Topic 6 Part 3 *[713 marks]*

1. Consider the functions f and g defined by

   $f(x) = 2^{\frac{1}{x}}$ and

   $g(x) = 4 - 2^{\frac{1}{x}}$,

   $x \neq 0$.

   (a) Find the coordinates of P, the point of intersection of the graphs of f and g .

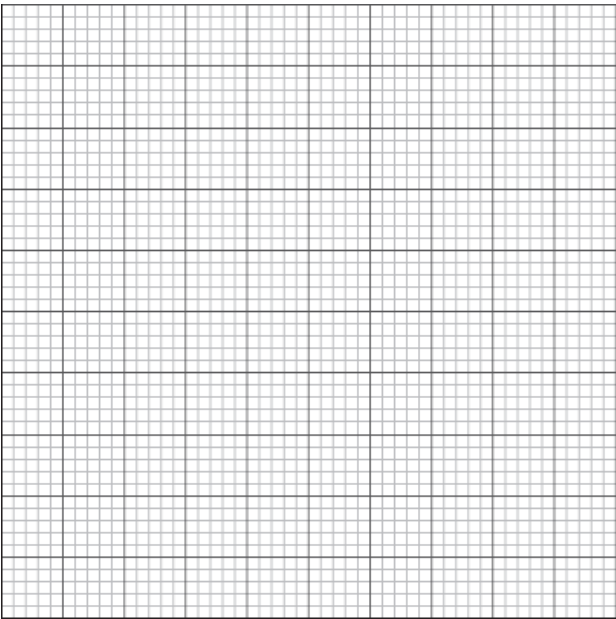   (b) Find the equation of the tangent to the graph of f at the point P.

2. (a) Let

   $a > 0$. Draw the graph of

   $y = \left| x - \frac{a}{2} \right|$ for

   $-a \leqslant x \leqslant a$ on the grid below.

   (b) Find k such that

   $\int_{-a}^{0} \left| x - \frac{a}{2} \right| dx = k \int_{0}^{a} \left| x - \frac{a}{2} \right| dx$.

3. Let f be a function defined by

$f(x) = x - \arctan x$,

$x \in \mathbb{R}$.

(a)   Find

$f(1)$ and

$f\left(-\sqrt{3}\right)$.

(b)   Show that

$f(-x) = -f(x)$, for

$x \in \mathbb{R}$.

(c)   Show that

$x - \frac{\pi}{2} < f(x) + \frac{\pi}{2}$, for

$x \in \mathbb{R}$.

(d)   Find expressions for

$f'(x)$ and

$f''(x)$. Hence describe the behaviour of the graph of f at the origin and justify your answer.

(e)   Sketch a graph of f, showing clearly the asymptotes.

(f)   Justify that the inverse of f is defined for all
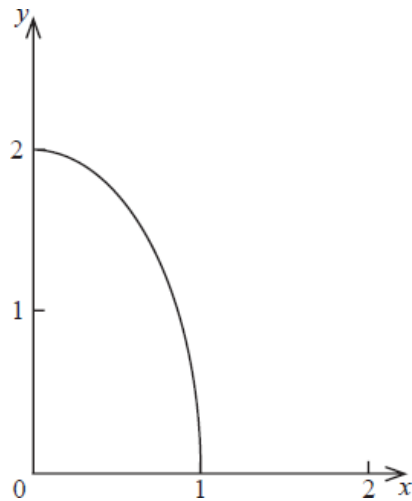
$x \in \mathbb{R}$ and sketch its graph.

4. (a)   Show that

$\frac{3}{x+1} + \frac{2}{x+3} = \frac{5x+11}{x^2+4x+3}$.

(b)   Hence find the value of k such that

$\int_0^2 \frac{5x+11}{x^2+4x+3} \mathrm{d}x = \ln k$.

5. Consider the part of the curve

$4x^2 + y^2 = 4$ shown in the diagram below.

(a) Find an expression for

$\frac{dy}{dx}$ in terms of x and y .

(b) Find the gradient of the tangent at the point

$\left(\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$.

(c) A bowl is formed by rotating this curve through

2π radians about the x-axis.

Calculate the volume of this bowl.

6. A function is defined as

$$f(x) = k\sqrt{x}, \text{ with}$$

$$k > 0 \text{ and}$$

$$x \geqslant 0 .$$

(a) Sketch the graph of

$$y = f(x) .$$

(b) Show that f is a one-to-one function.

(c) Find the inverse function,

$$f^{-1}(x) \text{ and state its domain.}$$

(d) If the graphs of

$$y = f(x) \text{ and}$$

$$y = f^{-1}(x) \text{ intersect at the point (4, 4) find the value of } k .$$

(e) Consider the graphs of

$$y = f(x) \text{ and}$$

$$y = f^{-1}(x) \text{ using the value of } k \text{ found in part (d).}$$

  (i) Find the area enclosed by the two graphs.

