Topic 6 Part 2 *[456 marks]*

1.

Markscheme

$V = 0.5\pi r^2$ *(A1)*

EITHER

$\frac{dV}{dr} = \pi r$ *A1*

$\frac{dV}{dt} = 4$ *(A1)*

applying chain rule   *M1*

for example

$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

OR

$\frac{dV}{dt} = \pi r \frac{dr}{dt}$ *M1A1*

$\frac{dV}{dt} = 4$ *(A1)*

THEN

$\frac{dr}{dt} = 4 \times \frac{1}{\pi r}$ *A1*

when

$r = 20, \frac{dr}{dt} = \frac{4}{20\pi}$ or $\frac{1}{5\pi}$ (cm s^{-1}) *A1*

Note: Allow h instead of 0.5 up until the final *A1*.

[6 marks]

Examiners report

There was a large variety of methods used in this question, with most candidates choosing to implicitly differentiate the expression for volume in terms of r.

2.

Markscheme

$8y \times \frac{1}{x} + 8\frac{dy}{dx}\ln x - 4x + 8y\frac{dy}{dx} = 0$ *M1A1A1*

Note: *M1* for attempt at implicit differentiation. *A1* for differentiating

$8y\ln x$, *A1* for differentiating the rest.

when

$x = 1, 8y \times 0 - 2 \times 1 + 4y^2 = 7$ *(M1)*

$y^2 = \frac{9}{4} \Rightarrow y = \frac{3}{2}$ (as $y > 0$)   *A1*

at

$(1, \frac{3}{2}) \frac{dy}{dx} = -\frac{2}{3}$ *A1*
or
$y - \frac{3}{2} = -\frac{2}{3}(x - 1)$
$y = -\frac{2}{3}x + \frac{13}{6}$ *A1*
[7 marks]

[2 marks]

Markscheme

$$\sin(\pi x^{-1}) = 0 \quad \text{(A1)} \quad \frac{\pi}{x} = \pi, \, 2\pi(\dots)$$

$$x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10} \quad \text{A1}$$

[2 marks]

[3 marks]

Markscheme

$$\left[\cos(\pi x^{-1}) \right]_{\frac{1}{n+1}}^{\frac{1}{n}} \quad \text{M1}$$

$$= \cos(\pi n) - \cos(\pi(n+1)) \quad \text{A1}$$

$= 2$ when n is even and $= -2$ when n is odd   *A1*

[3 marks]

[2 marks]

Markscheme

$$\int_{\frac{1}{10}}^{1} \left| \pi x^{-2} \sin(\pi x^{-1}) \right| \, dx = 2 + 2 + \dots + 2 = 18 \quad \text{(M1)A1}$$

[2 marks]

4a.

Markscheme

(i)   attempt at chain rule   *(M1)*

$f'(x) = \frac{2\ln x}{x}$ *A1*

(ii)   attempt at chain rule   *(M1)*

$g'(x) = \frac{2}{x\ln x}$ *A1*

(iii)

$g'(x)$ is positive on $]1, \infty[$ *A1*

so $g(x)$ is increasing on $]1, \infty[$ *AG*

[5 marks]

Examiners report

[N/A]

Markscheme

(i)   rearrange in standard form:

$$\frac{dy}{dx} + \frac{2}{x\ln x}y = \frac{2x-1}{(\ln x)^2}, \quad x > 1 \quad \textbf{(A1)}$$

integrating factor: **(M1)**

$$e^{\int \frac{2}{x\ln x}dx}$$

$$= e^{\ln\left((\ln x)^2\right)}$$

$$= (\ln x)^2 \quad \textbf{(A1)}$$

multiply by integrating factor   **(M1)**

$$(\ln x)^2\frac{dy}{dx} + \frac{2\ln x}{x}y = 2x - 1 \quad \textbf{M1}$$

$$\frac{d}{dx}\left(y(\ln x)^2\right) = 2x - 1 \quad \textbf{M1} \quad \left(\text{or } y(\ln x)^2 = \int 2x - 1\,dx\right)$$

attempt to integrate: **M1**

$$(\ln x)^2 y = x^2 - x + c \quad \textbf{A1}$$

$$y = \frac{x^2 - x + c}{(\ln x)^2}$$

(ii)   attempt to use the point

$$\left(e, e^2\right)$$

to determine c: **M1**

eg,

or

$$(\ln e)^2 e^2 = e^2 - e + c$$

or

$$e^2 = \frac{e^2 - e + c}{(\ln e)^2}$$

$$e^2 = e^2 - e + c \quad \textbf{A1}$$

$$c = e \quad \textbf{AG}$$

(iii)

$$y = \frac{x^2 - x + e}{(\ln x)^2}$$
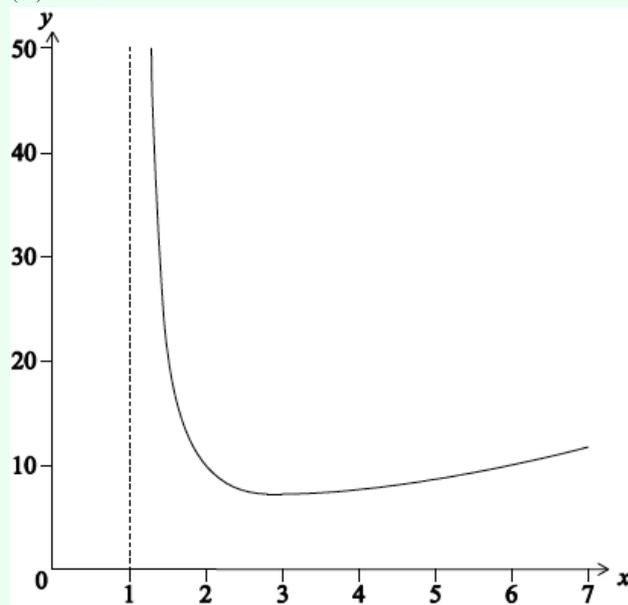
graph with correct shape   **A1**

minimum at

(accept answers to a minimum of 2 s.f)   **A1**

$$x = 3.1$$

asymptote shown at

A1

$$x = 1$$

Note: *y*-coordinate of minimum not required for **A1**;

Equation of asymptote not required for **A1** if VA appears on the sketch.

Award **A0** for asymptotes if more than one asymptote are shown

[12 marks]

5a.

Markscheme

(a)

A1A1
$\frac{\pi}{2}(1.57)$, $\frac{3\pi}{2}(4.71)$
hence the coordinates are

A1
$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$
[3 marks]

5b.

Markscheme

(i)

A1A1A1
$\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(x^2 - (x + 2\cos x)^2 \right) \, \mathrm{d}x$

Note: Award *A1* for

, *A1* for correct limits and *A1* for
$x^2 - (x + 2\cos x)^2$
.
π

(ii)

A2
$6\pi^2 \ (= 59.2)$

Notes: Do not award **ft** from (b)(i).

[5 marks]

Markscheme

METHOD 1

sketch showing where the lines cross or zeros of

(M1)
$y = x(x+2)^6 - x$
(A1)
$x = 0$
and
$x = -1$
(A1)
$x = -3$
the solution is

or
$-3 < x < -1$
A1A1
$x > 0$

Note: Do not award either final *A1* mark if strict inequalities are not given.

METHOD 2

separating into two cases

and
$x > 0$
(M1)
$x < 0$
if

then
$x > 0$
always true   *(M1)*
$(x+2)^6 > 1 \Rightarrow$
if

then
$x < 0$
(M1)
$(x+2)^6 < 1 \Rightarrow -3 < x < -1$
so the solution is

or
$-3 < x < -1$
A1A1
$x > 0$

Note: Do not award either final *A1* mark if strict inequalities are not given.

METHOD 3

(A1)
$f(x) = x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x$
solutions to

are   *(M1)*
$x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 63x = 0$
and
$x = 0, x = -1$
(A1)
$x = -3$
so the solution is

or
$-3 < x < -1$
A1A1
$x > 0$

Note: Do not award either final *A1* mark if strict inequalities are not given.

METHOD 4

when
$f(x) = x$

$x(x+2)^6 = x$
either

or
$x = 0$
(A1)
$(x+2)^6 = 1$
if

then
$(x+2)^6 = 1$
so
$x + 2 = \pm 1$
or
$x = -1$
(M1)(A1)

$x = -3$
the solution is

or

$-3 < x < -1$ **A1A1**

$x > 0$

Note: Do not award either final **A1** mark if strict inequalities are not given.

[5 marks]

Examiners report

[N/A]

6b. *[5 marks]*

Markscheme

METHOD 1 (by substitution)

substituting

$u = x + 2$ **(M1)**

$\mathrm{d}u = \mathrm{d}x$ **M1A1**

$\int (u - 2)u^6 \, \mathrm{d}u$ **(A1)**

$= \frac{1}{8}u^8 - \frac{2}{7}u^7 (+c)$ **A1**

$= \frac{1}{8}(x+2)^8 - \frac{2}{7}(x+2)^7 (+c)$

METHOD 2 (by parts)

$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1, \ \frac{\mathrm{d}v}{\mathrm{d}x} = (x+2)^6 \Rightarrow v = \frac{1}{7}(x+2)^7$ **(M1)(A1)**

$\int x(x+2)^6 \, \mathrm{d}x = \frac{1}{7}x(x+2)^7 - \frac{1}{7}\int (x+2)^7 \, \mathrm{d}x$ **M1**

$= \frac{1}{7}x(x+2)^7 - \frac{1}{56}(x+2)^8 (+c)$ **A1A1**

METHOD 3 (by expansion)

$\int f(x)\,\mathrm{d}x = \int \left(x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x\right) \mathrm{d}x$ **M1A1**

$= \frac{1}{8}x^8 + \frac{12}{7}x^7 + 10x^6 + 32x^5 + 60x^4 + 64x^3 + 32x^2 (+c)$ **M1A2**

Note: Award **M1A1** if at least four terms are correct.

[5 marks]

Examiners report

[N/A]

7a. *[4 marks]*

Markscheme

so

$x \to -\infty \Rightarrow y \to -\frac{1}{2}$ **(M1)A1**

is an asymptote

$y = -\frac{1}{2}$

so

$e^x - 2 = 0 \Rightarrow x = \ln 2$ **(M1)A1**

is an asymptote

$x = \ln 2 \ (= 0.693)$

[4 marks]

Examiners report

[N/A]

7b.

Markscheme

(i)

$$f'(x) = \frac{2(e^x-2)e^{2x}-(e^{2x}+1)e^x}{(e^x-2)^2}$$ **M1A1**

(ii) $$= \frac{e^{3x}-4e^{2x}-e^x}{(e^x-2)^2}$$

when $f'(x)=0$

$$e^{3x}-4e^{2x}-e^x=0$$ **M1**

$$e^x\left(e^{2x}-4e^x-1\right)=0$$

$$e^x=0,\ e^x=-0.236,\ e^x=4.24\ (\text{or } e^x=2\pm\sqrt{5})$$ **A1A1**

Note: Award *A1* for zero, *A1* for other two solutions.

Accept any answers which show a zero, a negative and a positive.

as

exactly one solution **R1**
$e^x>0$

Note: Do not award marks for purely graphical solution.

(iii) (1.44, 8.47) **A1A1**

[8 marks]

Examiners report

[N/A]

7c.

Markscheme

$f'(0)=-4$ **(A1)**
so gradient of normal is

$\frac{1}{4}$ **(M1)**
$f(0)=-2$ **(A1)**
so equation of

L_1 is

$y=\frac{1}{4}x-2$ **A1**
[4 marks]

Examiners report

[N/A]

7d. [5 marks]

Markscheme

$f'(x) = \frac{1}{4}$ *M1*

so

$x = 1.46$ *(M1)A1*

$f(1.46) = 8.47$ *(A1)*

equation of

is

L_2

$y - 8.47 = \frac{1}{4}(x - 1.46)$ *A1*

(or

)

$y = \frac{1}{4}x + 8.11$

[5 marks]

Examiners report

[N/A]

8a. [2 marks]

Markscheme

(a)

$\frac{(x^2+1)-2x(x+1)}{(x^2+1)^2} \left(= \frac{-x^2-2x+1}{(x^2+1)^2} \right)$ *M1A1*

[2 marks]

Examiners report

[N/A]

8b. [1 mark]

Markscheme

$\frac{-x^2-2x+1}{(x^2+1)^2} = 0$

$x = -1 \pm \sqrt{2}$ *A1*

[1 mark]

Examiners report

[N/A]

8c.

