Topic 9 Part 2 *[514 marks]*

1a. *[7 marks]*

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}t} = -v^2 - 1$

attempt to separate the variables ***M1***

$\int \frac{1}{1+v^2}\mathrm{d}v = \int -1\mathrm{d}t$ ***A1***

$\arctan v = -t + k$ ***A1A1***

Note: Do not penalize the lack of constant at this stage.

when $t = 0$, $v = 1$ ***M1***

$\Rightarrow k = \arctan 1 = \left(\frac{\pi}{4}\right) = (45^\circ)$ ***A1***

$\Rightarrow v = \tan\left(\frac{\pi}{4} - t\right)$ ***A1***

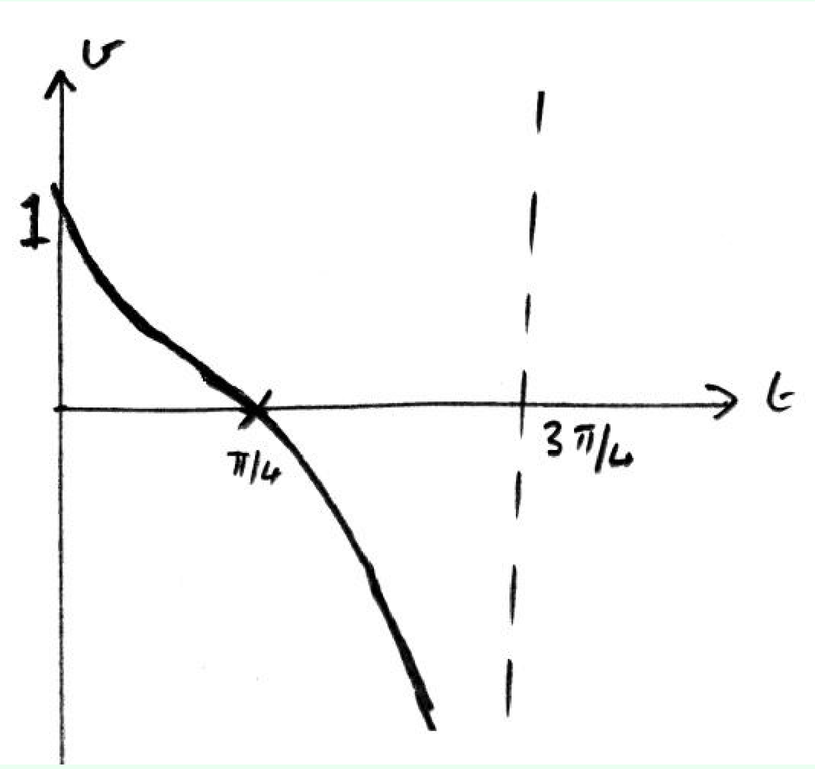
[7 marks]

Examiners report

This proved to be the most challenging question in section B with only a very small number of candidates producing fully correct answers. Many candidates did not realise that part (a) was a differential equation that needed to be solved using a method of separating the variables. Without this, further progress with the question was difficult. For those who did succeed in part (a), parts (b) and (c) were relatively well done. For the minority of candidates who attempted parts (d) and (e) only the best recognised the correct methods.

1b. *[3 marks]*

Markscheme



A1A1A1

Note: Award ***A1*** for general shape,

A1 for asymptote,

A1 for correct t and v intercept.

Note: Do not penalise if a larger domain is used.

[3 marks]

Examiners report

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1c. *[3 marks]*

Markscheme

(i)

$T = \frac{\pi}{4}$ ***A1***

(ii) area under curve

$= \int_0^{\frac{\pi}{4}} \tan\left(\frac{\pi}{4} - t\right) \mathrm{d}t$ ***(M1)***

$= 0.347 \left(= \frac{1}{2} \ln 2\right)$ ***A1***

[3 marks]

Examiners report

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1d. *[5 marks]*

Markscheme

$v = \tan\left(\frac{\pi}{4} - t\right)$

$s = \int \tan\left(\frac{\pi}{4} - t\right) \mathrm{d}t$ ***M1***

$\int \frac{\sin\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)} \mathrm{d}t$ ***(M1)***

$= \ln \cos\left(\frac{\pi}{4} - t\right) + k$ ***A1***

when

$t = 0, \ s = 0$

$k = -\ln \cos \frac{\pi}{4}$ ***A1***

$s = \ln \cos\left(\frac{\pi}{4} - t\right) - \ln \cos \frac{\pi}{4} \left(= \ln\left[\sqrt{2} \cos\left(\frac{\pi}{4} - t\right)\right]\right)$ ***A1***

[5 marks]

Examiners report

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1e. *[4 marks]*

Markscheme

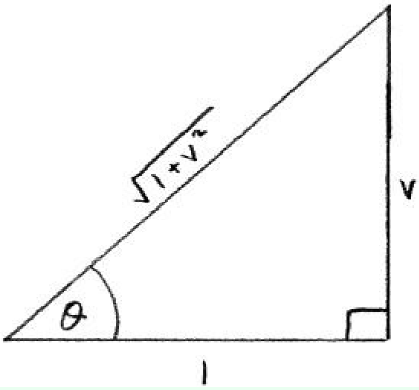
METHOD 1

$\frac{\pi}{4} - t = \arctan v$ ***M1***

$t = \frac{\pi}{4} - \arctan v$

$s = \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{\pi}{4} + \arctan v\right)\right]$

$s = \ln\left[\sqrt{2}\cos(\arctan v)\right]$ ***M1A1***



$s = \ln\left[\sqrt{2}\cos\left(\arccos\frac{1}{\sqrt{1+v^2}}\right)\right]$ ***A1***

$= \ln\frac{\sqrt{2}}{\sqrt{1+v^2}}$

$= \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

METHOD 2

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$

$= -\ln\sec\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1 + \tan^2\left(\frac{\pi}{4} - t\right)} - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1+v^2} - \ln\cos\frac{\pi}{4}$ ***A1***

$= \ln\frac{1}{\sqrt{1+v^2}} + \ln\sqrt{2}$ ***A1***

$= \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

METHOD 3

$v\frac{dv}{ds} = -v^2 - 1$ ***M1***

$\int \frac{v}{v^2+1}dv = -\int 1 ds$ ***M1***

$\frac{1}{2}\ln(v^2+1) = -s + k$ ***A1***

when

$s = 0,\ t = 0 \Rightarrow v = 1$

$\Rightarrow k = \frac{1}{2}\ln 2$ ***A1***

$\Rightarrow s = \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

[4 marks]

Examiners report

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2.

[6 marks]

Markscheme

apply l'Hôpital's Rule to a $0/0$ type limit

$\lim\limits_{x \to 0} \frac{e^x - 1 - x\cos x}{\sin^2 x} = \lim\limits_{x \to 0} \frac{e^x - \cos x + x\sin x}{2\sin x\cos x}$ ***M1A1***

noting this is also a $0/0$ type limit, apply l'Hôpital's Rule again ***(M1)***

obtain

$\lim\limits_{x \to 0} \frac{e^x + \sin x + x\cos x + \sin x}{2\cos 2x}$ ***A1***

substitution of $x = 0$ ***(M1)***

$= 0.5$ ***A1***

[6 marks]

Examiners report

The vast majority of candidates were familiar with L'Hôpitals rule and were also able to apply the technique twice as required by the problem. The errors that occurred were mostly due to difficulty in applying the differentiation rules correctly or errors in algebra. A small minority of candidates tried to use the quotient rule but it seemed that most candidates had a good understanding of L'Hôpital's rule and its application to finding a limit.

3a.

[7 marks]

Markscheme

attempt the first step of $y_{n+1} = y_n + (0.1)f(x_n, y_n)$ with $y_0 = 1,\ x_0 = 0$ ***(M1)***

$y_1 = 1.1$ ***A1***

$y_2 = 1.1 + (0.1)\frac{1.1^2}{1.1} = 1.21$ ***(M1)A1***

$y_3 = 1.332(0)$ ***(A1)***

$y_4 = 1.4685$ ***(A1)***

$y_5 = 1.62$ ***A1***

[7 marks]

Examiners report

Most candidates had a good knowledge of Euler's method and were confident in applying it to the differential equation in part (a). A few candidates who knew the Euler's method completed one iteration too many to arrive at an incorrect answer but this was rare. Nearly all candidates who applied the correct technique in part (a) correctly calculated the answer. Most candidates were able to attempt part (b) but some lost marks due to a lack of rigour by not clearly showing the implicit differentiation in the first line of working. Part (c) was reasonably well attempted by many candidates and many could solve the integrals although some did not find the arbitrary constant meaning that it was not possible to solve (ii) of the part (c).

3b.

[8 marks]

Markscheme

(i) recognition of both quotient rule and implicit differentiation ***M1***

$\frac{d^2y}{dx^2} = \frac{(1+x)2y\frac{dy}{dx} - y^2 \times 1}{(1+x)^2}$ ***A1A1***

Note: Award ***A1*** for first term in numerator, ***A1*** for everything else correct.

$= \frac{(1+x)2y\frac{y^2}{1+x} - y^2 \times 1}{(1+x)^2}$ ***M1A1***

$= \frac{2y^3 - y^2}{(1+x)^2}$ ***AG***

(ii) attempt to use $y = y(0) + x\frac{dy}{dx}(0) + \frac{x^2}{2!}\frac{d^2y}{dx^2}(0) + \ldots$ ***(M1)***

$= 1 + x + \frac{x^2}{2}$ ***A1A1***

Note: Award ***A1*** for correct evaluation of $y(0),\ \frac{dy}{dx}(0),\ \frac{d^2y}{dx^2}(0)$, ***A1*** for correct series.

[8 marks]

Examiners report

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3c. *[6 marks]*

Markscheme

(i) separating the variables

$\int \frac{1}{y^2}\mathrm{d}y = \int \frac{1}{1+x}\mathrm{d}x$ ***M1***

obtain

$-\frac{1}{y} = \ln(1+x) + (c)$ ***A1***

impose initial condition

$-1 = \ln 1 + c$ ***M1***

obtain

$y = \frac{1}{1-\ln(1+x)}$ ***A1***

(ii)

if $y \to \infty$ $\ln(1+x) \to 1$, so a = e – 1 ***(M1)A1***

Note: To award ***A1*** must see either $x \to e - 1$ or $a = e - 1$. Do not accept $x = e - 1$.