  (ii) The line $x = c$ cuts the graphs of

$$y = f(x) \text{ and}$$

$$y = f^{-1}(x) \text{ at the points P and Q respectively. Given that the tangent to}$$

$$y = f(x) \text{ at point P is parallel to the tangent to}$$

$$y = f^{-1}(x) \text{ at point Q find the value of } c .$$

7a. Calculate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x .$$
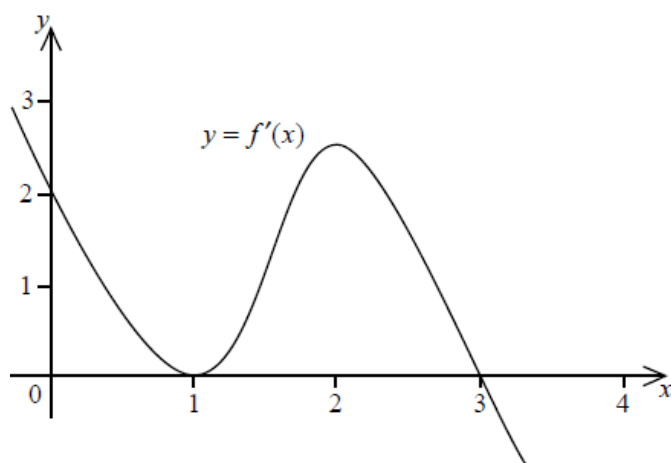
7b. Find

$$\int \tan^3 x \mathrm{d}x .$$

8. The diagram below shows a sketch of the gradient function

   $f'(x)$ of the curve

   $f(x)$.

On the graph below, sketch the curve

$y = f(x)$ given that

$f(0) = 0$. Clearly indicate on the graph any maximum, minimum or inflexion points.

9. A drinking glass is modelled by rotating the graph of

   $y = e^x$ about the y-axis, for

   $1 \leqslant y \leqslant 5$. Find the volume of the glass.

10. A tangent to the graph of

$y = \ln x$ passes through the origin.

(a)   Sketch the graphs of

$y = \ln x$ and the tangent on the same set of axes, and hence find the equation of the tangent.

(b)   Use your sketch to explain why

$\ln x \leqslant \frac{x}{e}$ for

$x > 0$.

(c)   Show that

$x^e \leqslant e^x$ for

$x > 0$.

(d)   Determine which is larger,

π^e or

e^π .

11. The region enclosed between the curves

$y = \sqrt{x}\,e^x$ and

$y = e\sqrt{x}$ is rotated through

2π about the x-axis. Find the volume of the solid obtained.

12. (a)   Given that

$\alpha > 1$, use the substitution

$u = \frac{1}{x}$ to show that

$$\int_1^\alpha \frac{1}{1+x^2}\,dx = \int_{\frac{1}{\alpha}}^1 \frac{1}{1+u^2}\,dx.$$

(b)   **Hence** show that

$\arctan \alpha + \arctan \frac{1}{\alpha} = \frac{\pi}{2}$.

13. Consider

$f(x) = \frac{x^2 - 5x + 4}{x^2 + 5x + 4}$.

(a)   Find the equations of all asymptotes of the graph of f.

(b)   Find the coordinates of the points where the graph of f meets the x and y axes.

(c)   Find the coordinates of

(i)   the maximum point and justify your answer;

(ii)   the minimum point and justify your answer.

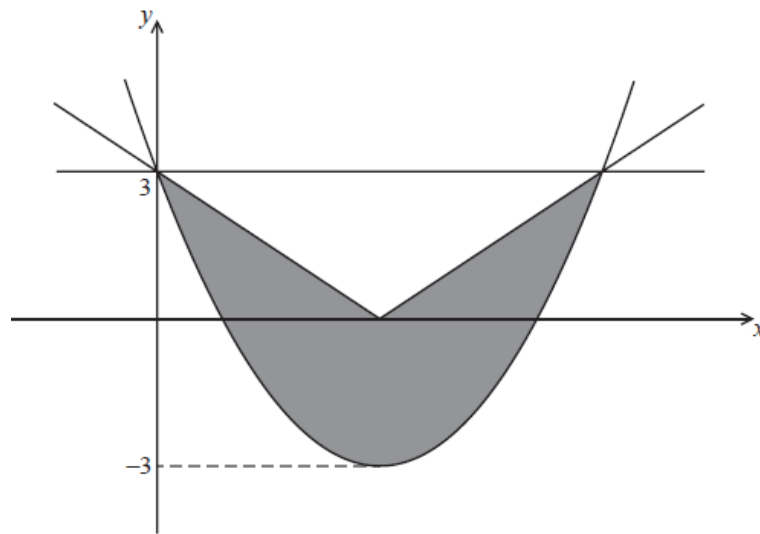(d)   Sketch the graph of f, clearly showing all the features found above.

(e)   **Hence**, write down the number of points of inflexion of the graph of f.

14. The diagram below shows the graphs of

$y = \left|\frac{3}{2}x - 3\right|$, $y = 3$ and a quadratic function, that all intersect in the same two points.

Given that the minimum value of the quadratic function is −3, find an expression for the area of the shaded region in the form

$\int_0^t (ax^2 + bx + c)\,\mathrm{d}x$, where the constants a, b, c and t are to be determined. (Note: The integral does not need to be evaluated.)

15. A body is moving through a liquid so that its acceleration can be expressed as

$$\left(-\frac{v^2}{200} - 32\right) \mathrm{ms}^{-2},$$

where

$v\ \mathrm{ms}^{-1}$ is the velocity of the body at time t seconds.