Markscheme

$$f''(x) = \frac{\overset{\textbf{A1A1}}{(-2x-2)(x^2+1)^2 - 2(2x)(x^2+1)(-x^2-2x+1)}}{(x^2+1)^4}$$

Note: Award **A1** for

or equivalent.
$$(-2x-2)(x^2+1)^2$$

Note: Award **A1** for

or equivalent.
$$-2(2x)(x^2+1)(-x^2-2x+1)$$

$$= \frac{(-2x-2)(x^2+1) - 4x(-x^2-2x+1)}{(x^2+1)^3}$$

$$\overset{\textbf{A1}}{=} \frac{2x^3+6x^2-6x-2}{(x^2+1)^3}$$

$$\left(= \frac{2(x^3+3x^2-3x-1)}{(x^2+1)^3} \right)$$

[3 marks]

Examiners report

[N/A]

8d.

Markscheme

recognition that

is a factor   **(R1)**
$$(x-1)$$

$$\overset{\textbf{M1}}{(x-1)}(x^2+bx+c) = (x^3+3x^2-3x-1)$$

$$\overset{\textbf{A1}}{\Rightarrow} x^2 + 4x + 1 = 0$$

$$\overset{\textbf{A1}}{x} = -2 \pm \sqrt{3}$$

Note: Allow long division / synthetic division.

[4 marks]

Examiners report

[N/A]

8e.

Markscheme

$$\int_{-1}^{0} \frac{x+1}{x^2+1} dx \quad \textbf{M1}$$

$$\int \frac{x+1}{x^2+1} dx = \int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx \quad \textbf{M1}$$

$$= \tfrac{1}{2}\ln(x^2+1) + \arctan(x) \quad \textbf{A1A1}$$

$$= \left[\tfrac{1}{2}\ln(x^2+1) + \arctan(x) \right]_{-1}^{0} = \tfrac{1}{2}\ln 1 + \arctan 0 - \tfrac{1}{2}\ln 2 - \arctan(-1) \quad \textbf{M1}$$

$$= \tfrac{\pi}{4} - \ln\sqrt{2} \quad \textbf{A1}$$

[6 marks]

## Examiners report

[N/A]

9a.
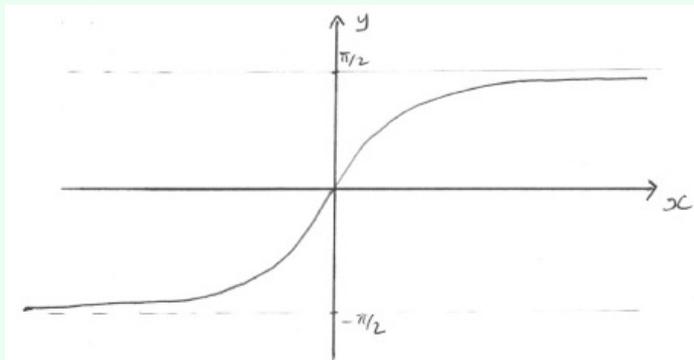
Markscheme

A1A1

Note: *A1* for correct shape, *A1* for asymptotic behaviour at

.
$y = \pm\frac{\pi}{2}$

[2 marks]

Examiners report

[N/A]

9b.

Markscheme

A1
$h \circ g(x) = \arctan\left(\frac{1}{x}\right)$
domain of

 is equal to the domain of
$h \circ g$
A1
$g : x \in \circ, \ x \neq 0$
[2 marks]

Examiners report

[N/A]

9c.

Markscheme

(i)

$f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$ ***M1A1***

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$ ***(A1)***

$f'(x) = \frac{1}{1+x^2} + \frac{\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$ ***A1***

$= 0$

(ii)   **METHOD 1**

f is a constant   ***R1***

when

$x > 0$ ***M1A1***

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$ ***AG***

METHOD 2

from diagram   ***A1***

$\theta = \arctan\frac{1}{x}$ ***A1***

$\alpha = \arctan x$ ***R1***

$\theta + \alpha = \frac{\pi}{2}$

hence

$f(x) = \frac{\pi}{2}$ ***AG***

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ ***M1***

$= \frac{x + \frac{1}{x}}{\left(\frac{1}{x}\right)}$ ***A1***
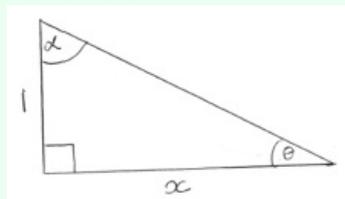
denominator = 0, so   ***R1***

$f(x) = \frac{\pi}{2}$ (for $x > 0$)

[7 marks]

Examiners report

[N/A]

9d.

Markscheme

(i)   Nigel is correct.   ***A1***

METHOD 1

$\arctan(x)$ is an odd function and
$\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function   ***R1***

METHOD 2

$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$ ***R1***

therefore f is an odd function.

(ii)

$f(x) = -\frac{\pi}{2}$ ***A1***

[3 marks]

Examiners report

[N/A]

Markscheme

$x_\text{A} = 2.87$ *A1*
$x_\text{B} = 6.78$ *A1*

[2 marks]

Examiners report

[N/A]

Markscheme

$\int_{2.8\ldots}^{6.7\ldots} 1 - 2\sin x - x^2 \text{e}^{-x} \text{d}x$ *(M1)(A1)*
$= 6.76$ *A1*

Note: Award *(M1)* for definite integral and *(A1)* for a correct definite integral.

[3 marks]

Examiners report

[N/A]

11.

Markscheme

METHOD 1

volume of a cone is

$V = \frac{1}{3}\pi r^2 h$
given

M1

$h = r$, $V = \frac{1}{3}\pi h^3$
(*A1*)

$\frac{dV}{dh} = \pi h^2$
when

M1A1

$h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$ (using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$)
A1

$\frac{dV}{dt} = 8\pi \, (= 25.1) \, (\text{cm}^3 \, \text{min}^{-1})$

METHOD 2

volume of a cone is

$V = \frac{1}{3}\pi r^2 h$
given

M1

$h = r$, $V = \frac{1}{3}\pi h^3$
A1

$\frac{dV}{dt} = \frac{1}{3}\pi \times 3h^2 \times \frac{dh}{dt}$
when

M1A1

$h = 4$, $\frac{dV}{dt} = \pi \times 4^2 \times 0.5$
A1

$\frac{dV}{dt} = 8\pi \, (= 25.1) \, (\text{cm}^3 \, \text{min}^{-1})$

METHOD 3

$V = \frac{1}{3}\pi r^2 h$
M1A1

$\frac{dV}{dt} = \frac{1}{3}\pi \left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt} \right)$

Note: Award ***M1*** for attempted implicit differentiation and ***A1*** for each correct term on the RHS.

when

M1A1

$h = 4$, $r = 4$, $\frac{dV}{dt} = \frac{1}{3}\pi \left(2 \times 4 \times 4 \times 0.5 + 4^2 \times 0.5 \right)$
A1

$\frac{dV}{dt} = 8\pi \, (= 25.1) \, (\text{cm}^3 \, \text{min}^{-1})$
[5 marks]

Examiners report

[N/A]

12a. [5 marks]

Markscheme

METHOD 1

expanding the brackets first:

M1

$$x^4 + 2x^2y^2 + y^4 = 4xy^2$$

M1A1A1

$$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 4y^2 + 8xy\frac{dy}{dx}$$

Note: Award *M1* for an attempt at implicit differentiation.

Award *A1* for each side correct.

or equivalent *A1*
$$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{x^2y + y^3}$$

METHOD 2

M1A1A1

$$2\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right) = 4y^2 + 8xy\frac{dy}{dx}$$

Note: Award *M1* for an attempt at implicit differentiation.

Award *A1* for each side correct.

METHOD 3

$$\left(x^2 + y^2\right)\left(x + y\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$$

$$x^3 + x^2y\frac{dy}{dx} + y^2x + y^3\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$$

or equivalent *A1*
$$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{x^2y + y^3}$$

[5 marks]

Examiners report

[N/A]

12b. [3 marks]

Markscheme

METHOD 1

at (1, 1),

$\frac{dy}{dx}$ is undefined *M1A1*

A1

$y = 1$

METHOD 2

gradient of normal

M1

$$= -\frac{1}{\frac{dy}{dx}} = -\frac{(yx^2 - 2xy + y^3)}{(x^3 - xy^2 + y^2)}$$

at $(1, 1)$ gradient

A1

$= 0$

A1

$y = 1$

[3 marks]

Examiners report

[N/A]

13.

Markscheme

(a)

$$C = \text{AX} \times 5k + \text{XB} \times k \quad \textbf{(M1)}$$

Note: Award **(M1)** for attempting to express the cost in terms of AX, XB and k.

$$= 5k\sqrt{450^2 + x^2} + (1000 - x)k \quad \textbf{A1}$$
$$= 5k\sqrt{202\,500 + x^2} + (1000 - x)k \quad \textbf{AG}$$

[2 marks]

(b)   (i)

$$\frac{\mathrm{d}C}{\mathrm{d}x} = k\left[\frac{5 \times 2x}{2\sqrt{202\,500 + x^2}} - 1\right] = k\left(\frac{5x}{\sqrt{202\,500 + x^2}} - 1\right) \quad \textbf{M1A1}$$

Note: Award **M1** for an attempt to differentiate and **A1** for the correct derivative.

(ii)   attempting to solve

$$\frac{\mathrm{d}C}{\mathrm{d}x} = 0 \quad \textbf{M1}$$

$$\frac{5}{\sqrt{202\,500 + x^2}} = 1 \quad \textbf{(A1)}$$

$$x = 91.9\left(= \frac{75\sqrt{6}}{2}\,(\text{m})\right) \quad \textbf{A1}$$

METHOD 1

for example,

at

$$x = 91 \quad \frac{\mathrm{d}C}{\mathrm{d}x} = -0.00895k < 0 \quad \textbf{M1}$$

$$x = 92 \quad \frac{\mathrm{d}C}{\mathrm{d}x} = 0.001506k > 0 \quad \textbf{A1}$$

Note: Award **M1** for attempting to find the gradient either side of $x = 91.9$ and **A1** for two correct values.

thus $x = 91.9$ gives a minimum   **AG**

METHOD 2

$$\frac{\mathrm{d}^2 C}{\mathrm{d}x^2} = \frac{1\,012\,500k}{(x^2 + 202\,500)^{\frac{3}{2}}} \quad \textbf{(M1)A1}$$

at $x = 91.9 \quad \frac{\mathrm{d}^2 C}{\mathrm{d}x^2} = 0.010451k > 0$

Note: Award **M1** for attempting to find the second derivative and **A1** for the correct value.

Note: If $\frac{\mathrm{d}^2 C}{\mathrm{d}x^2}$ is obtained and its value at $x = 91.9$ is not calculated, award **(M1)A1** for correct reasoning *eg*, both numerator and denominator are positive at $x = 91.9$.

thus $x = 91.9$ gives a minimum   **AG**

METHOD 3

Sketching the graph of either C versus x or $\frac{\mathrm{d}C}{\mathrm{d}x}$ versus x.   **M1**

Clearly indicating that $x = 91.9$ gives the minimum on their graph.   **A1**

(c)

A1

$C_{\min} = 3205k$

Note: Accept $3200k$.

Accept $3204k$.

