Topic 2 Part 1 *[373 marks]*

The functions f and g are defined by $f(x) = ax^2 + bx + c,\ x \in \mathbb{R}$ and $g(x) = p\sin x + qx + r,\ x \in \mathbb{R}$ where $a,\ b,\ c,\ p,\ q,\ r$ are real constants.

1a. Given that f is an even function, show that $b = 0$. *[2 marks]*

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1b. Given that g is an odd function, find the value of r. *[2 marks]*

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1c. The function h is both odd and even, with domain $\mathbb{R}$. *[2 marks]*

Find $h(x)$.

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A function f is defined by $f(x) = \frac{3x-2}{2x-1},\ x \in \mathbb{R},\ x \neq \frac{1}{2}$.

2a. Find an expression for $f^{-1}(x)$. *[4 marks]*

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2b. Given that $f(x)$ can be written in the form $f(x) = A + \frac{B}{2x-1}$, find the values of the constants A and B. *[2 marks]*

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2c. Hence, write down $\int \frac{3x-2}{2x-1} \mathrm{d}x$. *[1 mark]*

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Let $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36,\ x \in \mathbb{R}$.

3a. For the polynomial equation $p(x) = 0$, state *[3 marks]*

(i) the sum of the roots;

(ii) the product of the roots.

3b. A new polynomial is defined by $q(x) = p(x + 4)$. *[2 marks]*

Find the sum of the roots of the equation $q(x) = 0$.

The functions f and g are defined by $f(x) = 2x + \frac{\pi}{5},\ x \in \mathbb{R}$ and $g(x) = 3\sin x + 4,\ x \in \mathbb{R}$.

4a. Show that $g \circ f(x) = 3\sin\left(2x + \frac{\pi}{5}\right) + 4$. *[1 mark]*

4b. Find the range of $g \circ f$. *[2 marks]*

4c. Given that $g \circ f\left(\frac{3\pi}{20}\right) = 7$, find the next value of x, greater than $\frac{3\pi}{20}$, for which $g \circ f(x) = 7$. *[2 marks]*

4d. The graph of $y = g \circ f(x)$ can be obtained by applying four transformations to the graph of $y = \sin x$. State what the four transformations represent geometrically and give the order in which they are applied. *[4 marks]*

Let $y(x) = xe^{3x},\ x \in \mathbb{R}$.

5a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[2 marks]*

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5b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. *[7 marks]*

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5c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. *[5 marks]*

Justify whether any such point is a maximum or a minimum.

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5d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. *[5 marks]*

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6a. State the set of values of a for which the function $x \mapsto \log_a x$ exists, for all $x \in \mathbb{R}^+$. *[2 marks]*

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6b. Given that
$\log_x y = 4\log_y x$, find all the possible expressions of y as a function of x. *[6 marks]*

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The function f is defined by $f(x) = \frac{3x}{x-2},\ x \in \mathbb{R},\ x \neq 2.$

7a. Sketch the graph of $y = f(x)$, indicating clearly any asymptotes and points of intersection with the x and y axes. *[4 marks]*

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7b. Find an expression for $f^{-1}(x)$. *[4 marks]*

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7c. Find all values of x for which $f(x) = f^{-1}(x)$. *[3 marks]*

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7d. Solve the inequality $|f(x)| < \frac{3}{2}$. *[4 marks]*

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7e. Solve the inequality $f(|x|) < \frac{3}{2}$. *[2 marks]*

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Consider the functions $f(x) = \tan x,\ 0 \leq\ x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \neq 1$.

8a. Find an expression for $g \circ f(x)$, stating its domain. *[2 marks]*

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8b. Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. *[2 marks]*

8c. Let $y = g \circ f(x)$, find an exact value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. *[6 marks]*

8d. Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. *[6 marks]*

The cubic equation $x^3 + px^2 + qx + c = 0$, has roots $\alpha,\ \beta,\ \gamma$. By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that

9a. (i) $p = -(\alpha + \beta + \gamma)$; *[3 marks]*

(ii) $q = \alpha\beta + \beta\gamma + \gamma\alpha$;

(iii) $c = -\alpha\beta\gamma$.

9b. It is now given that $p = -6$ and $q = 18$ for parts (b) and (c) below. *[5 marks]*

(i) In the case that the three roots $\alpha,\ \beta,\ \gamma$ form an arithmetic sequence, show that one of the roots is 2.

(ii) Hence determine the value of c.