[6 marks]

Examiners report

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4. *[7 marks]*

Markscheme

recognise equation as first order linear and attempt to find the IF ***M1***

$\mathrm{IF} = \mathrm{e}^{\int \frac{2}{t}\mathrm{d}t} = t^2$ ***A1***

solution

$yt^2 = \int t\cos t\,\mathrm{d}t$ ***M1A1***

using integration by parts with the correct choice of u and v ***(M1)***

$\int t\cos t\,\mathrm{d}t = t\sin t + \cos t(+C)$ ***A1***

obtain

$y = \frac{\sin t}{t} + \frac{\cos t + C}{t^2}$ ***A1***

[7 marks]

Examiners report

Perhaps a small number of candidates were put off by the unusual choice of variables but in most instances it seemed that candidates who recognised the need for an integration factor could make a good attempt at this problem. Candidates who were not able to simplify the integrating factor from $e^{2\ln t}$ to t^2 rarely gained full marks. A significant number of candidates did not gain the final mark due to a lack of an arbitrary constant or not dividing the constant by the integration factor.

5a.

[3 marks]

Markscheme

$u_n = \frac{3+\frac{2}{n}}{2-\frac{1}{n}}$ or $\frac{3}{2} + \frac{A}{2n-1}$ ***M1***

using $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ ***(M1)***

obtain $\lim\limits_{n\to\infty} u_n = \frac{3}{2} = L$ ***A1 N1***

[3 marks]

Examiners report

The “show that” in part (a) of this problem was not adequately dealt with by a significant minority of candidates and simply stating the limit and not demonstrating its existence lost marks. Part (b), whilst being possible without significant knowledge of limits, seemed to intimidate some candidates due to its unfamiliarity and the notation. Part (c) was somewhat disappointing as many candidates attempted to apply rules on the convergence of series to solve a problem that was dealing with the limits of sequences. The same confusion was seen on part (d) where also some errors in algebra prevented candidates from achieving full marks.

5b.

[4 marks]

Markscheme

$u_n - L = \frac{7}{2(2n-1)}$ ***(A1)***

$|u_n - L| < \varepsilon \Rightarrow n > \frac{1}{2}\left(1 + \frac{7}{2\varepsilon}\right)$ ***(M1)***

(i)

$\varepsilon = 0.1 \Rightarrow N = 18$ ***A1***

(ii)

$\varepsilon = 0.00001 \Rightarrow N = 175000$ ***A1***

[4 marks]

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5c.

[6 marks]

Markscheme

$u_n \to L$ and $\frac{1}{n} \to 0$ ***M1***

$\Rightarrow \frac{u_n}{n} \to (L \times 0) = 0$, hence converges ***A1***

$2u_n - 2 \to 2L - 2 = 1 \Rightarrow \frac{1}{2u_n - 2} \to 1$, hence converges ***M1A1***

Note: To award ***A1*** the value of the limit and a statement of convergence must be clearly seen for each sequence.

$(-1)^n u_n$ does not converge ***A1***

The sequence alternates (or equivalent wording) between values close to $\pm L$ ***R1***

[6 marks]

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5d.

[2 marks]

Markscheme

$u_n - L > \frac{7}{4n}$ (re: harmonic sequence) ***M1***

$\Rightarrow \sum_{n=1}^{\infty}(u_n - L)$ diverges by the comparison theorem ***R1***

Note: Accept alternative methods.

[2 marks]

Examiners report

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6a.

[6 marks]

Markscheme

consider the limit as $R \to \infty$ of the (proper) integral

$\int_2^R \frac{\mathrm{d}x}{x(\ln x)^k}$ ***(M1)***

substitute $u = \ln x,\ \mathrm{d}u = \frac{1}{x}\mathrm{d}x$ ***(M1)***

obtain $\int_{\ln 2}^{\ln R} \frac{1}{u^k}\mathrm{d}u = \left[-\frac{1}{k-1}\frac{1}{u^{k-1}}\right]_{\ln 2}^{\ln R}$ ***A1***

Note: Ignore incorrect limits or omission of limits at this stage.

or $[\ln u]_{\ln 2}^{\ln R}$ if $k = 1$ ***A1***

Note: Ignore incorrect limits or omission of limits at this stage.

because $\ln R$ (and $\ln\ln R$) $\to \infty$ as $R \to \infty$ ***(M1)***

converges in the limit if $k > 1$ ***A1***

[6 marks]

Examiners report

A good number of candidates were able to find the integral in part (a) although the vast majority did not consider separately the integral when $k = 1$. Many candidates did not explicitly set a limit for the integral to let this limit go to infinity in the anti – derivative and it seemed that some candidates were “substituting for infinity”. This did not always prevent candidates finding a correct final answer but the lack of good technique is a concern. In part (b) many candidates seemed to have some knowledge of the relevant test for convergence but this test was not always rigorously applied. In showing that the series was not absolutely convergent candidates were often not clear in showing that the function being tested had to meet a number of criteria and in so doing lost marks.

6b.

[5 marks]

Markscheme

C:

terms $\to 0$ as $r \to \infty$ ***A1***

$|u_{n+1}| < |u_n|$ for all r ***A1***

convergence by alternating series test ***R1***

AC:

$(x \ln x)^{-1}$ is positive and decreasing on $[2, \infty)$ ***A1***

not absolutely convergent by integral test using part (a) for $k = 1$ ***R1***

[5 marks]

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7a. *[5 marks]*

Markscheme

apply the limit comparison test with

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ ***M1***

$\lim_{n \to \infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{n(n+1)} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} = 1$ ***M1A1***

(since the limit is finite and $\neq 0$) both series do the same ***R1***

we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and hence $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ also converges ***R1AG***

[5 marks]

Examiners report

Candidates and teachers need to be aware that the Limit comparison test is distinct from the comparison test. Quite a number of candidates lost most of the marks for this part by doing the wrong test.

Some candidates failed to state that because the result was finite and not equal to zero then the two series converge or diverge together. Others forgot to state, with a reason, that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

7b. *[3 marks]*

Markscheme

$(1+x)\ln(1+x) = (1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\ldots\right)$ ***A1***

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\ldots\right) + \left(x^2 - \frac{x3}{2} + \frac{x^4}{3} - \frac{x^5}{4}\ldots\right)$

EITHER

$= x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n}$ ***A1***

$= x + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n+1} \left(\frac{-1}{n+1} + \frac{1}{n}\right)$ ***M1***

OR

$x + \left(1 - \frac{1}{2}\right)x^2 - \left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{1}{3} - \frac{1}{4}\right)x^4 - \ldots$ ***A1***

$= x + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n+1} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ ***M1***

$= x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)}$ ***AG***

[3 marks]

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Some candidates failed to state that because the result was finite and not equal to zero then the two series converge or diverge together. Others forgot to state, with a reason, that

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In part (b) finding the partial fractions was well done. The second part involving the use of telescoping series was less well done, and students were clearly not as familiar with this technique as with some others.

Part (c) was the least well done of all the questions. It was expected that students would use explicitly the result from the first part of 4(b) or show it once again in order to give a complete answer to this question, rather than just assuming that a pattern spotted in the first few terms would continue.

Candidates need to be informed that unless specifically told otherwise they may use without proof any of the Maclaurin expansions given in the Information Booklet. There were many candidates who lost time and gained no marks by trying to derive the expansion for

$\ln(1+x)$.

8a. *[8 marks]*

Markscheme

prove that

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}$

for $n = 1$

$\text{LHS}=1,\ \text{RHS}=4-\frac{1+2}{2^0}=4-3=1$

so true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

so

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

now for $n = k + 1$

LHS:

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$ ***A1***

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

(or equivalent)

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$ ***A1***

(accept $=4-\frac{(k+1)+2}{2^{(k+1)-1}}$)

$4-\frac{k+3}{2^k}$ ***A1***

Therefore if it is true for $n = k$ it is true for $n = k + 1$. It has been shown to be true for $n = 1$ so it is true for all $n\ (\in \mathbb{Z}^+)$. ***R1***

Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.

[8 marks]

Examiners report

Part A: Given that this question is at the easier end of the ‘proof by induction’ spectrum, it was disappointing that so many candidates failed to score full marks. The $n = 1$ case was generally well done. The whole point of the method is that it involves logic, so ‘let n = k’ or ‘put n = k’, instead of ‘assume ... to be true for n = k’, gains no marks. The algebraic steps need to be more convincing than some candidates were able to show. It is astonishing that the R1 mark for the final statement was so often not awarded.

8b. [17 marks]

Markscheme

(a)

METHOD 1

M1A1A1

$\int e^{2x}\sin x dx = -\cos x e^{2x} + \int 2e^{2x}\cos x dx$

A1A1

$= -\cos x e^{2x} + 2e^{2x}\sin x - \int 4e^{2x}\sin x dx$

M1

$5\int e^{2x}\sin x dx = -\cos x e^{2x} + 2e^{2x}\sin x$

AG

$\int e^{2x}\sin x dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$

METHOD 2

M1A1A1

$\int \sin x e^{2x} dx = \frac{\sin x e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2} dx$

A1A1

$= \frac{\sin x e^{2x}}{2} - \cos x \frac{e^{2x}}{4} - \int \sin x \frac{e^{2x}}{4} dx$

M1

$\frac{5}{4}\int e^{2x}\sin x dx = \frac{e^{2x}\sin x}{2} - \frac{\cos x e^{2x}}{4}$

AG

$\int e^{2x}\sin x dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$

[6 marks]

(b)