[1 mark]

(d)

M1

$\arctan\left(\frac{450}{91.855865K}\right) = 78.463K^\circ$

$180 - 78.463K = 101.537K$

A1

$= 102^\circ$

[2 marks]

(e) (i) when

A1

$\theta = 120^\circ$, $x = 260$ (m) $\left(\frac{450}{\sqrt{3}} \text{ (m)}\right)$

M1

$\frac{133.728K}{3204.5407685K} \times 100\%$

A1

$= 4.17\ (\%\ )$

[3 marks]

Total [15 marks]

Examiners report

14a.   *[2 marks]*

Markscheme

(a)

A1
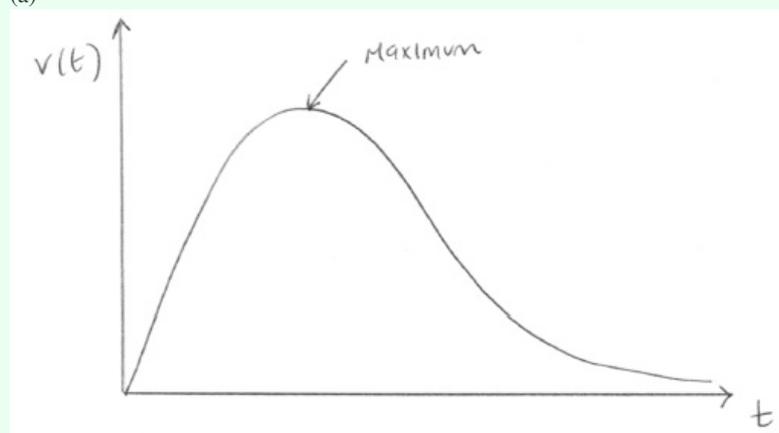
A1 for correct shape and correct domain

A1

$(1.41, 0.0884)$ $\left(\sqrt{2}, \frac{\sqrt{2}}{16}\right)$

[2 marks]

Examiners report

14b. [4 marks]

Markscheme

EITHER

$u = t^2$ ***A1***

$\frac{du}{dt} = 2t$

OR

$t = u^{\frac{1}{2}}$ ***A1***

$\frac{dt}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

THEN

M1

$\int \frac{t}{12+t^4}dt = \frac{1}{2}\int \frac{du}{12+u^2}$ ***M1***

$= \frac{1}{2\sqrt{12}}\arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$ ***A1***

or equivalent

$\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$

[4 marks]

Examiners report

[N/A]

14c. [3 marks]

Markscheme

$\int_0^6 \frac{t}{12+t^4}dt$ ***(M1)***

$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$ ***M1***

A1

$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)\ \left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)\ (\text{m})$

Note: Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

[3 marks]

Examiners report

[N/A]

14d. [3 marks]

Markscheme

$\frac{dv}{ds} = \frac{1}{2\sqrt{s(1-s)}}$ ***(A1)***

$a = v\frac{dv}{ds}$ ***(M1)***

$a = \arcsin\left(\sqrt{s}\right) \times \frac{1}{2\sqrt{s(1-s)}}$

$a = \arcsin\left(\sqrt{0.1}\right) \times \frac{1}{2\sqrt{0.1 \times 0.9}}$ ***A1***

$a = 0.536\ (\text{ms}^{-2})$

[3 marks]

Examiners report

[N/A]

15.

Markscheme

M1M1A1
$$3x^2y^2 + 2x^3y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 9\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Note: First *M1* for attempt at implicit differentiation, second *M1* for use of product rule.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}(A1)\frac{3x^2y^2+3x^2}{3y^2-2x^3y-9}\right)$$
$$\Rightarrow 3x^2 + 3x^2y^2 = 0$$
$$\Rightarrow 3x^2\left(1 + y^2\right) = 0$$
A1
$$x = 0$$

Note: Do not award *A1* if extra solutions given *eg*

$$y = \pm 1$$

substituting

into original equation   *(M1)*
$$x = 0$$
$$y^3 - 9y = 0$$
$$y(y + 3)(y - 3) = 0$$
$$y = 0, \; y = \pm 3$$
coordinates

A1
$$(0, 0), (0, 3), (0, -3)$$
[7 marks]

Examiners report

The majority of candidates were able to apply implicit differentiation and the product rule correctly to obtain

. The better then recognised that
$$3x^2\left(1 + y^2\right) = 0$$
was the only possible solution. Such candidates usually went on to obtain full marks. A number decided that
$$x = 0$$
though then made no further progress. The solution set
$$y = \pm 1$$
and
$$x = 0$$
was also occasionally seen. A small minority found the correct x and y values for the three co-ordinates but then surprisingly
$$y = \pm i$$
expressed them as

and
$$(0, 0), (3, 0)$$
$$(-3, 0)$$

16a.

Markscheme

(i)

M1A1
$$f'(x) = \mathrm{e}^{-x} - x\mathrm{e}^{-x}$$
(ii)

$$f'(x) = 0 \Rightarrow x = 1$$
coordinates

A1
$$\left(1, \mathrm{e}^{-1}\right)$$
[3 marks]

Examiners report

Part a) proved to be an easy start for the vast majority of candidates.

16b.

Markscheme

$f''(x) = -e^{-x} - e^{-x} + xe^{-x} \ (= -e^{-x}(2-x))$ **A1**

substituting

into

$x = 1$ **M1**

$f''(x)$

hence maximum **R1AG**

$f''(1) \left(= -e^{-1} \right) < 0$

[3 marks]

Examiners report

Full marks for part b) were again likewise seen, though a small number shied away from considering the sign of their second derivative, despite the question asking them to do so.

Part c) again proved to be an easily earned 2 marks.

16c.

Markscheme

$f''(x) = 0 \ (\Rightarrow x = 2)$ **M1**

coordinates

$\left(2, \, 2e^{-2}\right)$ **A1**

[2 marks]

Examiners report

Full marks for part b) were again likewise seen, though a small number shied away from considering the sign of their second derivative, despite the question asking them to do so.

Part c) again proved to be an easily earned 2 marks.

$f''(x) = -e^{-x} - e^{-x} + xe^{-x} \ (= -e^{-x}(2-x))$

substituting

into

$x = 1$

$f''(x)$

hence maximum

$f''(1) \left(= -e^{-1} \right) < 0$

Markscheme

(i)

A1

$g(x) = \frac{x}{2}e^{-\frac{x}{2}}$

(ii) coordinates of maximum

A1

$(2, e^{-1})$

(iii) equating

and attempting to solve

$f(x) = g(x)$

$xe^{-x} = \frac{x}{2}e^{-\frac{x}{2}}$

(A1)

$\Rightarrow x\left(2e^{\frac{x}{2}} - e^x\right) = 0$

A1

$\Rightarrow x = 0$

or

$2e^{\frac{x}{2}} = e^x$

$\Rightarrow e^{\frac{x}{2}} = 2$

$\Rightarrow x = 2\ln 2$

A1

$(\ln 4)$

Note: Award first *(A1)* only if factorisation seen or if two correct

solutions are seen.

Examiners report

Many candidates lost their way in part d). A variety of possibilities for

were suggested, commonly

$g(x)$

,

$2xe^{-2x}$

or similar variations. Despite section ii) being worth only one mark, (and 'state' being present in the question), many laborious

xe^{-1}

attempts at further differentiation were seen. Part diii was usually answered well by those who gave the correct function for

.

$g(x)$
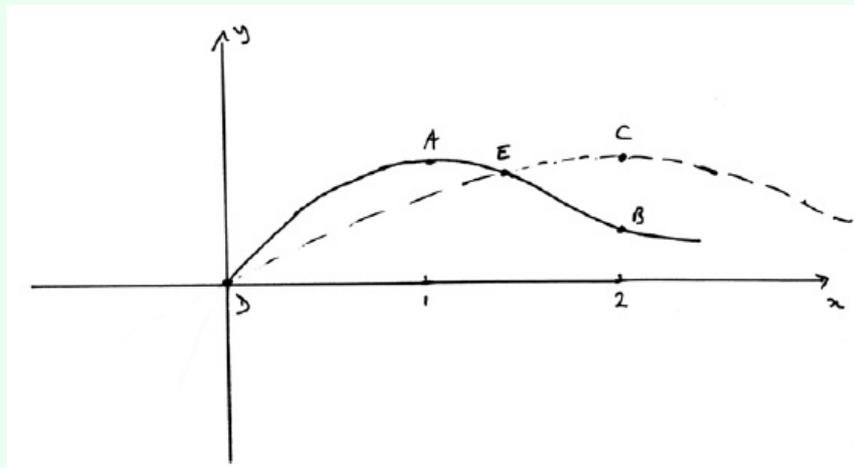
Markscheme

A4

Note: Award *A1* for shape of

, including domain extending beyond
f
.
$x = 2$
Ignore any graph shown for

.
$x < 0$
Award *A1* for A and B correctly identified.

Award *A1* for shape of

, including domain extending beyond
g
.
$x = 2$
Ignore any graph shown for

. Allow follow through from
$x < 0$
.
f
Award *A1* for C, D and E correctly identified (D and E are interchangeable).

[4 marks]

Examiners report

Part e) was also answered well by those who had earned full marks up to that point.

Markscheme

$$A = \int_0^1 \frac{x}{2} e^{-\frac{x}{2}} \, dx \quad \textbf{\textit{M1}} \atop \textbf{\textit{A1}}$$
$$= \left[-x e^{-\frac{x}{2}} \right]_0^1 - \int_0^1 -e^{-\frac{x}{2}} \, dx$$

Note: Condone absence of limits or incorrect limits.

$$= -e^{-\frac{1}{2}} - \left[2 e^{-\frac{x}{2}} \right]_0^1 \quad \textbf{\textit{A1}}$$
$$= -e^{-\frac{1}{2}} - \left(2 e^{-\frac{1}{2}} - 2 \right) = 2 - 3 e^{-\frac{1}{2}} \quad \textit{[3 marks]}$$

Examiners report

While the integration by parts technique was clearly understood, it was somewhat surprising how many careless slips were seen in this part of the question. Only a minority gained full marks for part f).

17a.

Markscheme

M1
$$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$$
A1
$$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$$
AG
$$= 2\cos n\theta$$
[2 marks]

Examiners report

Part a) has appeared several times before, though with it again being a 'show that' question, some candidates still need to be more aware of the need to show every step in their working, including the result that

$$\sin(-n\theta) = -\sin(n\theta)$$

17b.

Markscheme

(b)

A1
$$\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\tfrac{1}{z}\right) + 6z^2\left(\tfrac{1}{z^2}\right) + 4z\left(\tfrac{1}{z^3}\right) + \tfrac{1}{z^4}$$

Note: Accept

$$\left(z + z^{-1}\right)^4 = 16\cos^4\theta$$

[1 mark]

Examiners report

Part b) was usually answered correctly.

17c.

Markscheme

METHOD 1

M1
$$\left(z + z^{-1}\right)^4 = \left(z^4 + \tfrac{1}{z^4}\right) + 4\left(z^2 + \tfrac{1}{z^2}\right) + 6$$
A1A1
$$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$$

Note: Award **A1** for RHS, **A1** for LHS, independent of the **M1**.

A1
$$\cos^4\theta = \tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}$$
$$\left(\text{or } p = \tfrac{1}{8},\ q = \tfrac{1}{2},\ r = \tfrac{3}{8}\right)$$

METHOD 2

M1
$$\cos^4\theta = \left(\tfrac{\cos 2\theta + 1}{2}\right)^2$$
A1
$$= \tfrac{1}{4}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right)$$
A1
$$= \tfrac{1}{4}\left(\tfrac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$$
$$\cos^4\theta = \tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}$$
$$\left(\text{or } p = \tfrac{1}{8},\ q = \tfrac{1}{2},\ r = \tfrac{3}{8}\right)$$
[4 marks]

Examiners report

Part c) was again often answered correctly, though some candidates often less successfully utilised a trig-only approach rather than taking note of part b).

17d.

[3 marks]

Markscheme

M1

$$\left(z + z^{-1}\right)^6 = z^6 + 6z^5\left(\tfrac{1}{z}\right) + 15z^4\left(\tfrac{1}{z^2}\right) + 20z^3\left(\tfrac{1}{z^3}\right) + 15z^2\left(\tfrac{1}{z^4}\right) + 6z\left(\tfrac{1}{z^5}\right) + \tfrac{1}{z^6}$$

A1A1
$$\left(z + z^{-1}\right)^6 = \left(z^6 + \tfrac{1}{z^6}\right) + 6\left(z^4 + \tfrac{1}{z^4}\right) + 15\left(z^2 + \tfrac{1}{z^2}\right) + 20$$
$$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$$

Note: Award *A1* for RHS, *A1* for LHS, independent of the *M1*.