The initial velocity of the body was known to be

$40\ \mathrm{ms}^{-1}$.

(a) Show that the time taken, T seconds, for the body to slow to

$V\ \mathrm{ms}^{-1}$ is given by

$$T = 200 \int_V^{40} \frac{1}{v^2 + 80^2}\,\mathrm{d}v.$$

(b) (i) Explain why acceleration can be expressed as

$v\frac{\mathrm{d}v}{\mathrm{d}s}$, where s is displacement, in metres, of the body at time t seconds.

(ii) **Hence** find a similar integral to that shown in part (a) for the distance, S metres, travelled as the body slows to

$V\ \mathrm{ms}^{-1}$.

(c) **Hence**, using parts (a) and (b), find the distance travelled and the time taken until the body momentarily comes to rest.

16. The function f is defined by

$f(x) = e^{x^2 - 2x - 1.5}$.

(a)   Find

$f'(x)$.

(b)   You are given that

$y = \frac{f(x)}{x-1}$ has a local minimum at $x = a$, $a > 1$. Find the

value of a.

17. The normal to the curve

$xe^{-y} + e^y = 1 + x$, at the point $(c,$

$\ln c)$, has a y-intercept

$c^2 + 1$.

Determine the value of c.

18. Find the value of

$\int_0^1 t\ln(t+1)\mathrm{d}t$.
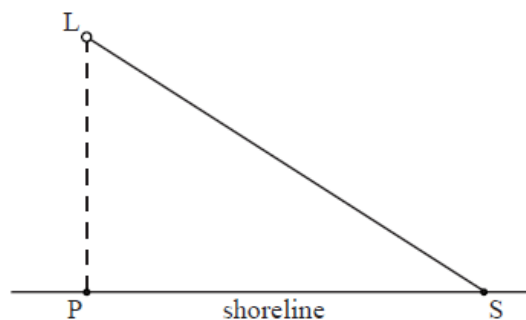
19. A lighthouse L is located offshore, 500 metres from the nearest point P on a long straight shoreline. The narrow beam of light

from the lighthouse rotates at a constant rate of

8π radians per minute, producing an illuminated spot S that moves along the shoreline. You may assume that the height of the lighthouse

can be ignored and that the beam of light lies in the horizontal plane defined by sea level.

When S is 2000 metres from P,

(a)   show that the speed of S, correct to three significant figures, is

214 000 metres per minute;

(b)   find the acceleration of S.

20. The function f is defined by

$$f(x) = (x^3 + 6x^2 + 3x - 10)^{\frac{1}{2}}, \text{ for } x \in D,$$

where

$D \subseteq \mathbb{R}$ is the greatest possible domain of f.

(a)   Find the roots of

$f(x) = 0$.

(b)   Hence specify the set D.

(c)   Find the coordinates of the local maximum on the graph

$y = f(x)$.

(d)   Solve the equation

$f(x) = 3$.

(e)   Sketch the graph of

$|y| = f(x)$, for $x \in D$.

(f)   Find the area of the region completely enclosed by the graph of

$|y| = f(x)$

[14 marks]

21a. A particle P moves in a straight line with displacement relative to origin given by

$$s = 2\sin(\pi t) + \sin(2\pi t), \ t \geqslant 0,$$

where t is the time in seconds and the displacement is measured in centimetres.

(i)   Write down the period of the function s.

(ii)   Find expressions for the velocity, v, and the acceleration, a, of P.

(iii)   Determine all the solutions of the equation $v = 0$ for

$0 \leqslant t \leqslant 4$.

[10 marks]

21b. Consider the function

$$f(x) = A\sin(ax) + B\sin(bx), \ A, \ a, \ B, \ b, \ x \in \mathbb{R}.$$

Use mathematical induction to prove that the

$(2n)^{\text{th}}$ derivative of f is given by

$(f^{(2n)})(x) = (-1)^n (Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx))$, for all

$n \in \mathbb{Z}^+$.

[8 marks]

22. Consider the curve

$y = xe^x$ and the line

$y = kx, \ k \in \mathbb{R}$.

(a) Let $k = 0$.

(i) Show that the curve and the line intersect once.

(ii) Find the angle between the tangent to the curve and the line at the point of intersection.

(b) Let $k = 1$. Show that the line is a tangent to the curve.

(c) (i) Find the values of k for which the curve

$y = xe^x$ and the line

$y = kx$ meet in two distinct points.

(ii) Write down the coordinates of the points of intersection.

(iii) Write down an integral representing the area of the region A enclosed by the curve and the line.

(iv) **Hence**, given that

$0 < k < 1$, show that

$A < 1$.

23. Find the equation of the normal to the curve

$x^3 y^3 - xy = 0$ at the point $(1, 1)$.

24. The line

$y = m(x - m)$ is a tangent to the curve

$(1 - x)y = 1$.

Determine m and the coordinates of the point where the tangent meets the curve.

25. Let

$f(x) = \frac{a + be^x}{ae^x + b}$, where

$0 < b < a$.