M1A1

$\int \frac{dy}{\sqrt{1-y^2}} = \int e^{2x}\sin x dx$

A1

$\arcsin y = \frac{1}{5}e^{2x}(2\sin x - \cos x)(+C)$

when

M1

$x = 0,\ y = 0 \Rightarrow C = \frac{1}{5}$

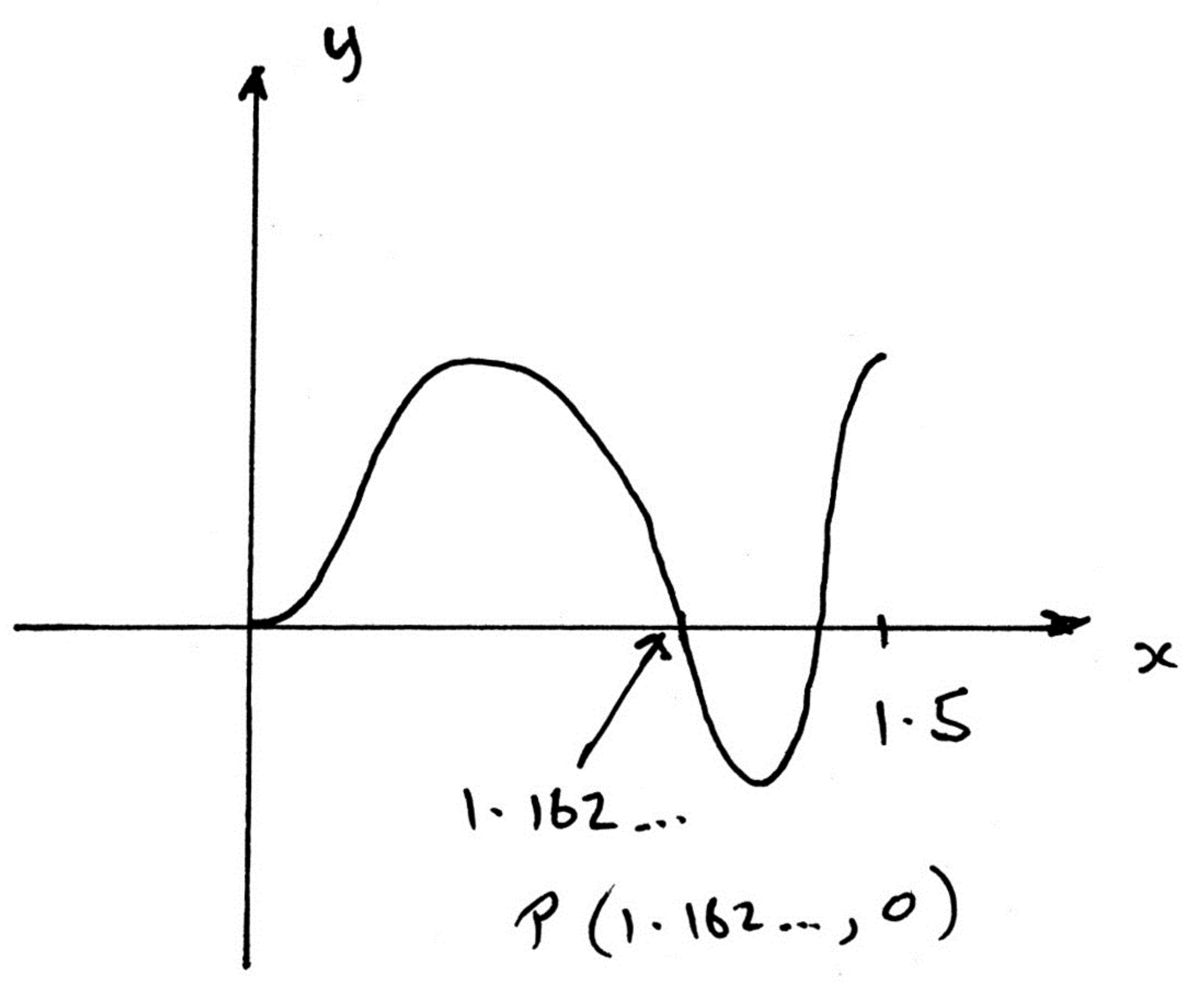
A1

$y = \sin\left(\frac{1}{5}e^{2x}(2\sin x - \cos x) + \frac{1}{5}\right)$

[5 marks]

(c)

(i)



A1

P is (1.16, 0) ***A1***

Note: Award ***A1*** for 1.16 seen anywhere, ***A1*** for complete sketch.

Note: Allow FT on their answer from (b)

(ii)

$V = \int_0^{1.162\ldots} \pi y^2 \mathrm{d}x$ ***M1A1***

$= 1.05$ ***A2***

Note: Allow FT on their answers from (b) and (c)(i).

[6 marks]

Examiners report

Part B: Part (a) was often well done, although some faltered after the first integration. Part (b) was also generally well done, although there were some errors with the constant of integration. In (c) the graph was often attempted, but errors in (b) usually led to manifestly incorrect plots. Many attempted the volume of integration and some obtained the correct value.

9a. *[6 marks]*

Markscheme

METHOD 1

A1
$f(x) = \ln(1 + e^x);\ f(0) = \ln 2$

A1
$f'(x) = \frac{e^x}{1+e^x};\ f'(0) = \frac{1}{2}$

Note: Award ***A0*** for

$f'(x) = \frac{1}{1+e^x};\ f'(0) = \frac{1}{2}$

M1A1
$f''(x) = \frac{e^x(1+e^x) - e^{2x}}{(1+e^x)^2};\ f''(0) = \frac{1}{4}$

Note: Award ***M0A0*** for

$f''(x)$ if $f'(x) = \frac{1}{1+e^x}$ is used

M1A1
$\ln(1 + e^x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$

[6 marks]

METHOD 2

M1A1
$\ln(1 + e^x) = \ln(1 + 1 + x + \frac{1}{2}x^2 + \ldots)$

A1
$= \ln 2 + \ln(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \ldots)$

A1
$= \ln 2 + \left(\frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right) - \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{4}x^2 + \ldots\right)^2 + \ldots$

A1
$= \ln 2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + \ldots$

A1
$= \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$

[6 marks]

Examiners report

In (a), candidates who found the series by successive differentiation were generally successful, the most common error being to state that the derivative of $\ln(1 + e^x)$ is $(1 + e^x)^{-1}$. Some candidates assumed the series for $\ln(1 + x)$ and e^x attempted to combine them. This was accepted as an alternative solution but candidates using this method were often unable to obtain the required series.

9b. *[4 marks]*

Markscheme

METHOD 1

$\lim_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2} = \lim_{x\to 0}\frac{2\ln 2+x+\frac{x^2}{4}+x^3 \text{ terms \& above}-x-\ln 4}{x^2}$ ***M1A1***

$= \lim_{x\to 0}\left(\frac{1}{4}+\text{powers of } x\right)=\frac{1}{4}$ ***M1A1***

Note: Accept $+\ldots$ as evidence of recognition of cubic and higher powers.

Note: Award ***M1A0M1A0*** for a solution which omits the cubic and higher powers.

[4 marks]

METHOD 2

using l'Hôpital's Rule

$\lim_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2} = \lim_{x\to 0}\frac{2e^x\div(1+e^x)-1}{2x}$ ***M1A1***

$= \lim_{x\to 0}\frac{2e^x\div(1+e^x)^2}{2}=\frac{1}{4}$ ***M1A1***

[4 marks]

Examiners report

In (b), candidates were equally split between using the series or using l'Hopital's rule to find the limit. Both methods were fairly successful, but a number of candidates forgot that if a series was used, there had to be a recognition that it was not a finite series.

10a. *[7 marks]*

Markscheme

use of $y\to y+h\frac{dy}{dx}$ ***(M1)***

x	y	$\frac{dy}{dx}$	$h\frac{dy}{dx}$	
0	1	1	0.1	***A1***
0.1	1.1	1.22	0.122	***A1***
0.2	1.222	1.533284	0.1533284	***A1***
0.3	1.3753284	1.981528208	0.1981528208	***A1***
0.4	1.573481221			***(A1)***

approximate value of $y = 1.57$ ***A1***

Note: Accept values in the tables correct to 3 significant figures.

[7 marks]

Examiners report

Most candidates were familiar with Euler's method. The most common way of losing marks was either to round intermediate answers to insufficient accuracy or simply to make an arithmetic error. Many candidates were given an accuracy penalty for not rounding their answer to three significant figures. Few candidates were able to answer (b) correctly with most believing incorrectly that the step length was a relevant factor.

10b.

[1 mark]

Markscheme

the approximate value is less than the actual value because it is assumed that $\frac{dy}{dx}$ remains constant throughout each interval whereas it is actually an increasing function ***R1***

[1 mark]

Examiners report

Most candidates were familiar with Euler's method. The most common way of losing marks was either to round intermediate answers to insufficient accuracy or simply to make an arithmetic error. Many candidates were given an accuracy penalty for not rounding their answer to three significant figures. Few candidates were able to answer (b) correctly with most believing incorrectly that the step length was a relevant factor.

11a.

[3 marks]

Markscheme

$\frac{dy}{dx} = \frac{e}{\ln e}(2+2) = 4e$ ***A1***

at (2, e) the tangent line is

$y - e = 4e(x-2)$ ***M1***

hence

$y = 4ex - 7e$ ***A1***

[3 marks]

Examiners report

Nearly always correctly answered.

11b. *[11 marks]*

Markscheme

M1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y}\mathrm{d}y = (x+2)\mathrm{d}x$

$\int \frac{\ln y}{y}\mathrm{d}y = \int (x+2)\mathrm{d}x$

using substitution

(M1)(A1)

$u = \ln y;\ \mathrm{d}u = \frac{1}{y}\mathrm{d}y$

(A1)

$\Rightarrow \int \frac{\ln y}{y}\mathrm{d}y = \int u\mathrm{d}u = \frac{1}{2}u^2$

A1A1

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$

at (2, e),

M1

$\frac{(\ln e)^2}{2} = 6 + c$

A1

$\Rightarrow c = -\frac{11}{2}$

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$

M1A1

$\ln y = \pm\sqrt{x^2 + 4x - 11} \Rightarrow y = e^{\pm\sqrt{x^2+4x-11}}$

since y > 1,

R1

$f(x) = e^{\sqrt{x^2+4x-11}}$

Note: ***M1*** for attempt to make *y* the subject.

[11 marks]

Examiners report

Most candidates separated the variables and attempted the integrals. Very few candidates made use of the condition *y* > 1, so losing 2 marks.

11c. *[6 marks]*

Markscheme

EITHER

A1

$x^2 + 4x - 11 > 0$

using the quadratic formula ***M1***

critical values are

A1

$\frac{-4\pm\sqrt{60}}{2}\ \left(= -2 \pm \sqrt{15}\right)$

using a sign diagram or algebraic solution ***M1***

A1A1

$x < -2 - \sqrt{15};\ x > -2 + \sqrt{15}$

OR

A1

$x^2 + 4x - 11 > 0$

by methods of completing the square ***M1***

A1

$(x+2)^2 > 15$

(M1)

$\Rightarrow x + 2 < -\sqrt{15}$ or $x + 2 > \sqrt{15}$

A1A1

$x < -2 - \sqrt{15};\ x > -2 + \sqrt{15}$

[6 marks]

Examiners report

Part (c) was often well answered, sometimes with follow through.