AG
$$\cos^6\theta = \tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16}$$

Note: Accept a purely trigonometric solution as for (c).

[3 marks]

Examiners report

Part d) was a good source of marks for those who kept with the spirit of using complex numbers for this type of question. Some limited attempts at trig-only solutions were seen, and correct solutions using this approach were extremely rare.

17e.

[3 marks]

Markscheme

$$\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16}\right) d\theta$$
M1A1
$$= \left[\tfrac{1}{192}\sin 6\theta + \tfrac{3}{64}\sin 4\theta + \tfrac{15}{64}\sin 2\theta + \tfrac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$$
A1
$$= \tfrac{5\pi}{32}$$

[3 marks]

Examiners report

Part e) was well answered, though numerical slips were often common. A small number integrated

as
$$\sin n\theta$$
$$n\cos n\theta$$
. A large number of candidates did not realise the help that part e) inevitably provided for part f). Some correctly expressed the volume

as

and thus gained the first 2 marks but were able to progress no further. Only a small number of able candidates were able to obtain the correct answer of
$$\pi\int \cos^4 x \, dx - \pi\int \cos^6 x \, dx$$

$$\tfrac{\pi^2}{32}$$

17f.

[4 marks]

Markscheme

M1
$$V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$$

M1
$$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x \, dx - \pi\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$$
A1
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \tfrac{3\pi}{16}$$
A1
$$V = \tfrac{3\pi^2}{16} - \tfrac{5\pi^2}{32} = \tfrac{\pi^2}{32}$$

Note: Follow through from an incorrect *r* in (c) provided the final answer is positive.

Examiners report

[N/A]

[3 marks]

Markscheme

(i) constant term =

$\binom{2k}{k}$ $= \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k})   *A1*

(ii)

$2^{2k} \int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, d\theta = \frac{(2k)!}{(k!)^2} \frac{\pi}{2}$ *A1*

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, d\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ *A1*

$\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$

[3 marks]

Examiners report

Part g) proved to be a challenge for the vast majority, though it was pleasing to see some of the highest scoring candidates gain all 3 marks.

[7 marks]

Markscheme

EITHER

$\frac{dx}{du} = 2\sec^2 u$ *A1*

$\int \frac{2\sec^2 u \, du}{4\tan^2 u \sqrt{4 + 4\tan^2 u}}$ *(M1)*

$\left(= \int \frac{2\sec^2 u \, du}{4\tan^2 u \times 2 \sec u} \text{ or } = \int \frac{2\sec^2 u \, du}{4\tan^2 u \sqrt{4\sec^2 u}}\right)$

OR $\int \frac{du}{4\sin^2 u \sqrt{\tan^2 u + 1}}$

$u = \arctan \frac{x}{2}$ *A1*

$\frac{du}{dx} = \frac{2}{4\tan^2 u + 4}$ *(M1)*

$\int \frac{\sqrt{4\tan^2 u + 4} \, du}{4\tan^2 u}$ *A1*

$\int \frac{2 \sec u \, du}{4\tan^2 u}$

THEN

$= \frac{1}{4} \int \frac{\sec u \, du}{\tan^2 u}$ *A1*

$= \frac{1}{4} \int \csc u \cot u \, du$ $\left(= \frac{1}{4} \int \frac{\cos u}{\sin^2 u} du\right)$ *A1*

$= -\frac{1}{4} \csc u (+C)$ $\left(= -\frac{1}{4\sin u}(+C)\right)$

use of either

or an appropriate trigonometric identity   *M1*

$u = \frac{x}{2}$
either

or

$\sin u = \frac{x}{\sqrt{x^2+4}}$
(or equivalent)   *A1*

$\csc u = \frac{\sqrt{x^2+4}}{x}$

$= \frac{-\sqrt{x^2+4}}{4x}(+C)$ *A1 AG*

[7 marks]

Examiners report

Most candidates found this a challenging question. A large majority of candidates were able to change variable from x to u but were not able to make any further progress.

Markscheme

(i)   either counterexample or sketch or

recognising that

intersects the graph of

$y = k \, (k > 1)$ twice   *M1*

$y = f(x)$, function is not

(does not obey horizontal line test)   *R1*

$1 - 1$ so

does not exist   *AG*

f^{-1}

(ii)

$f'(x) = \frac{1}{2}(e^x - e^{-x})$ *(A1)*

$f'(\ln 3) = \frac{4}{3} \, (= 1.33)$ *(A1)*

$m = -\frac{3}{4}$ *M1*   *A1*

$f(\ln 3) = \frac{5}{3} \, (= 1.67)$

EITHER

$\frac{y - \frac{5}{3}}{x - \ln 3} = -\frac{3}{4}$ *M1* *A1*

$4y - \frac{20}{3} = -3x + 3\ln 3$

OR

$\frac{5}{3} = -\frac{3}{4}\ln 3 + c$ *M1*

$c = \frac{5}{3} + \frac{3}{4}\ln 3$ *A1*

$y = -\frac{3}{4}x + \frac{5}{3} + \frac{3}{4}\ln 3$

$12y = -9x + 20 + 9\ln 3$

THEN

$9x + 12y - 9\ln 3 - 20 = 0$ *AG*

(iii)   The tangent at

has equation

$(a, \, f(a))$

.

$y - f(a) = f'(a)(x - a)$ *(M1)*

(or equivalent)   *(A1)*

$f'(a) = \frac{f(a)}{a}$

(or equivalent)   *A1*

$e^a - e^{-a} = \frac{e^a + e^{-a}}{a}$

attempting to solve for a *(M1)*

$a = \pm 1.20$ *A1A1*

[14 marks]

Examiners report

In part (a) (i), successful candidates typically sketched the graph of

, applied the horizontal line test to the graph and concluded that the function was not

$y = f(x)$ (it did not obey the horizontal line test).

$1 - 1$

In part (a) (ii), a large number of candidates were able to show that the equation of the normal at point P was

. A few candidates used the gradient of the tangent rather than using it to find the gradient of the normal.

$9x + 12y - 9\ln 3 - 20 = 0$

Part (a) (iii) challenged most candidates. Most successful candidates graphed

and

$y = f(x)$ on the same set of axes and found the x-coordinates of the intersection points.

$y = xf'(x)$

Markscheme

(i)

$2y = e^x + e^{-x}$ ***M1A1***

$e^{2x} - 2ye^x + 1 = 0$

Note: Award ***M1*** for either attempting to rearrange or interchanging x and y.

$e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$e^x = y \pm \sqrt{y^2 - 1}$ ***A1***

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

Note: Award ***A1*** for correct notation and for stating the positive "branch".

(ii)

$V = \pi \int_1^5 \left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 dy$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use

$V = \pi \int_c^d x^2 \, dy$

$= 37.1 \, \left(\text{units}^3\right)$ ***A1***

[8 marks]

Examiners report

Part (b) (i) challenged most candidates. While a large number of candidates seemed to understand how to find an inverse function, poor algebra skills (e.g. erroneously taking the natural logarithm of both sides) meant that very few candidates were able to form a quadratic in either

or
e^x
.
e^y

Markscheme

METHOD 1

$f'(x) = q - 2x = 0$ ***M1***

$f'(3) = q - 6 = 0$

$q = 6$ ***A1***

$f(3) = p + 18 - 9 = 5$ ***M1***

$p = -4$ ***A1***

METHOD 2

$f(x) = -(x - 3)^2 + 5$ ***M1A1***

$= -x^2 + 6x - 4$

$q = 6, p = -4$ ***A1A1***

[4 marks]

Examiners report

In general candidates handled this question well although a number equated the derivative to the function value rather than zero. Most recognised the shift in the second part although a number shifted only the squared value and not both x values.

20b. *[2 marks]*

Examiners report

In general candidates handled this question well although a number equated the derivative to the function value rather than zero. Most recognised the shift in the second part although a number shifted only the squared value and not both x values.

21a. *[4 marks]*

Examiners report

The whole of this question seemed to prove accessible to a high proportion of candidates.
(a)   was well answered by most, although a number of candidates gave only the x-values of the points or omitted the value at 0.
(b)   was successfully solved by the majority of candidates, who also found the correct equation of the normal in (c).
The last section proved more difficult for many candidates, the most common error being to use the wrong perpendicular sides. There were a number of different approaches here all of which were potentially correct but errors abounded.

21b. *[4 marks]*

$= \frac{3}{2}$

$y - 2 = \frac{3}{2}(x-1) \;\; \left(y = \frac{3}{2}x + \frac{1}{2}\right)$

$-2 = \frac{3}{2}(x-1)$
$x = -\frac{1}{3}$

$\left(-\frac{1}{3}, 0\right)$

21c.   *[7 marks]*

Markscheme

gradient of normal

$= -\frac{2}{3}$ *(A1)*

equation of normal is   *(M1)*

$y - 2 = -\frac{2}{3}(x - 1)$ $\left(y = -\frac{2}{3}x + \frac{8}{3}\right)$

at $x = 0$,

$y = \frac{8}{3}$ *A1*

Note: In the following, allow FT on incorrect coordinates of T and N.

lengths of

$PN = \sqrt{\frac{13}{9}}$ *A1*

$PT = \sqrt{\frac{52}{9}}$ *A1*

area of triangle   *M1*

$PTN = \frac{1}{2} \times \sqrt{\frac{13}{9}} \times \sqrt{\frac{52}{9}}$ (or equivalent *e.g.* $= \frac{13}{9}$)

$\frac{\sqrt{676}}{18}$ *A1*

[7 marks]

Markscheme

(i)
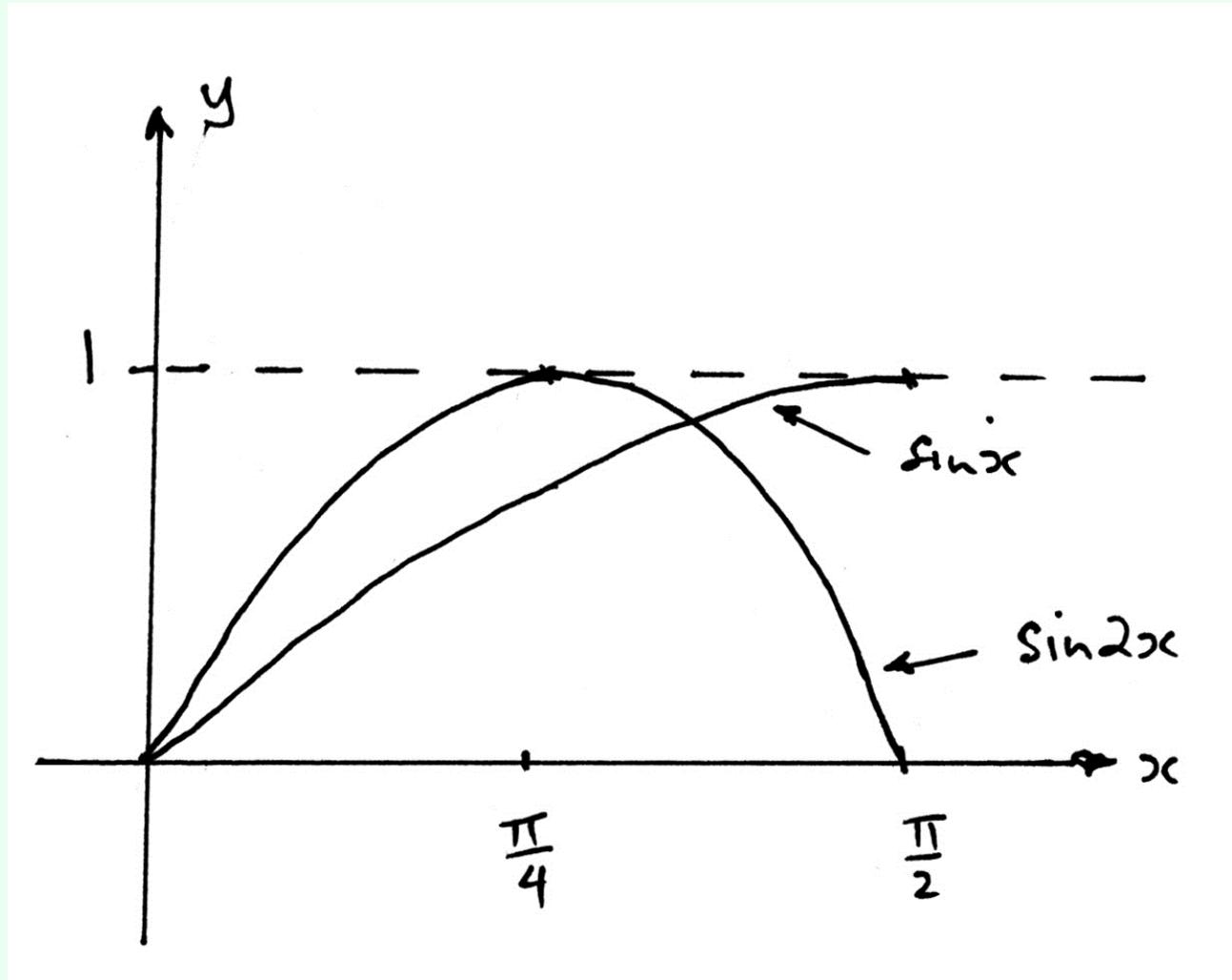
A2

Note: Award **A1** for correct

, **A1** for correct
$\sin x$
.
$\sin 2x$

Note: Award **A1A0** for two correct shapes with

and/or 1 missing.
$\frac{\pi}{2}$

Note: Condone graph outside the domain.