(a) Show that

$f'(x) = \frac{(b^2 - a^2)e^x}{(ae^x + b)^2}$.

(b) **Hence** justify that the graph of f has no local maxima or minima.

(c) Given that the graph of f has a point of inflexion, find its coordinates.

(d) Show that the graph of f has exactly two asymptotes.

(e) Let $a = 4$ and $b = 1$. Consider the region R enclosed by the graph of

$y = f(x)$, the y-axis and the line with equation
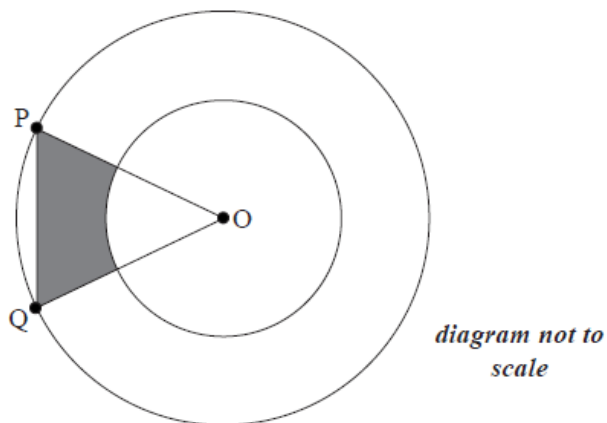
$y = \frac{1}{2}$.

Find the volume V of the solid obtained when R is rotated through

2π about the x-axis.

The diagram below shows two concentric circles with centre O and radii 2 cm and 4 cm.

The points P and Q lie on the larger circle and
$P\hat{O}Q = x$, where
$0 < x < \frac{\pi}{2}$.

diagram not to scale

26. (a) Show that the area of the shaded region is $8 \sin x - 2x$. [7 marks]

(b) Find the maximum area of the shaded region.

27. Find the gradient of the curve $e^{xy} + \ln(y^2) + e^y = 1 + e$ at the point $(0, 1)$. [7 marks]

Consider the function
f , defined by
$f(x) = x - a\sqrt{x}$, where
$x \geqslant 0,$
$a \in \mathbb{R}^+$.

28. (a) Find in terms of
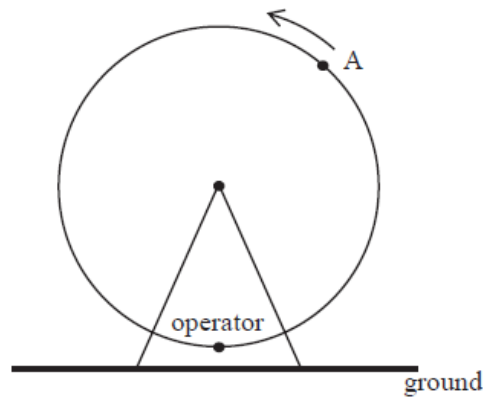a [11 marks]

(i) the zeros of
f ;

(ii) the values of
x for which
f is decreasing;

(iii) the values of
x for which
f is increasing;

(iv) the range of
f .

(b) State the concavity of the graph of
f .

Below is a sketch of a Ferris wheel, an amusement park device carrying passengers around the rim of the wheel.

29. (a)   The circular Ferris wheel has a radius of 10 metres and is revolving at a rate of 3 radians per minute. Determine how fast [10 marks] a passenger on the wheel is going vertically upwards when the passenger is at point A, 6 metres higher than the centre of the wheel, and is rising.

(b)   The operator of the Ferris wheel stands directly below the centre such that the bottom of the Ferris wheel is level with his eyeline. As he watches the passenger his line of sight makes an angle α with the horizontal. Find the rate of change of α at point A.

30. If

$$f(x) = x - 3x^{\frac{2}{3}} \,, \ x > 0 \,,$$

[5 marks]

(a)   find the x-coordinate of the point P where

$$f'(x) = 0 \,;$$

(b)   determine whether P is a maximum or minimum point.

31. Find the area between the curves

[7 marks]

$$y = 2 + x - x^2 \text{ and } y = 2 - 3x + x^2 \,.$$

32. The region bounded by the curve

[6 marks]

$$y = \frac{\ln(x)}{x}$$ and the lines $x = 1$, $x = e$, $y = 0$ is rotated through

2π radians about the x-axis.

Find the volume of the solid generated.

33. The function f is defined by

$f(x) = xe^{2x}$.

It can be shown that

$f^{(n)}(x) = (2^n x + n2^{n-1})e^{2x}$ for all

$n \in \mathbb{Z}^+$, where

$f^{(n)}(x)$ represents the

n^{th} derivative of

$f(x)$.

(a)   By considering

$f^{(n)}(x)$ for $n = 1$ and $n = 2$, show that there is one minimum point P on the graph of f, and find the coordinates of P.

(b)   Show that f has a point of inflexion Q at $x = -1$.

(c)   Determine the intervals on the domain of f where f is

(i)   concave up;

(ii)   concave down.

(d)   Sketch f , clearly showing any intercepts, asymptotes and the points P and Q.

(e)   Use mathematical induction to prove that

$f^{(n)}(x) = (2^n x + n2^{n-1})e^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}$ represents the nth derivative of $f(x)$.