11d. *[4 marks]*

Markscheme

M1

$f(x) = f'(x) \Rightarrow f(x) = \frac{f(x)}{\ln f(x)}(x+2)$

A1

$\Rightarrow \ln(f(x)) = x+2 \quad \left(\Rightarrow x+2 = \sqrt{x^2+4x-11}\right)$

A1

$\Rightarrow (x+2)^2 = x^2+4x-11 \Rightarrow x^2+4x+4 = x^2+4x-11$

R1AG

$\Rightarrow 4 = -11$, hence $f(x) \neq f'(x)$

[4 marks]

Examiners report

Only the best candidates were successful on part (d).

12. *[5 marks]*

Markscheme

using l'Hôpital's Rule ***(M1)***

A1A1

$\lim\limits_{x \to \frac{1}{2}} \left(\frac{\left(\frac{1}{4} - x^2\right)}{\cot \pi x} \right) = \lim\limits_{x \to \frac{1}{2}} \left[\frac{-2}{-\pi \text{cosec}^2 \pi x} \right]$

(M1)A1

$= \frac{-1}{-\pi \text{cosec}^2 \frac{\pi}{2}} = \frac{1}{\pi}$

[5 marks]

Examiners report

This question was accessible to the vast majority of candidates, who recognised that L'Hôpital's rule was required. However, some candidates omitted the factor π in the differentiation of $\cot \pi x$. Some candidates replaced $\cot \pi x$ by $\cos \pi x / \sin \pi x$, which is a valid method but the extra algebra involved often led to an incorrect answer. Many fully correct solutions were seen.

13a. *[2 marks]*

Markscheme

for

M1A1

$n \geqslant 1,\ n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1 \geqslant 2 \times 2 \times 2 \ldots 2 \times 2 \times 1 = 2^{n-1}$

AG

$\Rightarrow n! \geqslant 2^{n-1}$ for $n \geqslant 1$

[2 marks]

Examiners report

Part (a) of this question was found challenging by the majority of candidates, a fairly common ‘solution’ being that the result is true for $n = 1, 2, 3$ and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between

$\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

13b. *[3 marks]*

Markscheme

$n! \geqslant 2^{n-1} \Rightarrow \frac{1}{n!} \leqslant \frac{1}{2^{n-1}}$ for $n \geqslant 1$ ***A1***

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a positive converging geometric series ***R1***

hence $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test ***R1***

[3 marks]

Examiners report

Part (a) of this question was found challenging by the majority of candidates, a fairly common ‘solution’ being that the result is true for $n = 1, 2, 3$ and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between

$\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

14a. *[7 marks]*

Markscheme

using the ratio test (and absolute convergence implies convergence) ***(M1)***

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n x^n}{(n)2^n}}\right|$$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\times x^{n+1}\times n\times 2^n}{(-1)^n\times (n+1)\times 2^{n+1}\times x^n}\right|$$ ***(M1)***

$$= \lim_{n\to\infty}\frac{n}{2(n+1)}\left|x\right|$$ ***A1***

$$= \frac{|x|}{2}$$

for convergence we require

$$\frac{|x|}{2} < 1$$ ***M1***

$$\Rightarrow |x| < 2$$

hence radius of convergence is 2 ***A1***

[7 marks]

Examiners report

Most candidates were able to start (a) and a majority gained a fully correct answer. A number of candidates were careless with using the absolute value sign and with dealing with the negative signs and in the more extreme cases this led to candidates being penalised. Part (b) caused more difficulties, with many candidates appearing to know what to do, but then not succeeding in doing it or in not understanding the significance of the answer gained.

14b. *[4 marks]*

Markscheme

we now need to consider what happens when

$$x = \pm 2$$ ***(M1)***

when $x = 2$ we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ which is convergent (by the alternating series test) ***A1***

when $x = -2$ we have

$$\sum_{n=1}^{\infty}\frac{1}{n}$$ which is divergent ***A1***

hence interval of convergence is

$$]-2,\ 2]$$ ***A1***

[4 marks]

Examiners report

Most candidates were able to start (a) and a majority gained a fully correct answer. A number of candidates were careless with using the absolute value sign and with dealing with the negative signs and in the more extreme cases this led to candidates being penalised. Part (b) caused more difficulties, with many candidates appearing to know what to do, but then not succeeding in doing it or in not understanding the significance of the answer gained.

15a.

[4 marks]

Markscheme

(M1)(A1)

$\int \frac{1}{4x^2+1}\mathrm{d}x = \frac{1}{2}\arctan 2x + k$

Note: Do not penalize the absence of "+*k*".

(M1)

$\int_1^{\infty} \frac{1}{4x^2+1}\mathrm{d}x = \frac{1}{2}\lim_{a\to\infty}[\arctan 2x]_1^a$

Note: Accept

.

$\frac{1}{2}[\arctan 2x]_1^{\infty}$

A1

$= \frac{1}{2}\left(\frac{\pi}{2} - \arctan 2\right) \quad (= 0.232)$

hence the series converges ***AG***

[4 marks]

Examiners report

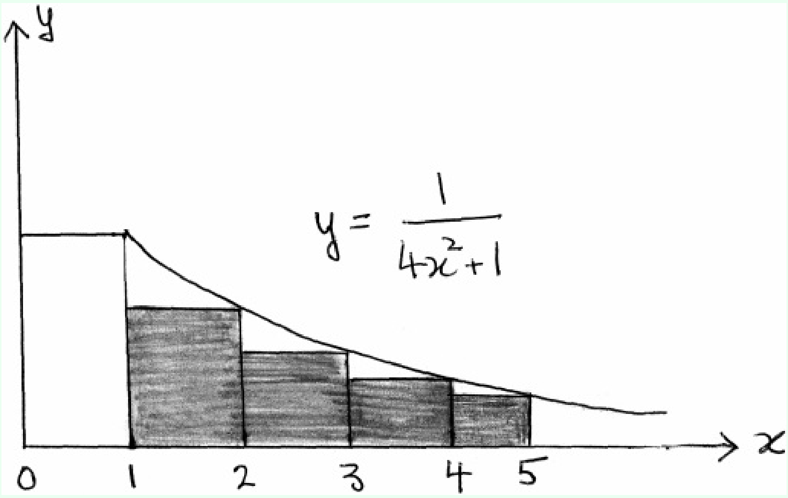
This proved to be a hard question for most candidates. A number of fully correct answers to (a) were seen, but a significant number were unable to integrate

$\frac{1}{4x^2+1}$ successfully. Part (b) was found the hardest by candidates with most candidates unable to draw a relevant diagram, without which the proof of the inequality was virtually impossible.

15b. *[4 marks]*

Markscheme

(i)



A2

The shaded rectangles lie within the area below the graph so that

$\sum\limits_{n=2}^{\infty}\frac{1}{4n^2+1}<\int_1^{\infty}\frac{1}{4x^2+1}\mathrm{d}x$. Adding the first term in the series, $\frac{1}{4\times1^2+1}$, gives

$\sum\limits_{n=1}^{\infty}\frac{1}{4n^2+1}<\frac{1}{4\times1^2+1}+\int_1^{\infty}\frac{1}{4x^2+1}\mathrm{d}x$ ***R1AG***

(ii) upper bound

$=\frac{1}{5}+\frac{1}{2}\left(\frac{\pi}{2}-\arctan 2\right)\quad(=0.432)$ ***A1***

[4 marks]

Examiners report

This proved to be a hard question for most candidates. A number of fully correct answers to (a) were seen, but a significant number were unable to integrate $\frac{1}{4x^2+1}$ successfully. Part (b) was found the hardest by candidates with most candidates unable to draw a relevant diagram, without which the proof of the inequality was virtually impossible.

16a. *[5 marks]*

Markscheme

METHOD 1

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$

$\frac{dy}{dx} = \frac{-2e^{-x}}{2(1+e^{-x})} = \frac{-e^{-x}}{1+e^{-x}}$ ***M1A1***

now

$\frac{1+e^{-x}}{2} = e^{y}$ ***M1***

$\Rightarrow 1 + e^{-x} = 2e^{y}$

$\Rightarrow e^{-x} = 2e^{y} - 1$ ***(A1)***

$\Rightarrow \frac{dy}{dx} = \frac{-2e^{y}+1}{2e^{y}}$ ***(A1)***

Note: Only one of the two above ***A1*** marks may be implied.

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} = -1$ ***AG***

Note: Candidates may find

$\frac{dy}{dx}$ as a function of x and then work backwards from the given answer. Award full marks if done correctly.

METHOD 2

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$

$\Rightarrow e^{y} = \frac{1+e^{-x}}{2}$ ***M1***

$\Rightarrow e^{-x} = 2e^{y} - 1$

$\Rightarrow x = -\ln(2e^{y} - 1)$ ***A1***

$\Rightarrow \frac{dx}{dy} = -\frac{1}{2e^{y}-1} \times 2e^{y}$ ***M1A1***

$\Rightarrow \frac{dy}{dx} = \frac{2e^{y}-1}{-2e^{y}}$ ***A1***

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} - 1$ ***AG***

[5 marks]

Examiners report

Many candidates were successful in (a) with a variety of methods seen. In (b) the use of the chain rule was often omitted when differentiating

e^{-y} with respect to *x*. A number of candidates tried to repeatedly differentiate the original expression, which was not what was asked for, although partial credit was given for this. In this case, they often found problems in simplifying the algebra.

16b. *[11 marks]*

Markscheme

METHOD 1

when

A1
$x = 0,\ y = \ln 1 = 0$

when

A1
$x = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} - 1 = -\frac{1}{2}$

M1A1
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\mathrm{e}^{-y}}{2}\frac{\mathrm{d}y}{\mathrm{d}x}$

when

A1
$x = 0,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

M1A1A1
$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{e}^{-y}}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{\mathrm{e}^{-y}}{2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

when

A1
$x = 0,\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{1}{2} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{4} = 0$

$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

(M1)A1
$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots$

two of the above terms are zero ***AG***

METHOD 2

when

A1
$x = 0,\ y = \ln 1 = 0$

when

A1
$x = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} - 1 = -\frac{1}{2}$

M1A1
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-\mathrm{e}^{-y}}{2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\mathrm{e}^{-y}}{2}\left(\frac{\mathrm{e}^{-y}}{2} - 1\right) = \frac{-\mathrm{e}^{2y}}{4} + \frac{\mathrm{e}^{-y}}{2}$

when

A1
$x = 0,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

M1A1A1
$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \left(\frac{\mathrm{e}^{-2y}}{2} - \frac{\mathrm{e}^{-y}}{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$

when

A1
$x = 0,\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\frac{1}{2} \times \left(\frac{1}{2} - \frac{1}{2}\right) = 0$

$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

(M1)A1
$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots$

two of the above terms are zero ***AG***

[11 marks]

Examiners report

Many candidates were successful in (a) with a variety of methods seen. In (b) the use of the chain rule was often omitted when differentiating

e^{-y} with respect to x. A number of candidates tried to repeatedly differentiate the original expression, which was not what was asked for, although partial credit was given for this. In this case, they often found problems in simplifying the algebra.