(ii)

,
$\sin 2x = \sin x$

$0 \leqslant x \leqslant \frac{\pi}{2}$ **M1**
$2 \sin x \cos x - \sin x = 0$

$\sin x(2 \cos x - 1) = 0$ **A1A1** **N1N1**
$x = 0, \frac{\pi}{3}$

(iii)   area

M1
$= \int_{0}^{\frac{\pi}{3}} (\sin 2x - \sin x) \mathrm{d}x$
Note: Award **M1** for an integral that contains limits, not necessarily correct, with

and
$\sin x$
subtracted in either order.
$\sin 2x$

A1
$$= \left[-\tfrac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$$
(M1)
$$= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0 + \cos 0\right)$$
$$= \tfrac{3}{4} - \tfrac{1}{2}$$
A1
$$= \tfrac{1}{4}$$

[9 marks]

Examiners report

A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.

Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the $x = 0$ value. Many recognised the value from the graph and corrected this in their final solution. The final part was done well by many candidates.

Many candidates found (b) challenging. Few were able to substitute the dx expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.

Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required. Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

22b. *[8 marks]*

Markscheme

M1A1A1
$$\int_0^1 \sqrt{\tfrac{x}{4-x}}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{\tfrac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\,d\theta$$
Note: Award **M1** for substitution and reasonable attempt at finding expression for dx in terms of $d\theta$, first **A1** for correct limits, second **A1** for correct substitution for dx.

A1
$$\int_0^{\frac{\pi}{6}} 8\sin^2\theta\,d\theta$$
M1
$$\int_0^{\frac{\pi}{6}} 4 - 4\cos 2\theta\,d\theta$$
A1
$$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{6}}$$
(M1)
$$= \left(\tfrac{2\pi}{3} - 2\sin\tfrac{\pi}{3}\right) - 0$$
A1
$$= \tfrac{2\pi}{3} - \sqrt{3}$$

[8 marks]

Examiners report

A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.

Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the $x = 0$ value. Many recognised the value from the graph and corrected this in their final solution. The final part was done well by many candidates.

Many candidates found (b) challenging. Few were able to substitute the dx expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.

Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required. Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

22c.

[8 marks]

Markscheme

(i)

M1

from the diagram above

the shaded area

R1
$$= \int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(y)\mathrm{d}y$$
AG
$$= ab - \int_0^b f^{-1}(x)\mathrm{d}x$$

(ii)

A1
$$f(x) = \arcsin\tfrac{x}{4} \Rightarrow f^{-1}(x) = 4\sin x$$
M1A1A1
$$\int_0^2 \arcsin\left(\tfrac{x}{4}\right)\mathrm{d}x = \tfrac{\pi}{3} - \int_0^{\frac{\pi}{6}} 4\sin x\,\mathrm{d}x$$
Note: Award ***A1*** for the limit

seen anywhere, ***A1*** for all else correct.
$\frac{\pi}{6}$

A1
$$= \tfrac{\pi}{3} - [-4\cos x]_0^{\frac{\pi}{6}}$$
A1
$$= \tfrac{\pi}{3} - 4 + 2\sqrt{3}$$
Note: Award no marks for methods using integration by parts.

[8 marks]

23a.   *[3 marks]*

Markscheme

$\quad$ *(M1)(A1)*
$a = 10\mathrm{e}^{-0.2t}$

at

$t = 10$
$\quad$ *A1*
$a = 1.35\ (\mathrm{ms}^{-2})\ \ (\text{accept } 10e^{-2})$

[3 marks]

23b.   *[3 marks]*

Markscheme

METHOD 1

$\quad$ *(M1)*
$d = \int_0^{10} 50(1 - \mathrm{e}^{-0.2t})\mathrm{d}t$
$\quad$ *A1*
$= 283.83\ldots$

so distance above ground

$\quad$ *A1*
$= 1720\ (\mathrm{m})\ (3\ \mathrm{sf})\ (\text{accept } 1716\ (\mathrm{m}))$

METHOD 2

$\quad$ *M1*
$s = \int 50(1 - \mathrm{e}^{-0.2t})\mathrm{d}t = 50t + 250\mathrm{e}^{-0.2t}(+c)$
Taking $s = 0$ when $t = 0$ gives $c = -250$ $\quad$ *M1*

So when $t = 10$, $s = 283.3\ldots$

so distance above ground

$\quad$ *A1*
$= 1720\ (\mathrm{m})\ (3\ \mathrm{sf})\ (\text{accept } 1716\ (\mathrm{m}))$

[3 marks]

Examiners report

Part (a) was generally correctly answered. A few candidates suffered the Arithmetic Penalty for giving their answer to more than 3sf. A smaller number were unable to differentiate the exponential function correctly. Part (b) was less well answered, many candidates not thinking clearly about the position and direction associated with the initial conditions.

Markscheme

let x = distance from observer to rocket

let h = the height of the rocket above the ground

METHOD 1

A1
$$\frac{dh}{dt} = 300 \text{ when } h = 800$$
M1
$$x = \sqrt{h^2 + 360\,000} = (h^2 + 360\,000)^{\frac{1}{2}}$$
A1
$$\frac{dx}{dh} = \frac{h}{\sqrt{h^2 + 360\,000}}$$
when h = 800

M1
$$\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$$
A1
$$= \frac{300h}{\sqrt{h^2 + 360\,000}}$$
A1
$$= 240 \ (\text{ms}^{-1})$$

[6 marks]

METHOD 2

M1
$$h^2 + 600^2 = x^2$$
A1
$$2h = 2x\frac{dx}{dh}$$

$$\frac{dx}{dh} = \frac{h}{x}$$
A1
$$= \frac{800}{1000} \left(= \frac{4}{5}\right)$$
A1
$$\frac{dh}{dt} = 300$$
M1
$$\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$$

$$= \frac{4}{5} \times 300$$
A1
$$= 240 \ (\text{ms}^{-1})$$

[6 marks]

METHOD 3

M1
$$x^2 = 600^2 + h^2$$
A1A1
$$2x\frac{dx}{dt} = 2h\frac{dh}{dt}$$
when $h = 800$, $x = 1000$

M1A1
$$\frac{dx}{dt} = \frac{800}{1000} \times \frac{dh}{dt}$$
A1
$$= 240 \ (\text{ms}^{-1})$$

[6 marks]

METHOD 4

Distance between the observer and the rocket

M1A1
$$= (600^2 + 800^2)^{\frac{1}{2}} = 1000$$
Component of the velocity in the line of sight

$$= \sin\theta \times 300$$
(where

 angle of elevation)   *M1A1*
$$\theta =$$

A1
...   800

$$\sin \theta = \frac{\cdots}{1000}$$
component

A1
$$= 240 \; (\text{ms}^{-1})$$
[6 marks]

Examiners report

Questions of this type are often open to various approaches, but most full solutions require the application of 'related rates of change'. Although most candidates realised this, their success rate was low. This was particularly apparent in approaches involving trigonometric functions. Some candidates assumed constant speed – this gained some small credit. Candidates should be encouraged to state what their symbols stand for.

25.

[8 marks]

Markscheme

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$$
M1
$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$
A1
$$\frac{dy}{dx} = -\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} = -\sqrt{\frac{y}{x}}$$
Note: Accept

$$\frac{dy}{dx} = 1 - \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$
from making y the subject of the equation, and all correct subsequent working

therefore the gradient at the point P is given by

A1
$$\frac{dy}{dx} = -\sqrt{\frac{q}{p}}$$
equation of tangent is

M1
$$y - q = -\sqrt{\frac{q}{p}}(x - p)$$
$$\left(y = -\sqrt{\frac{q}{p}}x + q + \sqrt{q}\sqrt{p}\right)$$
x-intercept: $y = 0$,

A1
$$n = \frac{q\sqrt{p}}{\sqrt{q}} + p = \sqrt{q}\sqrt{p} + p$$
y-intercept: $x = 0$,

A1
$$m = \sqrt{q}\sqrt{p} + q$$
M1
$$n + m = \sqrt{q}\sqrt{p} + p + \sqrt{q}\sqrt{p} + q$$

$$= 2\sqrt{q}\sqrt{p} + p + q$$
A1
$$= \left(\sqrt{p} + \sqrt{q}\right)^2$$
AG
$$= a$$
[8 marks]

Examiners report

Many candidates were able to perform the implicit differentiation. Few gained any further marks.

Markscheme

prove that

$$1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)^3 + \ldots + n\left(\tfrac{1}{2}\right)^{n-1} = 4 - \tfrac{n+2}{2^{n-1}}$$
for $n = 1$

LHS $= 1$, RHS $= 4 - \tfrac{1+2}{2^0} = 4 - 3 = 1$
so true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

so

$$1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)^3 + \ldots + k\left(\tfrac{1}{2}\right)^{k-1} = 4 - \tfrac{k+2}{2^{k-1}}$$
now for $n = k + 1$

LHS:

A1
$$1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)^3 + \ldots + k\left(\tfrac{1}{2}\right)^{k-1} + (k+1)\left(\tfrac{1}{2}\right)^k$$
M1A1
$$= 4 - \tfrac{k+2}{2^{k-1}} + (k+1)\left(\tfrac{1}{2}\right)^k$$
(or equivalent) ***A1***
$$= 4 - \tfrac{2(k+2)}{2^k} + \tfrac{k+1}{2^k}$$
(accept
$$= 4 - \tfrac{(k+1)+2}{2^{(k+1)-1}}$$
) ***A1***
$$4 - \tfrac{k+3}{2^k}$$

Therefore if it is true for $n = k$ it is true for $n = k + 1$. It has been shown to be true for $n = 1$ so it is true for all

.   ***R1***
$n \ (\in \mathbb{Z}^+)$

Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.

[8 marks]

Examiners report

Part A: Given that this question is at the easier end of the 'proof by induction' spectrum, it was disappointing that so many candidates failed to score full marks. The $n = 1$ case was generally well done. The whole point of the method is that it involves logic, so 'let n = k' or 'put n = k', instead of 'assume ... to be true for n = k', gains no marks. The algebraic steps need to be more convincing than some candidates were able to show. It is astonishing that the R1 mark for the final statement was so often not awarded.

