}{h'(0)} = \frac{0}{0}$, using l'Hôpital's Rule again, ***(M1)***

$\frac{g''(x)}{h''(x)} = \frac{6x^5 \cos(x^6)}{12x^5} = \frac{\cos(x^6)}{2}$ ***A1A1***

$\frac{g''(0)}{h''(0)} = \frac{1}{2}$, hence the limit is $\frac{1}{2}$ ***A1***

OR

So

$\lim\limits_{x\to0} \frac{1-\cos x^6}{x^{12}} = \lim\limits_{x\to0} \frac{\sin x^6}{2x^6}$ ***A1***

$= \frac{1}{2} \lim\limits_{x\to0} \frac{\sin x^6}{2x^6}$ ***A1***

$= \frac{1}{2}$ since $\lim\limits_{x\to0} \frac{\sin x^6}{2x^6} = 1$ ***A1 (R1)***

METHOD 2

substituting x^6 for x in the expansion

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} \ldots$ ***(M1)***

$\frac{1-\cos x^6}{x^{12}} = \frac{1-\left(1-\frac{x^{12}}{2}+\frac{x^{24}}{24}\right)\ldots}{x^{12}}$ ***M1A1***

$= \frac{1}{2} - \frac{x^{12}}{24} + \ldots$ ***A1A1***

$\lim\limits_{x\to0} \frac{1-\cos x^6}{x^{12}} = \frac{1}{2}$ ***M1A1***

Note: Accept solutions using Maclaurin expansions.

[7 marks]

Examiners report

Surprisingly, some weaker candidates were more successful in answering this question than stronger candidates. If candidates failed to simplify the expression after the first application of L'Hôpital's rule, they generally were not successful in correctly differentiating the expression a 2^{nd} time, hence could not achieve the final three A marks.

44.

[16 marks]

Markscheme

(a)

$\sum\limits_{n=0}^{\infty}\left(\sin\frac{n\pi}{2}-\sin\frac{(n+1)\pi}{2}\right)$ ***(M1)***

$=\left(\sin 0-\sin\frac{\pi}{2}\right)+\left(\sin\frac{\pi}{2}-\sin\pi\right)+\left(\sin\pi-\sin\frac{3\pi}{2}\right)+\left(\sin\frac{3\pi}{2}-\sin 2\pi\right)+\ldots$

the n^{th} term is ±1 for all n, *i.e.* the n^{th} term does not tend to 0 ***A1***

hence the series does not converge ***A1***

[3 marks]

(b) **EITHER**

using the ratio test ***(M1)***

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\left(\frac{e^{n+1}}{\pi^{n+1}}\right)\left(\frac{\pi^n}{e^n-1}\right)$ ***M1A1***

$\lim\limits_{n\to\infty}\left(\frac{e^{n+1}-1}{e^n-1}\right)\left(\frac{\pi^n}{\pi^{n+1}}\right)=\frac{e}{\pi}\ (\approx 0.865)$ ***M1A1***

$\frac{e}{\pi}<1$, hence the series converges ***R1A1***

OR

$\sum\limits_{n=1}^{\infty}\frac{e^n-1}{\pi^n}=\sum\limits_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n-\left(\frac{1}{\pi}\right)^n=\sum\limits_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n-\sum\limits_{n=1}^{\infty}\left(\frac{1}{\pi}\right)^n$ ***M1A1***

the series is the difference of two geometric series, with

$r=\frac{e}{\pi}\ (\approx 0.865)$ ***M1A1***

and

$\frac{1}{\pi}\ (\approx 0.318)$ ***A1***

for both $|r|<1$, hence the series converges ***R1A1***

OR

$\forall n,\ 0<\frac{e^n-1}{\pi^n}<\frac{e^n}{\pi^n}$ ***(M1)A1A1***

the series $\frac{e^n}{\pi^n}$ converges since it is a geometric series such that $|r|<1$ ***A1R1***

therefore, by the comparison test, $\frac{e^n-1}{\pi^n}$ converges ***R1A1***

[7 marks]