17. *[15 marks]*

Markscheme

$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+(x-y)=0$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-x}{x+y}$

let

M1

$y=vx$

A1

$\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$

(A1)

$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{vx-x}{x+vx}$

A1

$x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v-1}{v+1}-v=\frac{v-1-v^2-v}{v+1}=\frac{-1-v^2}{1+v}$

M1

$\int\frac{v+1}{1+v^2}\mathrm{d}v=-\int\frac{1}{x}\mathrm{d}x$

M1

$\int\frac{v}{1+v^2}\mathrm{d}v+\int\frac{1}{1+v^2}\mathrm{d}v=-\int\frac{1}{x}\mathrm{d}x$

A1A1

$\Rightarrow \frac{1}{2}\ln\left|1+v^2\right|+\arctan v=-\ln|x|+k$

Notes: Award ***A1*** for $\frac{1}{2}\ln\left|1+v^2\right|$, ***A1*** for the other two terms.

Do not penalize missing k or missing modulus signs at this stage.

M1

$\Rightarrow \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right|+\arctan\frac{y}{x}=-\ln|x|+k$

(M1)

$\Rightarrow \frac{1}{2}\ln 4+\arctan\sqrt{3}=-\ln 1+k$

A1

$\Rightarrow k=\ln 2+\frac{\pi}{3}$

$\Rightarrow \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right|+\arctan\frac{y}{x}=-\ln|x|+\ln 2+\frac{\pi}{3}$

attempt to combine logarithms ***M1***

(A1)

$\Rightarrow \frac{1}{2}\ln\left|\frac{y^2+x^2}{x^2}\right|+\frac{1}{2}\ln\left|x^2\right|=\ln 2+\frac{\pi}{3}-\arctan\frac{y}{x}$

(A1)

$\Rightarrow \frac{1}{2}\ln\left|y^2+x^2\right|=\ln 2+\frac{\pi}{3}-\arctan\frac{y}{x}$

$\Rightarrow \sqrt{y^2+x^2}=\mathrm{e}^{\ln 2+\frac{\pi}{3}\arctan\frac{y}{x}}$

A1

$\Rightarrow \sqrt{y^2+x^2}=\mathrm{e}^{\ln 2}\times\mathrm{e}^{\frac{\pi}{3}-\arctan\frac{y}{x}}$

AG

$\Rightarrow r=2\mathrm{e}^{\frac{\pi}{3}-\theta}$

[15 marks]

Examiners report

Most candidates realised that this was a homogeneous differential equation and that the substitution $y = vx$ was the way forward. Many of these candidates reached as far as separating the variables correctly but integrating $\frac{v+1}{v^2+1}$ proved to be too difficult for many candidates – most failed to realise that the expression had to be split into two separate integrals. Some candidates successfully evaluated the arbitrary constant but the combination of logs and the subsequent algebra necessary to obtain the final result proved to be beyond the majority of candidates.

18a. *[3 marks]*

Markscheme

volume $= \pi \int_0^h x^2 \mathrm{d}y$ ***(M1)***

$\pi \int_0^h y \mathrm{d}y$ ***M1***

$\pi \left[\frac{y^2}{2}\right]_0^h = \frac{\pi h^2}{2}$ ***A1***

[3 marks]

Examiners report

This question was found to be challenging by many candidates and there were very few completely correct solutions. Many candidates did not seem able to find the volume of revolution when taken about the *y*-axis in (a). Candidates did not always recognize that part (b) did not involve related rates. Those candidates who attempted the question made some progress by separating the variables and integrating in (c) but very few were able to identify successfully the values necessary to find the correct answer.

18b. *[6 marks]*

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3 \times$ surface area ***A1***

surface area $= \pi x^2$ ***(M1)***

$= \pi h$ ***A1***

$V = \frac{\pi h^2}{2} \Rightarrow h\sqrt{\frac{2V}{\pi}}$ ***M1A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\pi\sqrt{\frac{2V}{\pi}}$ ***A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$ ***AG***

Note: Assuming that $\frac{\mathrm{d}h}{\mathrm{d}t} = -3$ without justification gains no marks.

[6 marks]

Examiners report

This question was found to be challenging by many candidates and there were very few completely correct solutions. Many candidates did not seem able to find the volume of revolution when taken about the *y*-axis in (a). Candidates did not always recognize that part (b) did not involve related rates. Those candidates who attempted the question made some progress by separating the variables and integrating in (c) but very few were able to identify successfully the values necessary to find the correct answer.

18c. *[7 marks]*

Markscheme

$V_0 = 5000\pi$ (cm^3) ***A1***

$= 15700$

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$

attempting to separate variables ***M1***

EITHER

$\int \frac{\mathrm{d}V}{\sqrt{V}} = -3\sqrt{2\pi}\int \mathrm{d}t$ ***A1***

$2\sqrt{V} = -3\sqrt{2\pi}t + c$ ***A1***

$c = 2\sqrt{5000\pi}$ ***A1***

$V = 0$ ***M1***

$\Rightarrow t = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours ***A1***

OR

$\int_{5000\pi}^{0} \frac{\mathrm{d}V}{\sqrt{V}} = -3\sqrt{2\pi}\int_0^T \mathrm{d}t$ ***M1A1A1***

Note: Award ***M1*** for attempt to use definite integrals, ***A1*** for correct limits and ***A1*** for correct integrands.

$\left[2\sqrt{V}\right]_{5000\pi}^{0} = 3\sqrt{2\pi}T$ ***A1***

$T = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours ***A1***

[7 marks]

Examiners report

This question was found to be challenging by many candidates and there were very few completely correct solutions. Many candidates did not seem able to find the volume of revolution when taken about the *y*-axis in (a). Candidates did not always recognize that part (b) did not involve related rates. Those candidates who attempted the question made some progress by separating the variables and integrating in (c) but very few were able to identify successfully the values necessary to find the correct answer.

19. *[6 marks]*

Markscheme

Consider

$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{10}}{10^{n+1}} \times \frac{10^n}{n^{10}}$ ***M1A1***

$= \frac{1}{10}\left(1 + \frac{1}{n}\right)^{10}$ ***A1***

$\to \frac{1}{10}$ as $n \to \infty$ ***A1***

$\frac{1}{10} < 1$ ***R1***

So by the Ratio Test the series is convergent. ***R1***

[6 marks]

Examiners report

Most candidates used the Ratio Test successfully to establish convergence. Candidates who attempted to use Cauchy's (Root) Test were often less successful although this was a valid method.

20. *[9 marks]*

Markscheme

(a)

M1A1

$\lim\limits_{x\to\infty}\frac{x}{\mathrm{e}^x} = \lim\limits_{x\to\infty}\frac{1}{\mathrm{e}^x}$

$= 0$ ***AG***

[2 marks]

(b) Using integration by parts ***M1***

A1A1

$\int_0^a x\mathrm{e}^{-x}\mathrm{d}x = \left[-x\mathrm{e}^{-x}\right]_0^a + \int_0^a \mathrm{e}^{-x}\mathrm{d}x$

A1

$= -a\mathrm{e}^{-a} - \left[e^{-x}\right]_0^a$

A1

$= 1 - a\mathrm{e}^{-a} - \mathrm{e}^{-a}$

[5 marks]

(c) Since

e^{-a} and

$a\mathrm{e}^{-a}$ are both convergent (to zero), the integral is convergent. ***R1***

Its value is 1. ***A1***

[2 marks]

Total [9 marks]

Examiners report

Most candidates made a reasonable attempt at (a). In (b), however, it was disappointing to note that some candidates were unable to use integration by parts to perform the integration. In (c), while many candidates obtained the correct value of the integral, proof of its convergence was often unconvincing.

21. *[13 marks]*

Markscheme

(a) Rewrite the equation in the form

$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = \frac{x^2}{x^2+1}$ ***M1A1***

Integrating factor

$= \mathrm{e}^{\int -\frac{2}{x}\mathrm{d}x}$ ***M1***

$= \mathrm{e}^{-2\ln x}$ ***A1***

$= \frac{1}{x^2}$ ***A1***

Note: Accept $\frac{1}{x^3}$ as applied to the original equation.

[5 marks]

(b) Multiplying the equation,

$\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \frac{1}{x^2+1}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) = \frac{1}{x^2+1}$ ***(M1)(A1)***

$\frac{y}{x^2} = \int \frac{\mathrm{d}x}{x^2+1}$ ***M1***

$= \arctan x + C$ ***A1***

Substitute $x = 1,\ y = 1$. ***M1***

$1 = \frac{\pi}{4} + C \Rightarrow C = 1 - \frac{\pi}{4}$ ***A1***

$y = x^2\left(\arctan x + 1 - \frac{\pi}{4}\right)$ ***A1***

[8 marks]

Total [13 marks]

Examiners report

The response to this question was often disappointing. Many candidates were unable to find the integrating factor successfully.

22. *[15 marks]*

Markscheme

(a) The area under the curve is sandwiched between the sum of the areas of the lower rectangles and the upper rectangles. ***M2***

Therefore

$1 \times \frac{1}{4^3} + 1 \times \frac{1}{5^3} + 1 \times \frac{1}{6^3} + \ldots < \int_3^\infty \frac{\mathrm{d}x}{x^3} < 1 \times \frac{1}{3^3} + 1 \times \frac{1}{4^3} + 1 \times \frac{1}{5^3} + \ldots$ ***A1***

which leads to the printed result.