Markscheme

(a)

METHOD 1

$$\int e^{2x}\sin x\,dx \overset{\textit{M1A1A1}}{=} -\cos x e^{2x} + \int 2e^{2x}\cos x\,dx$$

$$\overset{\textit{A1A1}}{=} -\cos x e^{2x} + 2e^{2x}\sin x - \int 4e^{2x}\sin x\,dx$$

$$\overset{\textit{M1}}{5\int e^{2x}\sin x\,dx} = -\cos x e^{2x} + 2e^{2x}\sin x$$

$$\overset{\textit{AG}}{\int e^{2x}\sin x\,dx} = \tfrac{1}{5}e^{2x}(2\sin x - \cos x) + C$$

METHOD 2

$$\int \sin x e^{2x}\,dx \overset{\textit{M1A1A1}}{=} \frac{\sin x e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2}\,dx$$

$$\overset{\textit{A1A1}}{=} \frac{\sin x e^{2x}}{2} - \cos x \frac{e^{2x}}{4} - \int \sin x \frac{e^{2x}}{4}\,dx$$

$$\overset{\textit{M1}}{\tfrac{5}{4}\int e^{2x}\sin x\,dx} = \frac{e^{2x}\sin x}{2} - \frac{\cos x e^{2x}}{4}$$

$$\overset{\textit{AG}}{\int e^{2x}\sin x\,dx} = \tfrac{1}{5}e^{2x}(2\sin x - \cos x) + C$$

[6 marks]

(b)

$$\int \frac{dy}{\sqrt{1-y^2}} \overset{\textit{M1A1}}{=} \int e^{2x}\sin x\,dx$$

$$\overset{\textit{A1}}{\arcsin y} = \tfrac{1}{5}e^{2x}(2\sin x - \cos x)(+C)$$

when

$$x = 0,\ y = 0 \overset{\textit{M1}}{\Rightarrow} C = \tfrac{1}{5}$$

$$y \overset{\textit{A1}}{=} \sin\left(\tfrac{1}{5}e^{2x}(2\sin x - \cos x) + \tfrac{1}{5}\right)$$

[5 marks]

(c)

(i)

A1
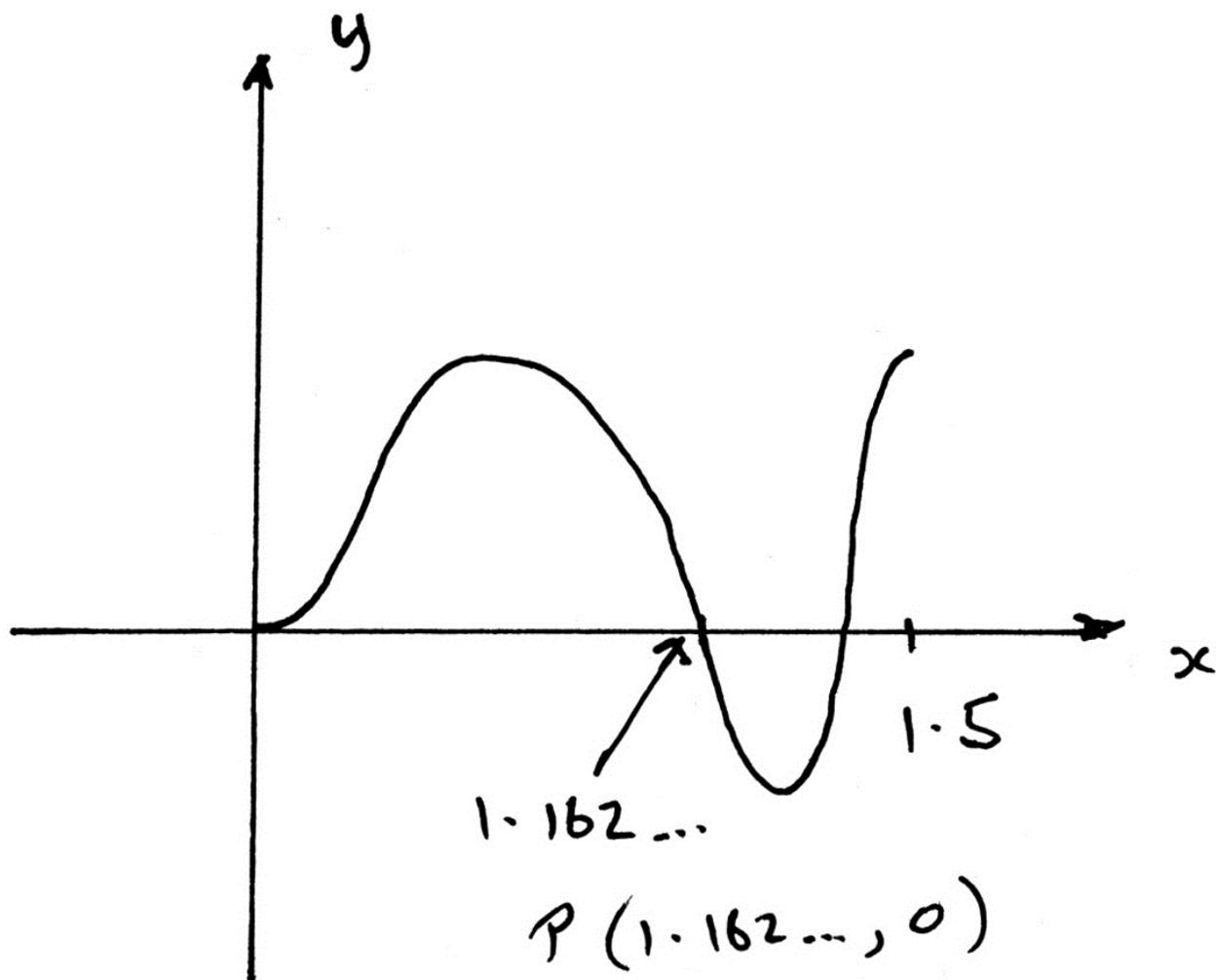
P is (1.16, 0)   *A1*

Note: Award *A1* for 1.16 seen anywhere, *A1* for complete sketch.

Note: Allow FT on their answer from (b)

(ii)

M1A1
$V = \int_0^{1.162...} \pi y^2 \, dx$
A2
$= 1.05$

Note: Allow FT on their answers from (b) and (c)(i).

[6 marks]

Examiners report

Part B: Part (a) was often well done, although some faltered after the first integration. Part (b) was also generally well done, although there were some errors with the constant of integration. In (c) the graph was often attempted, but errors in (b) usually led to manifestly incorrect plots. Many attempted the volume of integration and some obtained the correct value.

Markscheme

M1
$I_0 = \int_0^\pi \mathrm{e}^{-x} \sin x \,\mathrm{d}x$
Note: Award *M1* for

$I_0 = \int_0^\pi \mathrm{e}^{-x} |\sin x| \,\mathrm{d}x$

Attempt at integration by parts, even if inappropriate modulus signs are present.   *M1*

or
$= -[\mathrm{e}^{-x} \cos x]_0^\pi - \int_0^\pi \mathrm{e}^{-x} \cos x \,\mathrm{d}x$
      A1
$= -[\mathrm{e}^{-x} \sin x]_0^\pi - \int_0^\pi \mathrm{e}^{-x} \cos x \,\mathrm{d}x$
or
$= -[\mathrm{e}^{-x} \cos x]_0^\pi - [\mathrm{e}^{-x} \sin x]_0^\pi - \int_0^\pi \mathrm{e}^{-x} \sin x \,\mathrm{d}x$
      A1
$= -[\mathrm{e}^{-x} \sin x + \mathrm{e}^{-x} \cos x]_0^\pi - \int_0^\pi \mathrm{e}^{-x} \sin x \,\mathrm{d}x$
or
$= -[\mathrm{e}^{-x} \cos x]_0^\pi - [\mathrm{e}^{-x} \sin x]_0^\pi - I_0$
      M1
$-[\mathrm{e}^{-x} \sin x + \mathrm{e}^{-x} \cos x]_0^\pi - I_0$
Note: Do not penalise absence of limits at this stage

   A1
$I_0 = \mathrm{e}^{-\pi} + 1 - I_0$
   AG
$I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$
Note: If modulus signs are used around cos x , award no accuracy marks but do not penalise modulus signs around sin x .

[6 marks]

Examiners report

Part (a) is essentially core work requiring repeated integration by parts and many candidates realised that. However, some candidates

left the modulus signs in

 which invalidated their work. In parts (b) and (c) it was clear that very few candidates had a complete understanding of the
I_0
significance of the modulus sign and what conditions were necessary for it to be dropped. Overall, attempts at (b) and (c) were

disappointing with few correct solutions seen.

27b.

Markscheme

$I_n = \int_{n\pi}^{(n+1)\pi} e^{-x} |\sin x| dx$

Attempt to use the substitution

M1

$y = x - n\pi$

(putting

,

$y = x - n\pi$

and

$dy = dx$

)

$[n\pi, (n+1)\pi] \to [0, \pi]$

so

A1

$I_n = \int_0^\pi e^{-(y+n\pi)} |\sin(y + n\pi)| dy$

A1

$= e^{-n\pi} \int_0^\pi e^{-y} |\sin(y + n\pi)| dy$

A1

$= e^{-n\pi} \int_0^\pi e^{-y} \sin y \, dy$

AG

$= e^{-n\pi} I_0$

[4 marks]

Examiners report

Part (a) is essentially core work requiring repeated integration by parts and many candidates realised that. However, some candidates

left the modulus signs in

which invalidated their work. In parts (b) and (c) it was clear that very few candidates had a complete understanding of the
I_0
significance of the modulus sign and what conditions were necessary for it to be dropped. Overall, attempts at (b) and (c) were

disappointing with few correct solutions seen.

27c.

Markscheme

M1

$\int_0^\infty e^{-x} |\sin x| dx = \sum_{n=0}^\infty I_n$

(A1)

$= \sum_{n=0}^\infty e^{-n\pi} I_0$

the

term is an infinite geometric series with common ratio

$\sum$

(M1)

$e^{-\pi}$

therefore

(A1)

$\int_0^\infty e^{-x} |\sin x| dx = \frac{I_0}{1 - e^{-\pi}}$

A1

$= \frac{1 + e^{-\pi}}{2(1 - e^{-\pi})} \left(= \frac{e^\pi + 1}{2(e^\pi - 1)} \right)$

[5 marks]

28a.   *[10 marks]*

Markscheme

(i)   the period is 2   *A1*

(ii)

$v = \frac{ds}{dt} = 2\pi\cos(\pi t) + 2\pi\cos(2\pi t)$ *(M1)A1*

$a = \frac{dv}{dt} = -2\pi^2\sin(\pi t) - 4\pi^2\sin(2\pi t)$ *(M1)A1*

(iii)

$v = 0$

$2\pi\left(\cos(\pi t) + \cos(2\pi t)\right) = 0$

EITHER

$\cos(\pi t) + 2\cos^2(\pi t) - 1 = 0$ *M1*

$(2\cos(\pi t) - 1)(\cos(\pi t) + 1) = 0$ *(A1)*

$\cos(\pi t) = \frac{1}{2}$ or $\cos(\pi t) = -1$ *A1*

$t = \frac{1}{3}$, $t = 1$ *A1*

$t = \frac{5}{3}$, $t = \frac{7}{3}$, $t = \frac{11}{3}$, $t = 3$ *A1*

OR

$2\cos\left(\frac{\pi t}{2}\right)\cos\left(\frac{3\pi t}{2}\right) = 0$ *M1*

$\cos\left(\frac{\pi t}{2}\right) = 0$ or $\cos\left(\frac{3\pi t}{2}\right) = 0$ *A1A1*

$t = \frac{1}{3}$, 1 *A1*

$t = \frac{5}{3}$, $\frac{7}{3}$, 3, $\frac{11}{3}$ *A1*

[10 marks]

Examiners report

28b. [8 marks]

Markscheme

$P(n): f^{(2n)}(x) = (-1)^n \left(Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx)\right)$

M1
$P(1): f''(x) = \left(Aa\cos(ax) + Bb\cos(bx)\right)'$

$= -Aa^2\sin(ax) - Bb^2\sin(bx)$

A1
$= -1\left(Aa^2\sin(ax) + Bb^2\sin(bx)\right)$

true
$\therefore P(1)$
assume that

is true **M1**
$P(k): f^{(2k)}(x) = (-1)^k \left(Aa^{2k}\sin(ax) + Bb^{2k}\sin(bx)\right)$
consider

$P(k+1)$

M1A1
$f^{(2k+1)}(x) = (-1)^k \left(Aa^{2k+1}\cos(ax) + Bb^{2k+1}\cos(bx)\right)$

A1
$f^{(2k+2)}(x) = (-1)^k \left(-Aa^{2k+2}\sin(ax) - Bb^{2k+2}\sin(bx)\right)$

A1
$= (-1)^{k+1}\left(Aa^{2k+2}\sin(ax) + Bb^{2k+2}\sin(bx)\right)$

true implies
$P(k)$

true,
$P(k+1)$

true so
$P(1)$

true
$P(n)$

R1
$\forall n \in \mathbb{Z}^+$

Note: Award the final *R1* only if the previous three *M* marks have been awarded.