(c) by limit comparison test with $\frac{\sqrt{n}}{n^2}$, ***(M1)***

$\lim\limits_{n\to\infty}\left(\frac{\frac{\sqrt{n+1}}{n(n-1)}}{\frac{\sqrt{n}}{n^2}}\right)=\lim\limits_{n\to\infty}\left(\frac{\sqrt{n+1}}{n(n-1)}\times\frac{n^2}{\sqrt{n}}\right)=\lim\limits_{n\to\infty}\frac{n}{n-1}\sqrt{\frac{n+1}{n}}=1$ ***M1A1***

hence both series converge or both diverge ***R1***

by the p-test $\sum\limits_{n=1}^{\infty}\frac{\sqrt{n}}{n^2}=n^{-\frac{3}{2}}$ converges, hence both converge ***R1A1***

[6 marks]

Total [16 marks]

Examiners report

This was the least successfully answered question on the paper. Candidates often did not know which convergence test to use; hence very few full successful solutions were seen. The communication of the method used was often quite poor.

a) Many candidates failed to see that this is a telescoping series. If this was recognized then the question was fairly straightforward. Often candidates unsuccessfully attempted to apply the standard convergence tests.

b) Many candidates used the ratio test, but some had difficulty in simplifying the expression. Others recognized that the series is the difference of two geometric series, and although the algebraic work was done correctly, some failed to communicate the conclusion that since the absolute value of the ratios are less than 1, hence the series converges. Some candidates successfully used the comparison test.

c) Although the limit comparison test was attempted by most candidates, it often failed through an inappropriate selection of a series.

45. *[9 marks]*

Markscheme

(a)

$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

putting

$x = \frac{-x^2}{2}$ ***(M1)***

$\mathrm{e}^{-\frac{x^2}{2}} \approx 1 - \frac{x^2}{2} + \frac{x^4}{2^2 \times 2!} - \frac{x^6}{2^3 \times 3!} \approx \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right)$ ***A2***

[3 marks]

(b)

$\int_0^x \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u \approx \left[u - \frac{u^3}{3 \times 2} + \frac{u^5}{5 \times 2^2 \times 2!} - \frac{u^7}{7 \times 2^3 \times 3!}\right]_0^x$ ***M1(A1)***

$= x - \frac{x^3}{3 \times 2} + \frac{x^5}{5 \times 2^2 \times 2!} - \frac{x^7}{7 \times 2^3 \times 3!}$ ***A1***

$\left(= x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right)$

[3 marks]

(c) putting $x = 1$ in part (b) gives

$\int_0^1 \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x \approx 0.85535\ldots$ ***(M1)(A1)***

$\frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x \approx 0.341$ ***A1***

[3 marks]

Total [9 marks]

Examiners report

This was one of the most successfully answered questions. Some candidates however failed to use the data booklet for the expansion of the series, thereby wasting valuable time.

46. *[13 marks]*

Markscheme

writing the differential equation in standard form gives

M1

$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x-1}y = \mathrm{e}^{-x}$

M1A1

$\int \frac{x}{x-1}\mathrm{d}x = \int \left(1 + \frac{1}{x-1}\right)\mathrm{d}x = x + \ln(x-1)$

hence integrating factor is

M1A1

$\mathrm{e}^{x+\ln(x-1)} = (x-1)\mathrm{e}^x$

hence,

(A1)

$(x-1)\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x} + x\mathrm{e}^x y = x - 1$

(A1)

$\Rightarrow \frac{\mathrm{d}[(x-1)\mathrm{e}^x y]}{\mathrm{d}x} = x - 1$

A1

$\Rightarrow (x-1)\mathrm{e}^x y = \int (x-1)\mathrm{d}x$

A1

$\Rightarrow (x-1)\mathrm{e}^x y = \frac{x^2}{2} - x + c$

substituting (0, 1), $c = -1$ ***(M1)A1***

(A1)

$\Rightarrow (x-1)\mathrm{e}^x y = \frac{x^2-2x-2}{2}$

hence,

(or equivalent) ***A1***

$y = \frac{x^2-2x-2}{2(x-1)\mathrm{e}^x}$

[13 marks]

Examiners report

Apart from some candidates who thought the differential equation was homogenous, the others were usually able to make a good start, and found it quite straightforward. Some made errors after identifying the correct integrating factor, and so lost accuracy marks.