[3 marks]

(b) We note first that

$\int_3^\infty \frac{\mathrm{d}x}{x^3} = \left[-\frac{1}{2x^2}\right]_3^\infty = \frac{1}{18}$ ***M1A1***

Consider first

$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \left(\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \ldots\right)$ ***M1A1***

$< 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{18}$ ***M1A1***

(which is an upper bound) ***A1***

$= \frac{263}{216}\ (1.22)$

$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \left(\frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots\right)$ ***M1A1***

$> 1 + \frac{1}{8} + \frac{1}{18}$ ***M1A1***

(which is a lower bound) ***A1***

$= \frac{85}{72}\left(\frac{255}{216}\right)\ (1.18)$

[12 marks]

Total [15 marks]

Examiners report

Many candidates failed to give a convincing argument to establish the inequality. In (b), few candidates progressed beyond simply evaluating the integral.

23. *[17 marks]*

Markscheme

(a) Constant term = 0 ***A1***

[1 mark]

(b)

$f'(x) = \frac{1}{1-x}$ ***A1***

$f''(x) = \frac{1}{(1-x)^2}$ ***A1***

$f'''(x) = \frac{2}{(1-x)^3}$ ***A1***

$f'(0) = 1;\ f''(0) = 1;\ f'''(0) = 2$ ***A1***

Note: Allow ***FT*** on their derivatives.

$f(x) = 0 + \frac{1 \times x}{1!} + \frac{1 \times x^2}{2!} + \frac{2 \times x^3}{3!} + \ldots$ ***M1A1***

$= x + \frac{x^2}{2} + \frac{x^3}{3}$ ***AG***

[6 marks]

(c)

$\frac{1}{1-x} = 2 \Rightarrow x = \frac{1}{2}$ ***(A1)***

$\ln 2 \approx \frac{1}{2} + \frac{1}{8} + \frac{1}{24}$ ***M1***

$= \frac{2}{3}\ (0.667)$ ***A1***

[3 marks]

(d) Lagrange error

$= \frac{f^{(n+1)}(c)}{(n+1)!} \times \left(\frac{1}{2}\right)^{n+1}$ ***(M1)***

$= \frac{6}{(1-c)^4} \times \frac{1}{24} \times \left(\frac{1}{2}\right)^4$ ***A1***

$< \frac{6}{\left(1-\frac{1}{2}\right)^4} \times \frac{1}{24} \times \frac{1}{16}$ ***A2***

giving an upper bound of 0.25. ***A1***

[5 marks]

(e) Actual error

$= \ln 2 - \frac{2}{3} = 0.0265$ ***A1***

The upper bound calculated is much larger that the actual error therefore cannot be considered a good estimate. ***R1***

[2 marks]

Total [17 marks]

Examiners report

In (a), some candidates appeared not to understand the term ‘constant term’. In (b), many candidates found the differentiation beyond them with only a handful realising that the best way to proceed was to rewrite the function as $f(x) = -\ln(1-x)$. In (d), many candidates were unable to use the Lagrange formula for the upper bound so that (e) became inaccessible.

24a. *[3 marks]*

Markscheme

Using l'Hopital's rule,

M1A1

$$\lim_{x \to 1}\left(\frac{\ln x}{\sin 2\pi x}\right) = \lim_{x \to 1}\left(\frac{\frac{1}{x}}{2\pi \cos 2\pi x}\right)$$

A1

$$= \frac{1}{2\pi}$$

[3 marks]

Examiners report

Part (a) was well done but too often the instruction to use series in part (b) was ignored. When this hint was observed correct solutions followed.

24b. *[7 marks]*

Markscheme

M1A1A1

$$\lim_{x \to 0}\left(\frac{1-e^{x^2}}{1-\cos x}\right) = \lim_{x \to 0}\left(\frac{1-\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\ldots\right)}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)}\right)$$

Note: Award ***M1*** for evidence of using the two series.

A1

$$= \lim_{x \to 0}\left(\frac{\left(-x^2-\frac{x^4}{2!}-\frac{x^6}{3!}-\ldots\right)}{\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\ldots\right)}\right)$$

EITHER

M1A1

$$= \lim_{x \to 0}\left(\frac{\left(-1-\frac{x^2}{2!}-\frac{x^4}{3!}-\ldots\right)}{\left(\frac{1}{2!}-\frac{x^2}{4!}+\ldots\right)}\right)$$

A1

$$= \frac{-1}{\frac{1}{2}} = -2$$

OR

M1A1

$$= \lim_{x \to 0}\left(\frac{\left(-2x-\frac{4x^3}{2!}-\frac{6x^5}{3!}-\ldots\right)}{\left(\frac{2x}{2!}-\frac{4x^3}{4!}+\ldots\right)}\right)$$

$$= \lim_{x \to 0}\left(\frac{\left(-2-\frac{4x^2}{2!}-\frac{6x^4}{3!}-\ldots\right)}{\left(1-\frac{4x^2}{4!}+\ldots\right)}\right)$$

A1

$$= \frac{-2}{1} = -2$$

[7 marks]

Examiners report

Part (a) was well done but too often the instruction to use series in part (b) was ignored. When this hint was observed correct solutions followed.

25. *[9 marks]*

Markscheme

Let

M1A1

$\frac{1}{(x+2)(2x+1)} = \frac{A}{x+2} + \frac{B}{2x+1} = \frac{A(2x+1)+B(x+2)}{(x+2)(2x+1)}$

A1

$x = -2 \rightarrow A = -\frac{1}{3}$

A1 ***N3***

$x = -\frac{1}{2} \rightarrow B = \frac{2}{3}$

M1

$I = \frac{1}{3}\int_0^h \left[\frac{2}{(2x+1)} - \frac{1}{(x+2)}\right] \mathrm{d}x$

A1

$= \frac{1}{3}[\ln(2x+1) - \ln(x+2)]_0^h$

A1

$= \frac{1}{3}\left[\lim_{h\to\infty}\left(\ln\left(\frac{2h+1}{h+2}\right)\right) - \ln\frac{1}{2}\right]$

A1

$= \frac{1}{3}(\ln 2 - \ln\frac{1}{2})$

A1

$= \frac{2}{3}\ln 2$

Note: If the logarithms are not combined in the third from last line the last three ***A1*** marks cannot be awarded.

Total [9 marks]

Examiners report

Not a difficult question but combination of the logarithms obtained by integration was often replaced by a spurious argument with infinities to get an answer.

$\log(\infty + 1)$ was often seen.

26.

[14 marks]

Markscheme

(a)

(i)

$\frac{dy}{dx} = 2x(1 + x^2 - y)$

x_i	y_i	y_i'	Δy
1	2	0	0
1.1	2	0.4620	0.0462
1.2	2.0462	0.9451	0.0945
1.3	2.1407		

M1

A2

Note: Award ***A2*** for complete table.

Award ***A1*** for a reasonable attempt.

A1

$f(1.3) = 2.14 \quad (\text{accept } 2.141)$

(ii) Decrease the step size ***A1***

[5 marks]

(b)

$\frac{dy}{dx} = 2x(1 + x^2 - y)$

M1

$\frac{dy}{dx} + 2xy = 2x(1 + x^2)$

Integrating factor is

M1A1

$e^{\int 2x dx} = e^{x^2}$

So,

A1

$e^{x^2} y = \int (2xe^{x^2} + 2xe^{x^2} x^2) dx$

M1A1

$= e^{x^2} + x^2 e^{x^2} - \int 2xe^{x^2} dx$

$= e^{x^2} + x^2 e^{x^2} - e^{x^2} + k$

A1

$= x^2 e^{x^2} + k$

$y = x^2 + ke^{-x^2}$

M1

$x = 1,\ y = 2 \rightarrow 2 = 1 + ke^{-1}$

$k = e$

A1

$y = x^2 + e^{1-x^2}$

[9 marks]

Total [14 marks]

Examiners report

Some incomplete tables spoiled what were often otherwise good solutions. Although the intermediate steps were asked to four decimal places the answer was not and the usual degree of IB accuracy was expected.

Some candidates surprisingly could not solve what was a fairly easy differential equation in part (b).

27. *[14 marks]*

Markscheme

(a)

$f(x) = \ln \cos x$

M1A1

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$

M1

$f''(x) = -\sec^2 x$

A1

$f'''(x) = -2\sec x \sec x \tan x$

$f^{iv}(x) = -2\sec^2 x(\sec^2 x) - 2\tan x(2\sec^2 x \tan x)$

A1

$= -2\sec^4 x - 4\sec^2 x \tan^2 x$

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots$

M1

$f(0) = 0,$

$f'(0) = 0,$

$f''(0) = -1,$

$f'''(0) = 0,$

A1

$f^{iv}(0) = -2,$

Notes: Award the ***A1*** if all the substitutions are correct.

Allow ***FT*** from their derivatives.

A1

$\ln(\cos x) \approx -\frac{x^2}{2!} - \frac{2x^4}{4!}$

AG

$= -\frac{x^2}{2} - \frac{x^4}{12}$

[8 marks]

(b) Some consideration of the manipulation of ln 2 ***(M1)***

Attempt to find an angle ***(M1)***

EITHER

Taking

A1

$x = \frac{\pi}{3}$

A1

$\ln \frac{1}{2} \approx -\frac{\left(\frac{\pi}{3}\right)^2}{2!} - \frac{2\left(\frac{\pi}{3}\right)^4}{4!}$

A1

$-\ln 2 \approx -\frac{\frac{\pi^2}{9}}{2!} - \frac{2\frac{\pi^4}{81}}{4!}$

A1

$\ln 2 \approx \frac{\pi^2}{18} + \frac{\pi^4}{972} = \frac{\pi^2}{9}\left(\frac{1}{2} + \frac{\pi^2}{108}\right)$

OR

Taking

A1

$x = \frac{\pi}{4}$

A1

$\ln \frac{1}{\sqrt{2}} \approx -\frac{\left(\frac{\pi}{4}\right)^2}{2!} - \frac{2\left(\frac{\pi}{4}\right)^4}{4!}$

A1

$-\frac{1}{2}\ln 2 \approx -\frac{\frac{\pi^2}{16}}{2!} - \frac{2\frac{\pi^4}{256}}{4!}$

A1

$\ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536} = \frac{\pi^2}{8}\left(\frac{1}{2} + \frac{\pi^2}{192}\right)$

[6 marks]

Total [14 marks]

Examiners report

Some candidates had difficulty organizing the derivatives but most were successful in getting the series. Using the series to find the approximation for $\ln 2$ in terms of π was another story and it was rare to see a good solution.