34. Find the gradient of the tangent to the curve

$x^3 y^2 = \cos(\pi y)$ at the point $(-1,\ 1)$.

35. By using an appropriate substitution find

$$\int \frac{\tan(\ln y)}{y} dy,\ y > 0 .$$

36. A family of cubic functions is defined as
$f_k(x) = k^2 x^3 - kx^2 + x,\ k \in \mathbb{Z}^+$.
(a)   Express in terms of k
(i)
$f'_k(x)$ and $f''_k(x)$;
(ii)   the coordinates of the points of inflexion
P_k on the graphs of
f_k .
(b)   Show that all
P_k lie on a straight line and state its equation.
(c)   Show that for all values of k, the tangents to the graphs of
f_k at
P_k are parallel, and find the equation of the tangent lines.

37. Consider the curve with equation

$x^2 + xy + y^2 = 3$.

(a)   Find in terms of k, the gradient of the curve at the point $(-1, k)$.

(b)   Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k.

38. Show that

$$\int_0^{\frac{\pi}{6}} x \sin 2x \, dx = \frac{\sqrt{3}}{8} - \frac{\pi}{24}.$$
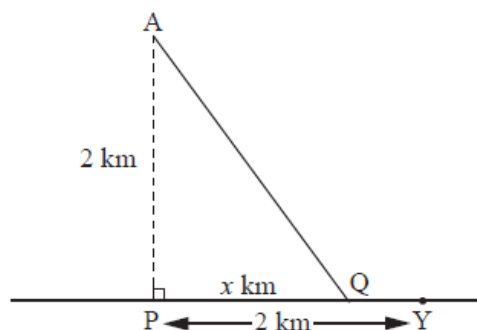
39. A normal to the graph of

$y = \arctan(x-1)$, for

$x > 0$, has equation

$y = -2x + c$, where

$x \in \mathbb{R}$.

Find the value of c.

40. André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.

When André swims he covers 1 km in
$5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a)   If PQ = x km,

$0 \leqslant x \leqslant 2$, find an expression for the time T minutes taken by André to reach point Y.

(b)   Show that
$\frac{dT}{dx} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5.$

(c)   (i)   Solve
$\frac{dT}{dx} = 0.$

(ii)   Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii)   Show that
$\frac{d^2T}{dx^2} = \frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum.

41. The curve

$y = e^{-x} - x + 1$ intersects the x-axis at P.

(a)   Find the x-coordinate of P.

(b)   Find the area of the region completely enclosed by the curve and the coordinate axes.

42. Consider the curve with equation

$f(x) = e^{-2x^2}$ for $x < 0$.

Find the coordinates of the point of inflexion and justify that it is a point of inflexion.

43. A particle moves in a straight line in a positive direction from a fixed point O.

The velocity v m

s^{-1} , at time t seconds, where

$t \geqslant 0$, satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{-v(1+v^2)}{50}.$$

The particle starts from O with an initial velocity of 10 m

s^{-1} .

(a) (i) Express as a definite integral, the time taken for the particle's velocity to decrease from 10 m

s^{-1} to 5 m

s^{-1} .

(ii) **Hence** calculate the time taken for the particle's velocity to decrease from 10 m

s^{-1} to 5 m

s^{-1} .

(b) (i) Show that, when

$v > 0$, the motion of this particle can also be described by the differential equation

$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-(1+v^2)}{50}$ where x metres is the displacement from O.

(ii) Given that $v =10$ when $x = 0$, solve the differential equation expressing x in terms of v.

(iii) **Hence** show that

$v = \frac{10 - \tan\frac{x}{50}}{1 + 10\tan\frac{x}{50}}.$

44. Calculate the exact value of

$\int_1^{e} x^2 \ln x \, \mathrm{d}x$.

45. Find the equation of the normal to the curve

$5xy^2 - 2x^2 = 18$ at the point $(1, 2)$.

46. A packaging company makes boxes for chocolates. An example of a box is shown below. This box is closed and the top and

bottom of the box are identical regular hexagons of side x cm.

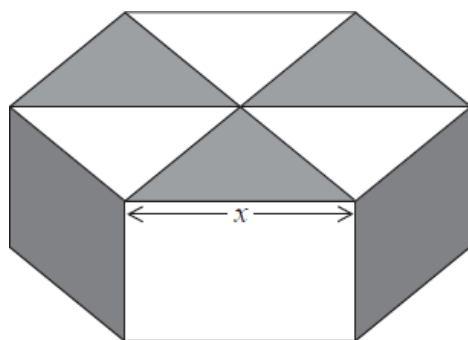diagram not to
scale

(a) Show that the area of each hexagon is

$\frac{3\sqrt{3}x^2}{2}$cm^2 .

(b) Given that the volume of the box is

90 cm^2 , show that when

$x = \sqrt[3]{20}$ the total surface area of the box is a minimum, justifying that this value gives a minimum.

$y = \ln\left(\frac{1}{3}(1 + e^{-2x})\right)$, show that

$\frac{dy}{dx} = \frac{2}{3}(e^{-y} - 3)$.

48. The function f is defined by

[21 marks]

$f(x) = x\sqrt{9 - x^2} + 2\arcsin\left(\frac{x}{3}\right)$.