[8 marks]

Examiners report

Solutions to (b) were disappointing in general with few candidates giving a correct solution.

$P(n): f^{(2n)}(x) = (-1)^n \left(Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx)\right)$

$P(1): f''(x) = \left(Aa\cos(ax) + Bb\cos(bx)\right)'$

$= -Aa^2\sin(ax) - Bb^2\sin(bx)$

Markscheme

A1A1A1
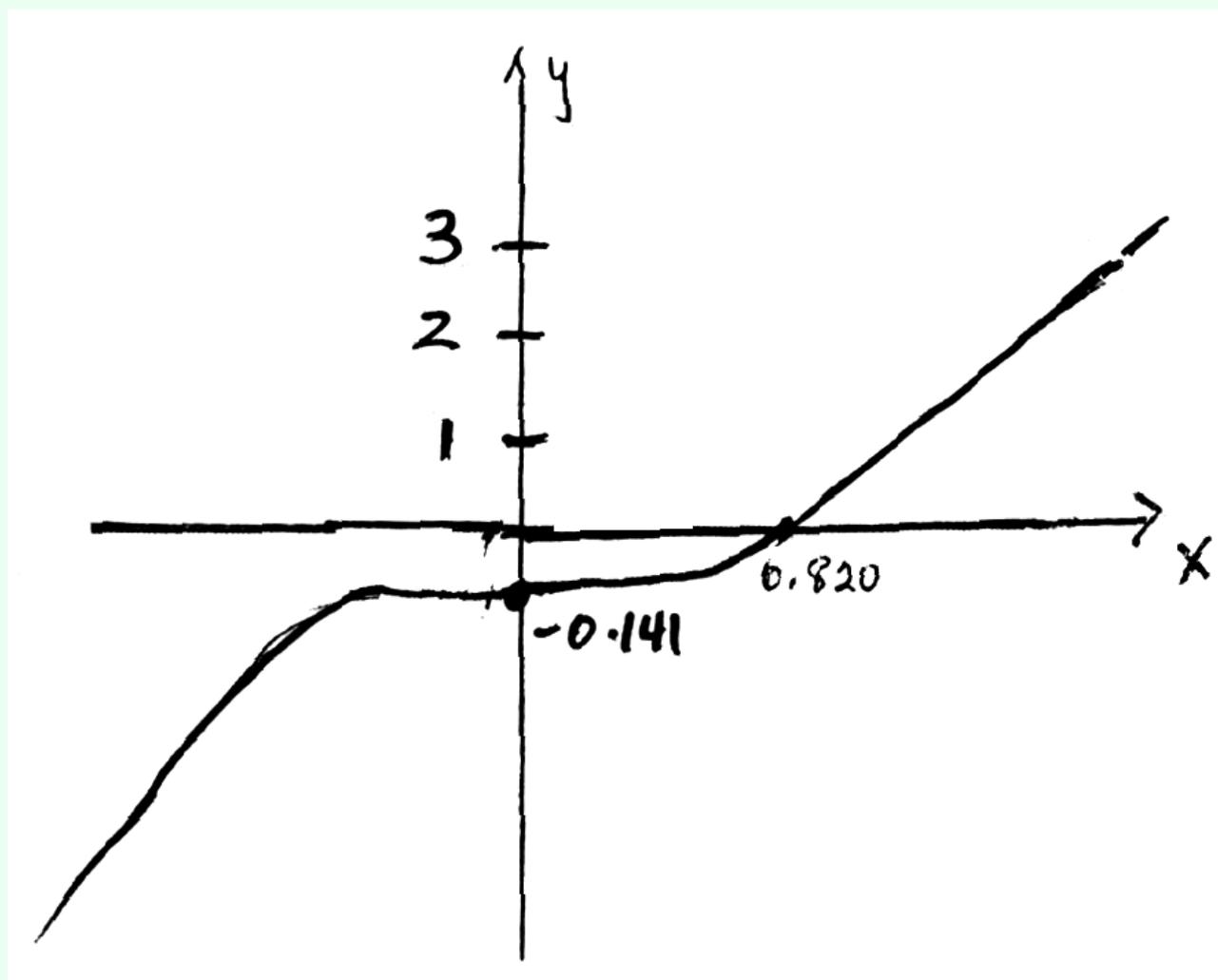
Note: Award *A1* for shape,

A1 for x-intercept is 0.820, accept

$\sin(-3)$ or $-\sin(3)$

A1 for y-intercept is –0.141.

[3 marks]

Examiners report

Many candidates attempted this question successfully. In (a), however, a large number of candidates did not use the zoom feature of the GDC to draw an accurate sketch of the given function. In (b), some candidates used the domain as the limits of the integral. Other candidates did not take the absolute value of the integral.

Markscheme

$A = \int_0^{0.8202} |x + \sin(x - 3)| \, dx \approx 0.0816$ sq units *(M1)A1*

[2 marks]

Examiners report

Many candidates attempted this question successfully. In (a), however, a large number of candidates did not use the zoom feature of the GDC to draw an accurate sketch of the given function. In (b), some candidates used the domain as the limits of the integral. Other candidates did not take the absolute value of the integral.

Markscheme

$V = \frac{\pi}{3} r^2 h$

$\frac{dV}{dt} = \frac{\pi}{3} \left[2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right]$ *M1A1A1*

at the given instant

$\frac{dV}{dt} = \frac{\pi}{3} \left[2(4)(200) \left(-\frac{1}{2} \right) + 40^2 (3) \right]$ *M1*

$= \frac{-3200\pi}{3} = -3351.03\ldots \approx 3350$ *A1*

hence, the volume is decreasing (at approximately 3350

mm^3 per century) *R1*

[6 marks]

Examiners report

Few candidates applied the method of implicit differentiation and related rates correctly. Some candidates incorrectly interpreted this question as one of constant linear rates.

Markscheme

METHOD 1

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$

$\frac{dy}{dx} = \frac{-2e^{-x}}{2(1+e^{-x})} = \frac{-e^{-x}}{1+e^{-x}}$ *M1A1*

now

$\frac{1+e^{-x}}{2} = e^y$ *M1*

$\Rightarrow 1 + e^{-x} = 2e^y$

$\Rightarrow e^{-x} = 2e^y - 1$ *(A1)*

$\Rightarrow \frac{dy}{dx} = \frac{-2e^y+1}{2e^y}$ *(A1)*

Note: Only one of the two above *A1* marks may be implied.

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} = -1$ *AG*

Note: Candidates may find

$\frac{dy}{dx}$

as a function of x and then work backwards from the given answer. Award full marks if done correctly.

METHOD 2

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$

$\Rightarrow e^y = \frac{1+e^{-x}}{2}$ *M1*

$\Rightarrow e^{-x} = 2e^y - 1$

$\Rightarrow x = -\ln(2e^y - 1)$ *A1*

$\Rightarrow \frac{dx}{dy} = -\frac{1}{2e^y-1} \times 2e^y$ *M1A1*

$\Rightarrow \frac{dy}{dx} = \frac{2e^y-1}{-2e^y}$ *A1*

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} - 1$ *AG*

[5 marks]

Examiners report

Many candidates were successful in (a) with a variety of methods seen. In (b) the use of the chain rule was often omitted when differentiating

with respect to *x*. A number of candidates tried to repeatedly differentiate the original expression, which was not what was asked for, e^{-y}
although partial credit was given for this. In this case, they often found problems in simplifying the algebra.

31b.

Markscheme

METHOD 1

when

$$x = 0, \ y = \ln 1 = 0 \quad \textbf{\textit{A1}}$$

when

$$x = 0, \ \frac{dy}{dx} = \frac{1}{2} - 1 = -\frac{1}{2} \quad \textbf{\textit{A1}}$$

$$\frac{d^2y}{dx^2} = -\frac{e^{-y}}{2}\frac{dy}{dx} \quad \textbf{\textit{M1A1}}$$

when

$$x = 0, \ \frac{d^2y}{dx^2} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \quad \textbf{\textit{A1}}$$

$$\frac{d^3y}{dx^3} = \frac{e^{-y}}{2}\left(\frac{dy}{dx}\right)^2 - \frac{e^{-y}}{2}\frac{d^2y}{dx^2} \quad \textbf{\textit{M1A1A1}}$$

when

$$x = 0, \ \frac{d^3y}{dx^3} = \frac{1}{2} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{4} = 0 \quad \textbf{\textit{A1}}$$

$$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots \quad \textbf{\textit{(M1)A1}}$$

two of the above terms are zero   **_AG_**

METHOD 2

when

$$x = 0, \ y = \ln 1 = 0 \quad \textbf{\textit{A1}}$$

when

$$x = 0, \ \frac{dy}{dx} = \frac{1}{2} - 1 = -\frac{1}{2} \quad \textbf{\textit{A1}}$$

$$\frac{d^2y}{dx^2} = \frac{-e^{-y}}{2}\frac{dy}{dx} = \frac{-e^{-y}}{2}\left(\frac{e^{-y}}{2} - 1\right) = \frac{-e^{2y}}{4} + \frac{e^{-y}}{2} \quad \textbf{\textit{M1A1}}$$

when

$$x = 0, \ \frac{d^2y}{dx^2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \quad \textbf{\textit{A1}}$$

$$\frac{d^3y}{dx^3} = \left(\frac{e^{-2y}}{2} - \frac{e^{-y}}{2}\right)\frac{dy}{dx} \quad \textbf{\textit{M1A1A1}}$$

when

$$x = 0, \ \frac{d^3y}{dx^3} = -\frac{1}{2} \times \left(\frac{1}{2} - \frac{1}{2}\right) = 0 \quad \textbf{\textit{A1}}$$

$$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots \quad \textbf{\textit{(M1)A1}}$$

two of the above terms are zero   **_AG_**

[11 marks]

32.   *[6 marks]*

Markscheme

METHOD 1

A1
area $= \int_{0}^{\sqrt{3}} \arctan x \, dx$
attempting to integrate by parts   *M1*

A1A1
$= [x \arctan x]_{0}^{\sqrt{3}} - \int_{0}^{\sqrt{3}} x \frac{1}{1+x^2} dx$
A1
$= [x \arctan x]_{0}^{\sqrt{3}} - \left[\frac{1}{2}\ln(1+x^2)\right]_{0}^{\sqrt{3}}$
Note: Award A1 even if limits are absent.

A1
$= \frac{\pi}{\sqrt{3}} - \frac{1}{2}\ln 4$

$\left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$

METHOD 2

M1A1A1
area $= \frac{\pi\sqrt{3}}{3} - \int_{0}^{\frac{\pi}{3}} \tan y \, dy$
M1A1
$= \frac{\pi\sqrt{3}}{3} + [\ln|\cos y|]_{0}^{\frac{\pi}{3}}$
A1
$= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2}$

$\left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$

[6 marks]

33.

Markscheme

attempt at implicit differentiation   *M1*

A1A1

$e^{(x+y)}\left(1+\frac{dy}{dx}\right) = -\sin(xy)\left(x\frac{dy}{dx}+y\right)$

let

$x = 0$ *M1*
$y = 0$

$e^0\left(1+\frac{dy}{dx}\right) = 0$ *A1*

$\frac{dy}{dx} = -1$

let

$x = \sqrt{2\pi}$
$y = -\sqrt{2\pi}$

so
$e^0\left(1+\frac{dy}{dx}\right) = -\sin(-2\pi)\left(x\frac{dy}{dx}+y\right) = 0$

$\frac{dy}{dx} = -1$ *A1*

since both points lie on the line

this is a common tangent   *R1*

$y = -x$

Note:

$y = -x$ must be seen for the final *R1*. It is not sufficient to note that the gradients are equal.

[7 marks]

Examiners report

Implicit differentiation was attempted by many candidates, some of whom obtained the correct value for the gradient of the tangent. However, very few noticed the need to go further and prove that both points were on the same line.

34a.