28a. *[6 marks]*

Markscheme

The ratio test gives

$$\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{(-1)^{n+1}x^{n+1}(n+1)3^n}{(n+2)3^{n+1}(-1)^n x^n}\right|$$ ***M1A1***

$$= \lim_{n \to \infty}\left|\frac{(n+1)x}{3(n+2)}\right|$$ ***A1***

$$= \frac{|x|}{3}$$ ***A1***

So the series converges for $\frac{|x|}{3} < 1$, ***A1***

the radius of convergence is 3 ***A1***

Note: Do not penalise lack of modulus signs.

[6 marks]

Examiners report

Some corners were cut in applying the ratio test and some candidates tried to use the comparison test. With careful algebra finding the radius of convergence was not too difficult. Often the interval of convergence was given instead of the radius.
Part (b) was done only by the best candidates. A little algebraic manipulation together with an auxiliary series soon gave the answer.

28b. *[7 marks]*

Markscheme

$u_n = \sqrt[3]{n^3+1} - n$

M1A1

$= n\left(\sqrt[3]{1+\frac{1}{n^3}} - 1\right)$

A1

$= n\left(1 + \frac{1}{3n^3} - \frac{1}{9n^6} + \frac{5}{81n^9} - \ldots - 1\right)$

using

$v_n = \frac{1}{n^2}$ as the auxilliary series, ***M1***

since

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$ and $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ converges ***M1A1***

then

$\sum u_n$ converges ***A1***

Note: Award ***M1A1A1M0M0A0A0*** to candidates attempting to use the integral test.

[7 marks]

Examiners report

Some corners were cut in applying the ratio test and some candidates tried to use the comparison test. With careful algebra finding the radius of convergence was not too difficult. Often the interval of convergence was given instead of the radius.

Part (b) was done only by the best candidates. A little algebraic manipulation together with an auxiliary series soon gave the answer.

29. *[12 marks]*

Markscheme

(a) **EITHER**

use the substitution $y = vx$

M1A1
$\frac{\mathrm{d}y}{\mathrm{d}x}x + v = v + 1$

$\int \mathrm{d}v = \int \frac{\mathrm{d}x}{x}$
by integration

A1
$v = \frac{y}{x} = \ln x + c$
OR

the equation can be rearranged as first order linear

M1
$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = 1$
the integrating factor I is

A1
$\mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x} = \mathrm{e}^{-\ln x} = \frac{1}{x}$
multiplying by I gives

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}y\right) = \frac{1}{x}$
A1
$\frac{1}{x}y = \ln x + c$
THEN

the condition gives $c = -1$

so the solution is

AG
$y = x(\ln x - 1)$
[5 marks]

(b) (i)

A1
$f'(x) = \ln x - 1 + 1 = \ln x$
A1
$f''(x) = \frac{1}{x}$
A1
$f'''(x) = -\frac{1}{x^2}$

(ii) the Taylor series about $x = 1$ starts

(M1)
$f(x) \approx f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!}$
A1A1A1
$= -1 + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!}$
[7 marks]

Total: [12 marks]

Examiners report

Part(a) was well done by many candidates. In part(b)(i), however, it was disappointing to see so many candidates unable to differentiate

$x(\ln x - 1)$

correctly. Again, too many candidates were able to quote the general form of a Taylor series expansion, but not how to apply it to the given function.

30a. *[8 marks]*

Markscheme

(i) the integrand is non-singular on the domain if $p > -1$ with the latter assumed, consider

M1A1

$\int_1^R \frac{1}{x(x+p)} \mathrm{d}x = \frac{1}{p} \int_1^R \frac{1}{x} - \frac{1}{x+p} \mathrm{d}x$

A1

$= \frac{1}{p} \left[\ln \left(\frac{x}{x+p} \right) \right]_1^R$, $p \neq 0$

this evaluates to

M1

$\frac{1}{p} \left(\ln \frac{R}{R+p} - \ln \frac{1}{1+p} \right)$, $p \neq 0$

A1

$\to \frac{1}{p} \ln(1+p)$

because

R1

$\frac{R}{R+p} \to 1 \text{ as } R \to \infty$

hence the integral is convergent ***AG***

(ii) the given series is

M1

$\sum_{n=1}^{\infty} f(n),\ f(n) = \frac{1}{n(n-0.5)}$

the integral test and $p = -0.5$ in (i) establishes the convergence of the series ***R1***

[8 marks]

Examiners report

Part(a)(i) caused problems for some candidates who failed to realize that the integral can only be tackled by the use of partial fractions. Even then, the improper integral only exists as a limit – too many candidates ignored or skated over this important point. Candidates must realize that in this type of question, rigour is important, and full marks will only be awarded for a full and clearly explained argument. This applies as well to part(b), where it was also noted that some candidates were confusing the convergence of the terms of a series to zero with convergence of the series itself.

30b. *[11 marks]*

Markscheme

(i) as we have a series of positive terms we can apply the comparison test, limit form

comparing with

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ ***M1***

$\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n(n+3)}\right)}{\frac{1}{n^2}} = 1$ ***M1A1***

as

$\sin\theta \approx \theta$ for small θ ***R1***

and

$\frac{n^2}{n(n+3)} \to 1$ ***R1***

(so as the limit (of 1) is finite and non-zero, both series exhibit the same behavior)

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so this series converges ***R1***

(ii) the general term is

$\sqrt{\frac{1}{n(n+1)}}$ ***A1***

$\sqrt{\frac{1}{n(n+1)}} > \sqrt{\frac{1}{(n+1)(n+1)}}$ ***M1***

$\sqrt{\frac{1}{(n+1)(n+1)}} = \frac{1}{n+1}$ ***A1***

the harmonic series diverges ***R1***

so by the comparison test so does the given series ***R1***

[11 marks]

Examiners report

Part(a)(i) caused problems for some candidates who failed to realize that the integral can only be tackled by the use of partial fractions. Even then, the improper integral only exists as a limit – too many candidates ignored or skated over this important point. Candidates must realize that in this type of question, rigour is important, and full marks will only be awarded for a full and clearly explained argument. This applies as well to part(b), where it was also noted that some candidates were confusing the convergence of the terms of a series to zero with convergence of the series itself.

31.

[12 marks]

Markscheme

(a) (i)

$f(x) = (1+ax)(1+bx)^{-1}$

M1A1

$= (1+ax)(1-bx+\ldots(-1)^n b^n x^n + \ldots$

it follows that

M1A1

$c_n = (-1)^n b^n + (-1)^{n-1} a b^{n-1}$

AG

$= (-b)^{n-1}(a-b)$

(ii)

A1

$\mathrm{R} = \frac{1}{|b|}$

[5 marks]

(b) to agree up to quadratic terms requires

M1A1A1

$1 = -b + a,\ \frac{1}{2} = b^2 - ab$

from which

A1

$a = -b = \frac{1}{2}$

[4 marks]

(c)

A1

$\mathrm{e}^x \approx \frac{1+0.5x}{1-0.5x}$

putting

M1

$x = \frac{1}{3}$

A1

$\mathrm{e}^{\frac{1}{3}} \approx \frac{\left(1+\frac{1}{6}\right)}{\left(1-\frac{1}{6}\right)} = \frac{7}{5}$

[3 marks]

Total [12 marks]

Examiners report

Most candidates failed to realize that the first step was to write *f*(*x*) as

$(1+ax)(1+bx)^{-1}$. Given the displayed answer to part(a), many candidates successfully tackled part(b). Few understood the meaning of the 'hence' in part(c).

32. *[17 marks]*

Markscheme

(a) this separable equation has general solution

(M1)(A1)

$\int \sec^2 y \mathrm{d}y = \int \cos x \mathrm{d}x$

A1

$\tan y = \sin x + c$

the condition gives

M1

$\tan \frac{\pi}{4} = \sin \pi + c \Rightarrow c = 1$

the solution is

A1

$\tan y = 1 + \sin x$

AG

$y = \arctan(1 + \sin x)$

[5 marks]

(b) the limit cannot exist unless

R1A1

$a = \arctan\left(1 + \sin \frac{\pi}{2}\right) = \arctan 2$

in that case the limit can be evaluated using l'Hopital's rule (twice) limit is

M1A1

$\lim\limits_{x \to \frac{\pi}{2}} \frac{(\arctan(1+\sin x))\ '}{2\left(x - \frac{\pi}{2}\right)} = \lim\limits_{x \to \frac{\pi}{2}} \frac{y'}{2\left(x - \frac{\pi}{2}\right)}$

where *y* is the solution of the differential equation

the numerator has zero limit (from the factor

in the differential equation) ***R1***

$\cos x$

so required limit is

M1A1

$\lim\limits_{x \to \frac{\pi}{2}} \frac{y''}{2}$

finally,

M1A1

$y'' = -\sin x \cos^2 y - 2\cos x \cos y \sin y \times y'(x)$

since

A1

$\cos y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{5}}$

A1

$y'' = -\frac{1}{5}$ at $x = \frac{\pi}{2}$

the required limit is

A1

$-\frac{1}{10}$

[12 marks]

Total [17 marks]

Examiners report

Many candidates successfully obtained the displayed solution of the differential equation in part(a). Few complete solutions to part(b) were seen which used the result in part(a). The problem can, however, be solved by direct differentiation although this is algebraically more complicated. Some successful solutions using this method were seen.

33.

[13 marks]

Markscheme

(a)

$\frac{\mathrm{d}V}{\mathrm{d}t} = cr$ ***A1***

$V = \frac{4}{3}\pi r^3$

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$ ***M1A1***

$\Rightarrow 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = cr$ ***M1***

$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{4\pi r}$ ***A1***

$= \frac{k}{r}$ ***AG***

[5 marks]

(b)

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r}$

$\Rightarrow \int r\mathrm{d}r = \int k\mathrm{d}t$ ***M1***

$\frac{r^2}{2} = kt + d$ ***A1***

An attempt to substitute either $t = 0$, $r = 8$ or $t = 30$, $r = 12$ ***M1***

When $t = 0$, $r = 8$

$\Rightarrow d = 32$ ***A1***

$\Rightarrow \frac{r^2}{2} = kt + 32$

When $t = 30$, $r = 12$

$\Rightarrow \frac{12^2}{2} = 30k + 32$

$\Rightarrow k = \frac{4}{3}$ ***A1***

$\therefore \frac{r^2}{2} = \frac{4}{3}t + 32$

When $t = 15$,

$\frac{r^2}{2} = \frac{4}{3}15 + 32$ ***M1***

$\Rightarrow r^2 = 104$ ***A1***

$r \approx 10 \text{ cm}$ ***A1***

Note: Award ***M0*** to incorrect methods using proportionality which give solution $r = 10$ cm .