(a)   Write down the largest possible domain, for each of the two terms of the function, f, and hence state the largest possible domain, D,

for f.

(b)   Find the volume generated when the region bounded by the curve $y = f(x)$, the x-axis, the y-axis and the line $x = 2.8$ is rotated

through

2π radians about the x-axis.

(c)   Find

$f'(x)$ in simplified form.

(d)   **Hence** show that

$\int_{-p}^{p} \frac{11 - 2x^2}{\sqrt{9 - x^2}}dx = 2p\sqrt{9 - p^2} + 4\arcsin\left(\frac{p}{3}\right)$, where

$p \in D$.

(e)   Find the value of p which maximises the value of the integral in (d).

(f)   (i)   Show that

$f''(x) = \frac{x(2x^2 - 25)}{(9 - x^2)^{\frac{3}{2}}}$.

(ii)   Hence justify that $f(x)$ has a point of inflexion at $x = 0$, but not at

$x = \pm\sqrt{\frac{25}{2}}$.

The function f is defined, for

$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, by

$f(x) = 2\cos x + x\sin x$.

49a. Determine whether f is even, odd or neither even nor odd.

[3 marks]

49b. Show that

[2 marks]

$f''(0) = 0$.

49c. John states that, because

[2 marks]

$f''(0) = 0$, the graph of f has a point of inflexion at the point $(0, 2)$. Explain briefly whether John's statement is correct or not.
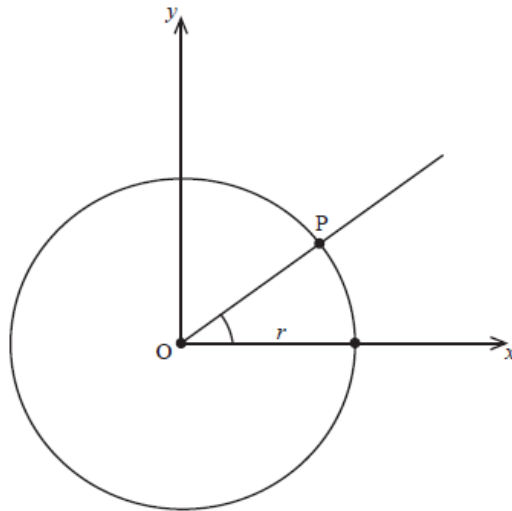
50. (a)   Integrate

[5 marks]

$\int \frac{\sin\theta}{1 - \cos\theta}d\theta$.

(b)   Given that

$\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1 - \cos\theta}d\theta = \frac{1}{2}$ and

$\frac{\pi}{2} < a < \pi$, find the value of

a .

The diagram below shows a circle with centre at the origin O and radius $r > 0$.

A point P(x, y), ($x > 0$, $y > 0$) is moving round the circumference of the circle.

Let $m = \tan\left(\arcsin\frac{y}{r}\right)$.

51. (a)   Given that $\frac{dy}{dt} = 0.001r$, show that $\frac{dm}{dt} = \left(\frac{r}{10\sqrt{r^2 - y^2}}\right)^3$.

[7 marks]

(b)   State the geometrical meaning of $\frac{dm}{dt}$.
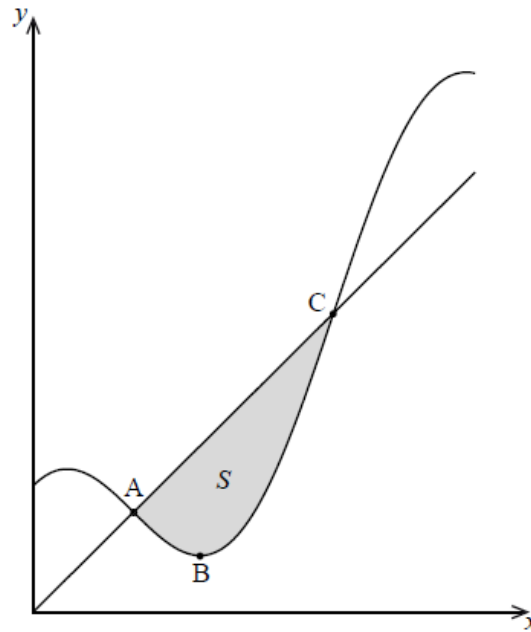
Let

f be a function defined by

$f(x) = x + 2\cos x$,

$x \in [0, 2\pi]$. The diagram below shows a region

S bound by the graph of

f and the line

$y = x$.

A and C are the points of intersection of the line

$y = x$ and the graph of

f , and B is the minimum point of

f .

52. (a)   If A, B and C have x-coordinates

$a\frac{\pi}{2}$,

$b\frac{\pi}{6}$ and

$c\frac{\pi}{2}$, where

a ,

b,

$c \in \mathbb{N}$, find the values of

a ,

b and

c .

(b)   Find the range of

f .

(c)   Find the equation of the normal to the graph of f at the point C, giving your answer in the form

$y = px + q$.

(d)   The region

S is rotated through

2π about the x-axis to generate a solid.

   (i)   Write down an integral that represents the volume

   V of this solid.

   (ii)   Show that

   $V = 6\pi^2$.