9c. In another case the three roots $\alpha,\ \beta,\ \gamma$ form a geometric sequence. Determine the value of c. *[6 marks]*

10a. Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0,\ n \in \mathbb{Z}$. *[2 marks]*

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10b. Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$. *[2 marks]*

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10c. Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2,\ n \in \mathbb{Z}$. *[9 marks]*

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11. A function f is defined by $f(x) = x^3 + \mathrm{e}^x + 1,\ x \in \mathbb{R}$. By considering $f'(x)$ determine whether f is a one-to-one or a many-to-one function. *[4 marks]*

12a. A function f is defined by $f(x) = (x+1)(x-1)(x-5),\ x \in \mathbb{R}$. *[3 marks]*

Find the values of x for which $f(x) < |f(x)|$.

12b. A function g is defined by $g(x) = x^2 + x - 6,\ x \in \mathbb{R}$. *[7 marks]*

Find the values of x for which $g(x) < \frac{1}{g(x)}$.

13. When the polynomial *[5 marks]*

$3x^3 + ax + b$ is divided by

$(x-2)$, the remainder is 2, and when divided by

$(x+1)$, it is 5. Find the value of a and the value of b.

14. The equation [6 marks]

$5x^3 + 48x^2 + 100x + 2 = a$ has roots

r_1,

r_2 and

r_3.

Given that

$r_1 + r_2 + r_3 + r_1 r_2 r_3 = 0$, find the value of a.

15. One root of the equation [4 marks]

$x^2 + ax + b = 0$ is

$2 + 3\mathrm{i}$ where

$a,\ b \in \mathbb{R}$. Find the value of

a and the value of

b.

Let

$f(x) = x(x+2)^6$.

16a. Solve the inequality [5 marks]

$f(x) > x$.

16b. Find [5 marks]

$\int f(x)\mathrm{d}x$.

Let

$f(x) = \frac{\mathrm{e}^{2x}+1}{\mathrm{e}^{x}-2}$.

17a. Find the equations of the horizontal and vertical asymptotes of the curve [4 marks]

$y = f(x)$.

17b. (i) Find [8 marks]

$f'(x)$.

(ii) Show that the curve has exactly one point where its tangent is horizontal.

(iii) Find the coordinates of this point.

17c. Find the equation of [4 marks]

L_1, the normal to the curve at the point where it crosses the y-axis.

The line
L_2 is parallel to
L_1 and tangent to the curve
$y = f(x)$.

17d. Find the equation of the line [5 marks]

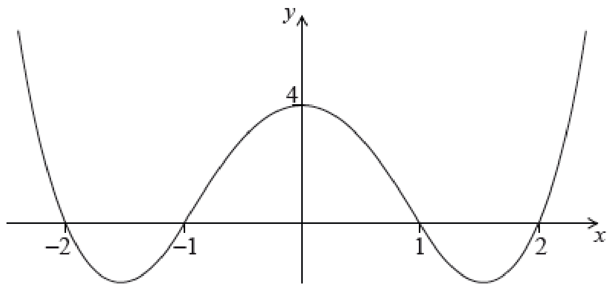
L_2.

18. Let [18 marks]

$f(x) = |x| - 1.$

(a) The graph of

$y = g(x)$ is drawn below.



(i) Find the value of

$(f \circ g)(1).$

(ii) Find the value of

$(f \circ g \circ g)(1).$

(iii) Sketch the graph of

$y = (f \circ g)(x).$

(b) (i) Sketch the graph of

$y = f(x).$

(ii) State the zeros of *f*.

(c) (i) Sketch the graph of

$y = (f \circ f)(x).$

(ii) State the zeros of

$f \circ f.$

(d) Given that we can denote

$\underbrace{f \circ f \circ f \circ \ldots \circ f}_{n \text{ times}}$ as

$f^n,$

(i) find the zeros of

$f^3;$

(ii) find the zeros of

$f^4;$

(iii) deduce the zeros of

$f^8.$

(e) The zeros of

f^{2n} are

$a_1, a_2, a_3, \ldots, a_N.$

(i) State the relation between *n* and *N*;

(ii) Find, and simplify, an expression for

$\sum_{r=1}^{N} |a_r|$ in terms of *n*.

19. The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β. *[6 marks]*

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$;

(b) find a quadratic equation with roots α^2 and β^2.

20a. Sketch the graph of $y = \left|\cos\left(\frac{x}{4}\right)\right|$ for $0 \leqslant x \leqslant 8\pi$. *[2 marks]*

20b. Solve $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$ for $0 \leqslant x \leqslant 8\pi$. *[3 marks]*

The function *f* is defined by

$$f(x) = \begin{cases} 1 - 2x, & x \le 2 \\ \frac{3}{4}(x-2)^2 - 3, & x > 2 \end{cases}$$

21a. Determine whether or not *f* is continuous. *[2 marks]*

21b. The graph of the function g is obtained by applying the following transformations to the graph of f: *[4 marks]*

a reflection in the y-axis followed by a translation by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Find $g(x)$.

Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$

$g(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$

22a. Sketch the graph of $y = h(x)$. *[2 marks]*

22b. Find an expression for the composite function $h \circ g(x)$ and state its domain. *[2 marks]*

22c. Given that

$f(x) = h(x) + h \circ g(x)$, *[7 marks]*

(i) find

$f'(x)$ in simplified form;

(ii) show that

$f(x) = \frac{\pi}{2}$ for

$x > 0$.

22d. Nigel states that *[3 marks]*

f is an odd function and Tom argues that

f is an even function.

(i) State who is correct and justify your answer.

(ii) Hence find the value of

$f(x)$ for

$x < 0$.

The graphs of

$y = x^2 e^{-x}$ and

$y = 1 - 2\sin x$ for

$2 \leqslant x \leqslant 7$ intersect at points A and B.

The x-coordinates of A and B are

x_A and

x_B.

23a. Find the value of *[2 marks]*

x_A and the value of

x_B.

23b. Find the area enclosed between the two graphs for *[3 marks]*

$x_A \leqslant x \leqslant x_B$.

The function f is defined by $f(x) = \frac{1}{x},\ x \neq 0$.

The graph of the function $y = g(x)$ is obtained by applying the following transformations to the graph of $y = f(x)$:

a translation by the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$; a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

24a. Find an expression for $g(x)$. *[2 marks]*

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24b. State the equations of the asymptotes of the graph of g. *[2 marks]*

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The quadratic equation $2x^2 - 8x + 1 = 0$ has roots α and β.

25a. Without solving the equation, find the value of *[2 marks]*

(i) $\alpha + \beta$;

(ii) $\alpha\beta$.

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25b. Another quadratic equation $x^2 + px + q = 0,\ p,\ q \in \mathbb{Z}$ has roots $\frac{2}{\alpha}$ and $\frac{2}{\beta}$. *[4 marks]*

Find the value of p and the value of q.

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The function f is defined as $f(x) = \mathrm{e}^{3x+1},\ x \in \mathbb{R}$.

26a. (i) Find $f^{-1}(x)$. *[4 marks]*

(ii) State the domain of f^{-1}.

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26b. The function g is defined as $g(x) = \ln x,\ x \in \mathbb{R}^{+}$. *[5 marks]*

The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

Find the coordinates of P.

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26c. The graph of $y = g(x)$ intersects the x-axis at the point Q. *[3 marks]*

Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$.

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26d. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. *[5 marks]*

Find the area of the region R.

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26e. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. *[6 marks]*

(i) Show that $g(x) \leq x - 1,\ x \in \mathbb{R}^+$.

(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \leq g(x),\ x \in \mathbb{R}^+$.

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27. Consider $p(x) = 3x^3 + ax + 5a, \quad a \in \mathbb{R}$. *[6 marks]*

The polynomial $p(x)$ leaves a remainder of -7 when divided by $(x - a)$.

Show that only one value of a satisfies the above condition and state its value.

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The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

28a. Show that $d = \frac{a}{2}$. *[3 marks]*

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28b. The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200. *[6 marks]*

Find the least value of n for which this occurs.

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Compactness is a measure of how compact an enclosed region is.

The compactness,
C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$, where
A is the area of the region and
d is the maximum distance between any two points in the region.

For a circular region, $C = 1$.

Consider a regular polygon of
n sides constructed such that its vertices lie on the circumference of a circle of diameter
x units.

29a. If $n > 2$ and even, show that $C = \frac{n}{2\pi} \sin \frac{2\pi}{n}$. *[3 marks]*

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29b. If $n > 1$ and odd, it can be shown that $C = \frac{n \sin \frac{2\pi}{n}}{\pi\left(1+\cos \frac{\pi}{n}\right)}$. *[4 marks]*

Find the regular polygon with the least number of sides for which the compactness is more than 0.99.