Markscheme

(i)

$f'(x) = \frac{x\frac{1}{x}-\ln x}{x^2}$ *M1A1*

$= \frac{1-\ln x}{x^2}$

so

when
$f'(x) = 0$

i.e.
$\ln x = 1$ *A1*

$x = e$

(ii)

when
$f'(x) > 0$

and
$x < e$

when
$f'(x) < 0$ *R1*

$x > e$

hence local maximum   *AG*

Note: Accept argument using correct second derivative.

(iii)

$y \leqslant \frac{1}{e}$ *A1*

[5 marks]

34b. *[5 marks]*

Markscheme

$f''(x) = \dfrac{x^2 \frac{-1}{x} - (1 - \ln x)2x}{x^4}$ **M1**

$= \dfrac{-x - 2x + 2x \ln x}{x^4}$

$= \dfrac{-3 + 2\ln x}{x^3}$ **A1**

Note: May be seen in part (a).

$f''(x) = 0$ **(M1)**

$-3 + 2\ln x = 0$

$x = e^{\frac{3}{2}}$

since

when $f''(x) < 0$ and $x < e^{\frac{3}{2}}$

when $f''(x) > 0$ and $x > e^{\frac{3}{2}}$ **R1**

then point of inflexion

$\left(e^{\frac{3}{2}}, \dfrac{3}{2e^{\frac{3}{2}}}\right)$ **A1**

[5 marks]

Markscheme

A1A1A1

Note: Award *A1* for the maximum and intercept, *A1* for a vertical asymptote and *A1* for shape (including turning concave up).

[3 marks]

Examiners report

Most candidates attempted parts (a), (b) and (c) and scored well, although many did not gain the reasoning marks for the justification of the existence of local maximum and inflexion point. The graph sketching was poorly done. A wide selection of range shapes were seen, in some cases showing little understanding of the relation between the derivatives of the function and its graph and difficulties with transformation of graphs. In some cases candidates sketched graphs consistent with their previous calculations but failed to label them properly.

Markscheme

(i)

A1A1
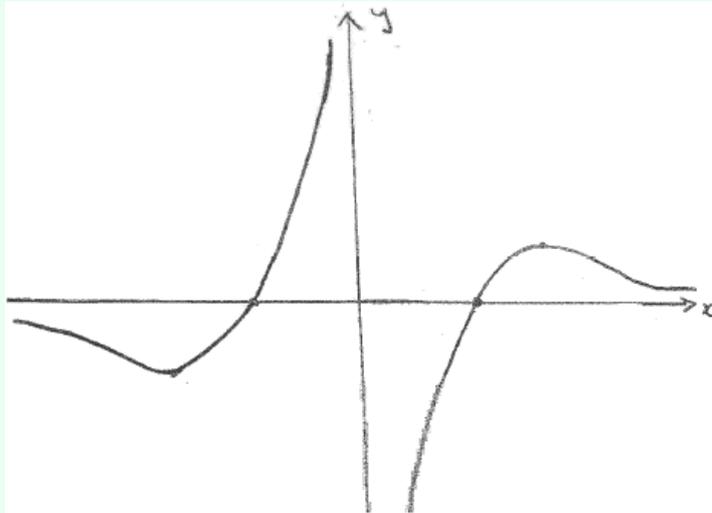
Note: Award *A1* for each correct branch.

(ii) all real values   *A1*

(iii)

(M1)(A1)
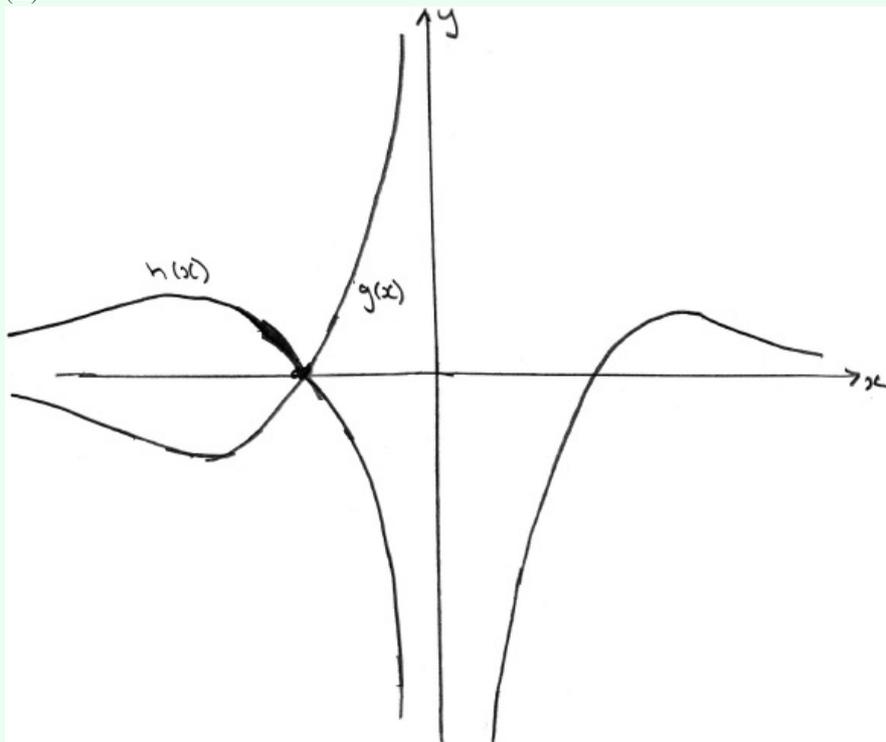
Note: Award *(M1)(A1)* for sketching the graph of *h*, ignoring any graph of *g*.

(accept
$y e^2$ *A1* $x < -1$
$x < -1$
[6 marks]

35a. [4 marks]

Markscheme

$f'(x) = 3x^2 - 6x - 9$ *(M1)*
$= 0$

$(x+1)(x-3) = 0$;
$x = -1$
$x = 3$
(max) $(-1, 15)$; (min) $(3, -17)$ *A1A1*

Note: The coordinates need not be explicitly stated but the values need to be seen.

$y = -8x + 7$ *A1   N2*
[4 marks]

35b. [2 marks]

Markscheme

inflexion $(1, -1)$ *A1*
$f''(x) = 6x - 6 = 0 \Rightarrow$
which lies on
$y = -8x + 7$ *R1AG*
[2 marks]

36a. [1 mark]

Markscheme

equation of line in graph
$a = -\frac{25}{60}t + 15$ *A1*

$\left(a = -\frac{5}{12}t + 15\right)$
[1 mark]

36b.

Markscheme

$\frac{dv}{dt} = -\frac{5}{12}t + 15$ *(M1)*

$v = -\frac{5}{24}t^2 + 15t + c$ *(A1)*

when $t = 0$,

$v = 125$

$v = -\frac{5}{24}t^2 + 15t + 125$ *A1*

from graph or by finding time when

$a = 0$

maximum

$= 395$ ms^{-1} *A1*

[4 marks]

Markscheme

EITHER

graph drawn and intersection with

ms^{-1} *(M1)(A1)*

$v = 295$

\(t = 57.91 - 14.09 = 43.8\)   *A1*

OR

$295 = -\frac{5}{24}t^2 + 15t + 125 \Rightarrow t = 57.91...$

14.09...
    A1

$t = 57.91... - 14.09... = 43.8\left(8\sqrt{30}\right)$

[3 marks]

Examiners report

This question was well answered by a large number of candidates and indicated a good understanding of calculus, kinematics and use of the graphing calculator. Some candidates worked in

and
rather than
y
and
but mostly obtained correct solutions. Although the majority of candidate used integration throughout the question some correct solutions were obtained by considering the areas in the diagram.

Markscheme

volume

(M1)

$= \pi \int_0^h x^2 \, dy$

M1

$\pi \int_0^h y \, dy$

A1

$\pi \left[\frac{y^2}{2} \right]_h = \frac{\pi h^2}{2}$

[3 marks]

37b. *[3 marks]*

Markscheme

volume

$$= \pi \int_0^h x^2 \, dy \quad \textit{(M1)}$$

$$\pi \int_0^h y \, dy \quad \textit{M1}$$

$$\pi \left[\frac{y^2}{2} \right]_0^h = \frac{\pi h^2}{2} \quad \textit{A1}$$

[3 marks]

37c. *[6 marks]*

Markscheme

surface area $= \pi x^2$ *A1*

$$\frac{dV}{dt} = -3 \times \text{surface area} \quad \textit{(M1)}$$

$$= \pi h \quad \textit{A1}$$

$$V = \frac{\pi h^2}{2} \Rightarrow h \sqrt{\frac{2V}{\pi}} \quad \textit{M1A1}$$

$$\frac{dV}{dt} = -3\pi \sqrt{\frac{2V}{\pi}} \quad \textit{A1}$$

$$\frac{dV}{dt} = -3\sqrt{2\pi V} \quad \textit{AG}$$

Note: Assuming that $\frac{dh}{dt} = -3$ without justification gains no marks.

[6 marks]

Markscheme

$\frac{dV}{dt} = -3 \times$ surface area   *A1*

surface area

$= \pi x^2$ *(M1)*

$= \pi h$ *A1*

$V = \frac{\pi h^2}{2} \Rightarrow h = \sqrt{\frac{2V}{\pi}}$ *M1A1*

$\frac{dV}{dt} = -3\pi \sqrt{\frac{2V}{\pi}}$ *A1*

$\frac{dV}{dt} = -3\sqrt{2\pi V}$ *AG*

Note: Assuming that $\frac{dh}{dt} = -3$ without justification gains no marks.

[6 marks]

Examiners report

This question was found to be challenging by many candidates and there were very few completely correct solutions. Many candidates did not seem able to find the volume of revolution when taken about the *y*-axis in (a). Candidates did not always recognize that part (b) did not involve related rates. Those candidates who attempted the question made some progress by separating the variables and integrating in (c) but very few were able to identify successfully the values necessary to find the correct answer.

Markscheme

$V \, (\text{m}^3) = 5000\pi$ *A1*

$= 15700$

$\frac{dV}{dt} = -3\sqrt{2\pi V}$

attempting to separate variables   *M1*

EITHER

$\int \frac{dV}{\sqrt{V}} = -3\sqrt{2\pi} \int dt$ *A1*

$2\sqrt{V} = -3\sqrt{2\pi} t + c$ *A1*

$c = 2\sqrt{5000\pi}$ *A1*

$V = 0$ *M1*

$\Rightarrow t = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours   *A1*

OR

$\int_{5000\pi}^{0} \frac{dV}{\sqrt{V}} = -3\sqrt{2\pi} \int_0^T dt$ *M1A1A1*

Note: Award *M1* for attempt to use definite integrals, *A1* for correct limits and *A1* for correct integrands.

$\left[2\sqrt{V}\right]_{5000\pi}^{0} = 3\sqrt{2\pi} T$ *A1*

$T = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours   *A1*

[7 marks]

Examiners report

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Markscheme

$V \text{ (m}^3) = 5000\pi$ *A1*
$= 15700$

$\frac{dV}{dt} = -3\sqrt{2\pi V}$
attempting to separate variables   *M1*

EITHER

$\int \frac{dV}{\sqrt{V}} = -3\sqrt{2\pi} \int dt$ *A1*
$2\sqrt{V} = -3\sqrt{2\pi}t + c$ *A1*
$c = 2\sqrt{5000\pi}$ *A1*
$V = 0$ *M1*

$\Rightarrow t = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours   *A1*

OR

$\int_{5000\pi}^{0} \frac{dV}{\sqrt{V}} = -3\sqrt{2\pi} \int_{0}^{T} dt$ *M1A1A1*
Note: Award *M1* for attempt to use definite integrals, *A1* for correct limits and *A1* for correct integrands.

$\left[2\sqrt{V}\right]_{5000\pi}^{0} = 3\sqrt{2\pi}T$ *A1*
$T = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours   *A1*

[7 marks]

Examiners report

This question was found to be challenging by many candidates and there were very few completely correct solutions. Many candidates did not seem able to find the volume of revolution when taken about the *y*-axis in (a). Candidates did not always recognize that part (b) did not involve related rates. Those candidates who attempted the question made some progress by separating the variables and integrating in (c) but very few were able to identify successfully the values necessary to find the correct answer.