[19 marks]

53. (a)   Differentiate
$f(x) = \arcsin x + 2\sqrt{1-x^2}$,
$x \in [-1,1]$.

[6 marks]

(b)   Find the coordinates of the point on the graph of
$y = f(x)$ in
$[-1,1]$, where the gradient of the tangent to the curve is zero.

The cubic curve
$y = 8x^3 + bx^2 + cx + d$ has two distinct points P and Q, where the gradient is zero.

54. (a)   Show that
$b^2 > 24c$.

[8 marks]

(b)   Given that the coordinates of P and Q are
$\left(\frac{1}{2}, -12\right)$ and
$\left(-\frac{3}{2}, 20\right)$ respectively, find the values of
b,
c and
d.

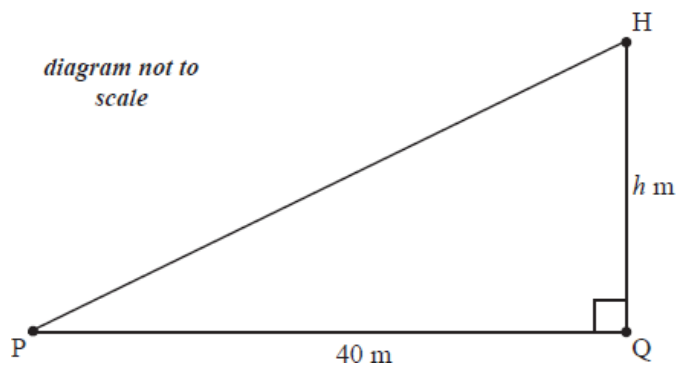55. Using the substitution
$x = 2\sin\theta$, show that

$$\int \sqrt{4-x^2}\,dx = Ax\sqrt{4-x^2} + B\arcsin\frac{x}{2} + \text{constant} ,$$

[8 marks]

where
A and
B are constants whose values you are required to find.

A helicopter H is moving vertically upwards with a speed of 10 ms^{-1} . The helicopter is
h m directly above the point Q which is situated on level ground. The helicopter is observed from the point P which is also at ground level and PQ
$= 40$ m. This information is represented in the diagram below.

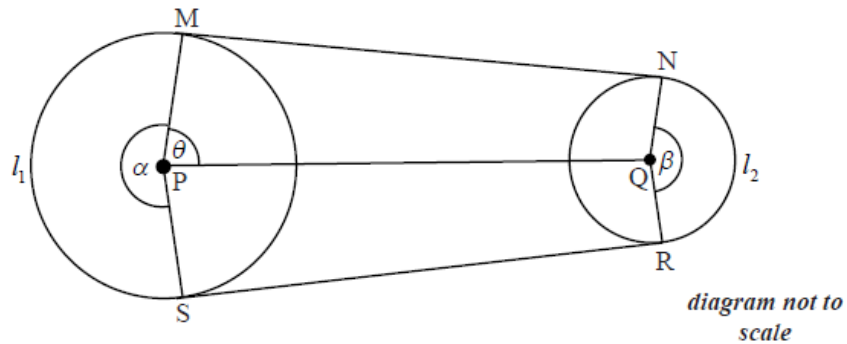diagram not to scale

H

h m

P   40 m   Q

When
$h = 30$,

56. (a)   show that the rate of change of
$H\hat{P}Q$ is
0.16 radians per second;

[7 marks]

(b)   find the rate of change of PH.

Two non-intersecting circles C_1 , containing points M and S , and C_2 , containing points N and R, have centres P and Q where PQ $= 50$. The line segments [MN] and [SR] are common tangents to the circles. The size of the reflex angle MPS is α, the size of the obtuse angle NQR is β , and the size of the angle MPQ is θ . The arc length MS is l_1 and the arc length NR is l_2 . This information is represented in the diagram below.

diagram not to scale

The radius of C_1 is x , where $x \geqslant 10$ and the radius of C_2 is 10.

57. (a)   Explain why $x < 40$.

(b)   Show that $\cos\theta = \dfrac{x - 10}{50}$.

(c)   (i)   Find an expression for MN in terms of x .

(ii)   Find the value of x that maximises MN.

(d)   Find an expression in terms of x for

(i) α ;

(ii) β .

(e)   The length of the perimeter is given by $l_1 + l_2 + MN + SR$.

(i)   Find an expression, $b(x)$, for the length of the perimeter in terms of x .

(ii)   Find the maximum value of the length of the perimeter.

(iii)   Find the value of x that gives a perimeter of length 200.

[18 marks]

Consider the graphs
$y = e^{-x}$ and
$y = e^{-x} \sin 4x$, for
$0 \leqslant x \leqslant \frac{5\pi}{4}$.

58. (a)   On the same set of axes draw, on graph paper, the graphs, for   [22 marks]
$0 \leqslant x \leqslant \frac{5\pi}{4}$. Use a scale of
1 cm to
$\frac{\pi}{8}$ on your
x-axis and
5 cm to
1 unit on your
y-axis.

(b)   Show that the
x-intercepts of the graph
$y = e^{-x} \sin 4x$ are
$\frac{n\pi}{4}$,
$n = 0,$
1,
2,
3,
4,
5.