47. *[15 marks]*

Markscheme

(a) applying the alternating series test as

$\forall n \geqslant 2, \frac{1}{n \ln n} \in \mathbb{R}^+$ ***M1***

$\forall n, \frac{1}{(n+1)\ln(n+1)} \leqslant \frac{1}{n \ln n}$ ***A1***

$\lim_{n \to \infty} \frac{1}{n \ln n} = 0$ ***A1***

hence, by the alternating series test, the series converges ***R1***

[4 marks]

(b) as $\frac{1}{x \ln x}$ is a continuous decreasing function, apply the integral test to determine if it converges absolutely ***(M1)***

$\int_2^\infty \frac{1}{x \ln x} \mathrm{d}x = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x} \mathrm{d}x$ ***M1A1***

let $u = \ln x$ then $\mathrm{d}u = \frac{1}{x} \mathrm{d}x$ ***(M1)A1***

$\int \frac{1}{u} \mathrm{d}u = \ln u$ ***(A1)***

hence, $\lim_{b \to \infty} \int_2^b \frac{1}{x \ln x} \mathrm{d}x = \lim_{b \to \infty} [\ln(\ln x)]_2^b$ which does not exist ***M1A1A1***

hence, the series does not converge absolutely ***(A1)***

the series converges conditionally ***A1***

[11 marks]

Total [15 marks]

Examiners report

Part (a) was answered well by many candidates who attempted this question. In part (b), those who applied the integral test were mainly successful, but too many failed to supply the justification for its use, and proper conclusions.

48.

[8 marks]

Markscheme

(a) rearrange $\frac{\cos^2 x}{e^y} - e^{e^y}\frac{dy}{dx} = 0$ to obtain $\cos^2 x dx = e^y e^{e^y} dy$ ***(M1)***

as $\int \cos^2 x dx = \int \frac{1+\cos(2x)}{2} dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C_1$ ***M1A1***

and $\int e^y e^{e^y} dy = e^{e^y} + C_2$ ***A1***

Note: The above two integrations are independent and should not be penalized for missing.

a general solution of $\frac{\cos^2 x}{e^y} - e^{e^y}\frac{dy}{dx} = 0$ is $\frac{1}{2}x + \frac{1}{4}\sin(2x) - e^{e^y} = C$ ***A1***

given that $y = 0$ when $x = \pi$, $C = \frac{\pi}{2} + \frac{1}{4}\sin(2\pi) - e^{e^0} = \frac{\pi}{2} - e$ (or -1.15) ***(M1)***

so, the required solution is defined by the equation

$\frac{1}{2}x + \frac{1}{4}\sin(2x) - e^{e^y} = \frac{\pi}{2} - e$ or $y = \ln\left(\ln\left(\frac{1}{2}x + \frac{1}{4}\sin(2x) + e^{e^y} - \frac{\pi}{2}\right)\right)$ (or equivalent) ***A1 N0***

(b) for $x = \frac{\pi}{2}$, $y = \ln\left(\ln\left(e - \frac{\pi}{4}\right)\right)$ (or -0.417) ***A1***

[8 marks]

Examiners report

This was a more difficult question and it was apparent that students did find it so. For those that managed to rearrange the equation to separate the variables, few could manage to successfully integrate both sides. The unfamiliarity of e^{e^y} seemed to disturb some students.

49.

[6 marks]

Markscheme

$\frac{dv}{dt} = -\frac{1}{2}v$ ***A1***

$\int \frac{dv}{v} = \int -\frac{1}{2}dt$ ***(A1)***

$\ln v = -\frac{1}{2}t + c$ ***(A1)***

$v = e^{-\frac{1}{2}t+c}$ $\left(= Ae^{-\frac{1}{2}t}\right)$ ***(A1)***

$t = 0$, $v = 40$ so $A = 40$ ***M1***

$v = 40e^{-\frac{1}{2}t}$ (or equivalent) ***A1***

[6 marks]

Examiners report

This was a poorly answered question which linked the topic of kinematics with that of first order differential equations. Many candidates seemed unaware that the acceleration is the time derivative of the velocity. This was often followed by a failure to recognize a separable differential equation and/or integration with respect to the wrong variable.