[8 marks]

Total [13 marks]

Examiners report

Candidates found this question quite difficult, with only the better students making appreciable progress on part (a). Relatively few candidates recognised that part (b) was asking them to solve a differential equation. Many students tried methods involving direct proportion, which did not lead anywhere.

34.

[8 marks]

Markscheme

Let the number of mosquitoes be y.

$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky$ ***M1***

$\int \frac{1}{y}\mathrm{d}y = \int -k\mathrm{d}t$ ***M1***

$\ln y = -kt + c$ ***A1***

$y = \mathrm{e}^{-kt+c}$

$y = A\mathrm{e}^{-kt}$

when

$t = 0,\ y = 500\,000 \Rightarrow A = 500\,000$ ***A1***

$y = 500\,000\mathrm{e}^{-kt}$

when

$t = 5,\ y = 400\,000$

$400\,000 = 500\,000\mathrm{e}^{-5k}$ ***M1***

$\frac{4}{5} = \mathrm{e}^{-5k}$

$-5k = \ln\frac{4}{5}$

$k = -\frac{1}{5}\ln\frac{4}{5}\ \ (= \ 0.0446)$ ***A1***

$250\,000 = 500\,000\mathrm{e}^{-kt}$ ***M1***

$\frac{1}{2} = \mathrm{e}^{-kt}$

$\ln\frac{1}{2} = -kt$

$t = \frac{5}{\ln\frac{4}{5}}\ln\frac{1}{2} = 15.5 \text{ years}$ ***A1***

[8 marks]

Examiners report

Some candidates assumed that the decrease in population size was exponential / geometric and were therefore unable to gain the first 4 marks. Apart from this, reasonably good attempts were made by many candidates.

35. *[20 marks]*

Markscheme

(a) Using an increment of 0.25 in the x-values ***A1***

n	x_n	y_n	$f(x_n, y_n)$	$hf(x_n, y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$	
0	1	−1	1	0.25	−0.75	***(M1)A1***
1	1.25	−0.75	0.68	0.17	−0.58	***A1***
2	1.5	−0.58	0.574756	0.143689	−0.4363...	***A1***
3	1.75	−0.436311	0.531080	0.132770	−0.3035...	***A1***

Note: The ***A1*** marks are awarded for final column.

A1

$\Rightarrow y(2) \approx -0.304$

[7 marks]

(b) (i) let $y = vx$ ***M1***

(A1)

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$

(M1)

$\Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2x^2+x^2}{2x^2}$

(A1)

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-2v+v^2}{2}$

A1

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{(1-v)^2}{2}$

M1

$\Rightarrow \int \frac{2}{(1-v)^2}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$

A1A1

$\Rightarrow 2(1-v)^{-1} = \ln x + c$

$\Rightarrow \frac{2}{1-\frac{y}{x}} = \ln x + c$

when

M1A1

$x = 1,\ y = -1 \Rightarrow c = 1$

$\Rightarrow \frac{2x}{x-y} = \ln x + 1$

M1A1

$\Rightarrow y = x - \frac{2x}{1+\ln x} \quad \left(= \frac{x\ln x - x}{1+\ln x}\right)$

(ii) when

A1

$x = 2,\ y = -0.362 \quad \left(\text{accept } 2 - \frac{4}{1+\ln 2}\right)$

[13 marks]

Total [20 marks]

Examiners report

Part (a) was well done by many candidates, but a number were penalised for not using a sufficient number of significant figures. Part (b) was started by the majority of candidates, but only the better candidates were able to reach the end. Many were unable to complete the question correctly because they did not know what to do with the substitution $y = vx$ and because of arithmetic errors and algebraic errors.

36a.

[4 marks]

Markscheme

$f'(x) = \frac{\cos x}{1+\sin x}$ ***A1***

$f''(x) = \frac{-\sin x(1+\sin x)-\cos x\cos x}{(1+\sin x)^2}$ ***M1A1***

$= \frac{-\sin x-(\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$ ***A1***

$= -\frac{1}{1+\sin x}$ ***AG***

[4 marks]

Examiners report

[N/A]

36b.

[7 marks]

Markscheme

(i)

$f'''(x) = \frac{\cos x}{(1+\sin x)^2}$ ***A1***

$f^{(4)}(x) = \frac{-\sin x(1+\sin x)^2-2(1+\sin x)\cos^2 x}{(1+\sin x)^4}$ ***M1A1***

$f(0) = 0,\ f'(0) = 1,\ f''(0) = -1$ ***M1***

$f'''(0) = 1,\ f^{(4)}(0) = -2$ ***A1***

$f(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$ ***A1***

(ii) the series contains even and odd powers of x ***R1***

[7 marks]

Examiners report

[N/A]

36c.

[3 marks]

Markscheme

$\lim\limits_{x\to 0} \frac{\ln(1+\sin x)-x}{x^2} = \lim\limits_{x\to 0} \frac{x-\frac{x^2}{2}+\frac{x^3}{6}+\ldots-x}{x^2}$ ***M1***

$= \lim\limits_{x\to 0} \frac{-\frac{1}{2}+\frac{x}{6}+\ldots}{1}$ ***(A1)***

$= -\frac{1}{2}$ ***A1***

Note: Use of l'Hopital's Rule is also acceptable.

[3 marks]

Examiners report

[N/A]

37a. *[1 mark]*

Markscheme

the equation can be rewritten as

A1
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+\sqrt{x^2-y^2}}{x} = \frac{y}{x} + \sqrt{1-\left(\frac{y}{x}\right)^2}$
so the differential equation is homogeneous ***AG***

[1 mark]

Examiners report

[N/A]

37b. *[7 marks]*

Markscheme

put $y = vx$ so that

M1A1
$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$
substituting,

M1
$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + \sqrt{1-v^2}$
M1
$\int \frac{\mathrm{d}v}{\sqrt{1-v^2}} = \int \frac{\mathrm{d}x}{x}$
A1
$\arcsin v = \ln x + C$
A1
$\frac{y}{x} = \sin(\ln x + C)$
A1
$y = x\sin(\ln x + C)$
[7 marks]

Examiners report

[N/A]

38a. *[6 marks]*

Markscheme

we note that

and
$y(0) = 1$
A1
$y'(0) = 2$
M1
$y'' = 2\mathrm{e}^x + y' \tan x + y\sec^2 x$
A1
$y''(0) = 3$
M1
$y''' = 2\mathrm{e}^x + y'' \tan x + 2y' \sec^2 x + 2y\sec^2 x \tan x$
A1
$y'''(0) = 6$
the maclaurin series solution is therefore

A1
$y = 1 + 2x + \frac{3x^2}{2} + x^3 + \ldots$
[6 marks]

Examiners report

[N/A]

38b. *[9 marks]*

Markscheme

(i)

$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x(\sin x+\cos x)) = \mathrm{e}^x(\sin x+\cos x)+\mathrm{e}^x(\cos x-\sin x)$ ***M1***

$= 2\mathrm{e}^x\cos x$ ***A1***

it follows that

$\int \mathrm{e}^x\cos x\mathrm{d}x = \frac{1}{2}\mathrm{e}^x(\sin x+\cos x)+c$ ***AG***

(ii) the differential equation can be written as

$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = 2\mathrm{e}^x$ ***M1***

$\mathrm{IF} = \mathrm{e}^{\int -\tan x\mathrm{d}x} = \mathrm{e}^{\ln\cos x} = \cos x$ ***M1A1***

$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} - y\sin x = 2\mathrm{e}^x\cos x$ ***M1***

integrating,

$y\cos x = \mathrm{e}^x(\sin x+\cos x)+C$ ***A1***

$y = 1$ when $x = 0$ gives $C = 0$ ***M1***

therefore

$y = \mathrm{e}^x(1+\tan x)$ ***A1***

[9 marks]

Examiners report

[N/A]

39a. *[7 marks]*

Markscheme

we note that

$f(0) = 0,\ f(x) = 3x$ for $x > 0$ and

$f(x) = x$ for $x < 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$ ***M1A1***

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x = 0$ ***A1***

since $f(0) = 0$, the function is continuous when $x = 0$ ***AG***

$\lim_{x\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{x\to 0^-}\frac{h}{h} = 1$ ***M1A1***

$\lim_{x\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{x\to 0^+}\frac{3h}{h} = 3$ ***A1***

these limits are unequal ***R1***

so f is not differentiable when $x = 0$ ***AG***

[7 marks]

Examiners report

[N/A]

39b.

[3 marks]

Markscheme

$\int_{-a}^{a} f(x)\mathrm{d}x = \int_{-a}^{0} x\mathrm{d}x + \int_{0}^{a} 3x\mathrm{d}x$ ***M1***

$= \left[\frac{x^2}{2}\right]_{-a}^{0} + \left[\frac{3x^2}{2}\right]_{0}^{a}$ ***A1***

$= a^2$ ***A1***

[3 marks]

Examiners report

[N/A]

40a.

[4 marks]

Markscheme

using the ratio test,

$\frac{u_{n+1}}{u_n} = \frac{nx^{n+1}}{(n+1)^2 2^{n+1}} \times \frac{n^2 2^n}{(n-1)x^n}$ ***M1***

$= \frac{n^3}{(n+1)^2(n-1)} \times \frac{x}{2}$ ***A1***

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{x}{2}$ ***A1***

the radius of convergence R satisfies

$\frac{R}{2} = 1$

so $R = 2$ ***A1***

[4 marks]

Examiners report

[N/A]

40b. *[9 marks]*

Markscheme

considering $x = 2$ for which the series is

$\sum_{n=1}^{\infty} \frac{(n-1)}{n^2}$

using the limit comparison test with the harmonic series ***M1***

$\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges

consider

A1

$\lim_{n \to \infty} \frac{u_n}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n-1}{n} = 1$

the series is therefore divergent for $x = 2$ ***A1***

when $x = -2$, the series is

$\sum_{n=1}^{\infty} \frac{(n-1)}{n^2} \times (-1)^n$

this is an alternating series in which the

n^{th} term tends to 0 as

A1

$n \to \infty$

consider

M1

$f(x) = \frac{x-1}{x^2}$

A1

$f'(x) = \frac{2-x}{x^3}$

this is negative for

$x > 2$ so the sequence

$\{|u_n|\}$ is eventually decreasing ***R1***

the series therefore converges when $x = -2$ by the alternating series test ***R1***

the interval of convergence is therefore [–2, 2[***A1***

[9 marks]

Examiners report

[N/A]