(c)   Find the
x-coordinates of the points at which the graph of
$y = e^{-x} \sin 4x$ meets the graph of
$y = e^{-x}$. Give your answers in terms of
π.

(d)   (i)   Show that when the graph of
$y = e^{-x} \sin 4x$ meets the graph of
$y = e^{-x}$, their gradients are equal.

  (ii)   Hence explain why these three meeting points are not local maxima of the graph
  $y = e^{-x} \sin 4x$.

(e)   (i)   Determine the
y-coordinates,
y_1 ,
y_2 and
y_3, where
$y_1 > y_2 > y_3$, of the local maxima of
$y = e^{-x} \sin 4x$ for
$0 \leqslant x \leqslant \frac{5\pi}{4}$. You do not need to show that they are maximum values, but the values should be simplified.

  (ii)   Show that
  y_1 ,
  y_2 and
  y_3 form a geometric sequence and determine the common ratio
  r .

The function f is defined on the domain

$x \geqslant 0$ by

$f(x) = e^x - x^e$.

(i)   Find an expression for

[3 marks]

$f'(x)$.

(ii)   Given that the equation

$f'(x) = 0$ has two roots, state their values.

Sketch the graph of f , showing clearly the coordinates of the maximum and minimum.

[3 marks]

Hence show that

[1 mark]

$e^\pi > \pi^e$.

Find the value of the integral

[7 marks]

$\int_0^{\sqrt{2}} \sqrt{4 - x^2}\, dx$.

Find the value of the integral

[5 marks]

$\int_0^{0.5} \arcsin x\, dx$.

Using the substitution

[7 marks]

$t = \tan\theta$, find the value of the integral

$$\int_0^{\frac{\pi}{4}} \frac{d\theta}{3\cos^2\theta + \sin^2\theta} \, .$$

The function f is defined by

$f(x) = e^x \sin x$.

Show that

[3 marks]

$f''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right)$.

Obtain a similar expression for

[4 marks]

$f^{(4)}(x)$.

Suggest an expression for

[8 marks]

$f^{(2n)}(x)$,

$n \in \mathbb{Z}^+$, and prove your conjecture using mathematical induction.

The function f is defined by

$$f(x) = \begin{cases} 2x - 1, & x \leqslant 2 \\ ax^2 + bx - 5, & 2 < x < 3 \end{cases}$$

where a ,

$b \in \mathbb{R}$.

62a. Given that f and its derivative,

f' , are continuous for all values in the domain of f , find the values of a and b .

[6 marks]

62b. Show that f is a one-to-one function.

[3 marks]

62c. Obtain expressions for the inverse function

f^{-1} and state their domains.

[5 marks]

The particle P moves along the x-axis such that its velocity,

v ms^{-1} , at time t seconds is given by

$v = \cos(t^2).$

63a. Given that P is at the origin O at time $t = 0$, calculate

(i)   the displacement of P from O after 3 seconds;

(ii)   the total distance travelled by P in the first 3 seconds.

[4 marks]

63b. Find the time at which the total distance travelled by P is 1 m.

[2 marks]

64. A ladder of length 10 m on horizontal ground rests against a vertical wall. The bottom of the ladder is moved away from the wall at a constant speed of

0.5 ms^{-1}. Calculate the speed of descent of the top of the ladder when the bottom of the ladder is 4 m away from the wall.

[7 marks]

The function f is defined on the domain [0, 2] by

$f(x) = \ln(x + 1)\sin(\pi x)$.

65a. Obtain an expression for

$f'(x)$.

[3 marks]

65b. Sketch the graphs of f and

f' on the same axes, showing clearly all x-intercepts.

[4 marks]

65c. Find the x-coordinates of the two points of inflexion on the graph of f .

[2 marks]

65d. Find the equation of the normal to the graph of f where $x = 0.75$, giving your answer in the form $y = mx + c$.

65e. Consider the points

$\quad$ A $(a$, $f(a))$,

$\quad$ B $(b$, $f(b))$ and

$\quad$ C $(c$, $f(c))$ where a , b and c

$\quad$ $(a < b < c)$ are the solutions of the equation

$\quad$ $f(x) = f'(x)$. Find the area of the triangle ABC.

$\quad$ The function f is defined on the domain

$\quad$ $\left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$ by $f(x) = \ln(1 + \sin x)$.

66a. Show that

$\quad$ $f''(x) = -\frac{1}{(1+\sin x)}$.

66b. (i)$\quad$ Find the Maclaurin series for

$\quad$ $f(x)$ up to and including the term in

x^4 .

$\quad$ (ii)$\quad$ Explain briefly why your result shows that f is neither an even function nor an odd function.

66c. Determine the value of

$\quad$ $\lim\limits_{x \to 0} \frac{\ln(1+\sin x) - x}{x^2}$.

$\quad$ Let

$\quad$ $f(x) = 2x + |x|$,

$\quad$ $x \in \mathbb{R}$.

67a. Prove that f is continuous but not differentiable at the point $(0, 0)$.

67b. Determine the value of

$\quad$ $\int_{-a}^{a} f(x)\mathrm{d}x$ where

$\quad$ $a > 0$.