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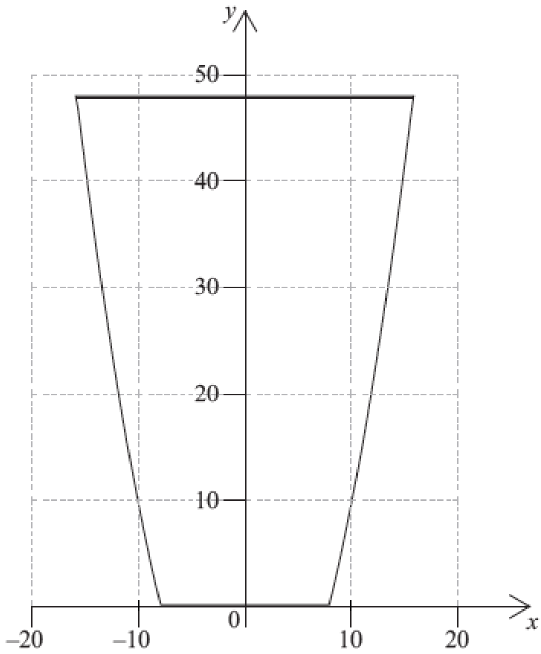
The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is 48 cm.

30a. If the container is filled with water to a depth of $h\,\mathrm{cm}$, show that the volume, $V\,\mathrm{cm}^3$, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. *[3 marks]*

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30b. Once empty, water is pumped back into the container at a rate of $8.5\ \mathrm{cm}^3\mathrm{s}^{-1}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\mathrm{cm}^3\mathrm{s}^{-1}$. *[3 marks]*

Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container.

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A random variable X has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ \frac{1}{4} & 1 \leq x < 3 \\ 0 & x \geq 3 \end{cases}$$

31a. Sketch the graph of $y = f(x)$. *[1 mark]*

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31b. Find the cumulative distribution function for X. *[5 marks]*

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31c. Find the interquartile range for X. *[3 marks]*

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The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 0,\ x < 0 \\ \frac{\sin x}{4},\ 0 \le x \le \pi \\ a(x-\pi),\ \pi < x \le 2\pi \\ 0,\ 2\pi < x \end{cases}.$$

32a. Sketch the graph $y = f(x)$. *[2 marks]*

32b. Find $\mathrm{P}(X \le \pi)$. *[2 marks]*

32c. Show that $a = \frac{1}{\pi^2}$. *[3 marks]*

32d. Write down the median of X. *[1 mark]*

32e. Calculate the mean of X. *[3 marks]*

32f. Calculate the variance of X. *[3 marks]*

32g. Find $\mathrm{P}\left(\frac{\pi}{2} \le X \le \frac{3\pi}{2}\right)$. *[2 marks]*

32h. Given that $\frac{\pi}{2} \leq X \leq \frac{3\pi}{2}$ find the probability that $\pi \leq X \leq 2\pi$. *[4 marks]*

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33a. (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$. *[6 marks]*

(ii) Hence use De Moivre's theorem to prove

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

(iii) State a similar expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

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Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^5 - 1 = 0$ which has the smallest positive argument.

33b. Find the value of r and the value of α. *[4 marks]*

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33c. Using (a) (ii) and your answer from (b) show that $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$. *[4 marks]*

33d. Hence express $\sin 72^\circ$ in the form $\frac{\sqrt{a+b\sqrt{c}}}{d}$ where $a,\ b,\ c,\ d \in \mathbb{Z}$. *[5 marks]*

34a. Sketch the graph of $y = (x-5)^2 - 2\,|x-5| - 9,\ \text{for } 0 \le x \le 10$. *[3 marks]*

$$(x-5)^2 - 2\,|x-5| - 9 = 0$$

The graph of $y = \ln(5x + 10)$ is obtained from the graph of $y = \ln x$ by a translation of a units in the direction of the x-axis followed by a translation of b units in the direction of the y-axis.

35a. Find the value of a and the value of b. *[4 marks]*

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35b. The region bounded by the graph of $y = \ln(5x + 10)$, the x-axis and the lines $x = \mathrm{e}$ and $x = 2\mathrm{e}$, is rotated through 2π radians about the x-axis. Find the volume generated. *[2 marks]*

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Farmer Bill owns a rectangular field, 10 m by 4 m. Bill attaches a rope to a wooden post at one corner of his field, and attaches the other end to his goat Gruff.

36a. Given that the rope is 5 m long, calculate the percentage of Bill’s field that Gruff is able to graze. Give your answer correct to the nearest integer. *[4 marks]*

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36b. Bill replaces Gruff's rope with another, this time of length $a,\ 4 < a < 10$, so that Gruff can now graze exactly one half of Bill's field. *[4 marks]*

Show that a satisfies the equation

$$a^2 \arcsin\left(\frac{4}{a}\right) + 4\sqrt{a^2 - 16} = 40.$$

36c. Find the value of a. *[2 marks]*

A particle moves in a straight line, its velocity $v\ \mathrm{ms}^{-1}$ at time t seconds is given by $v = 9t - 3t^2,\ 0 \le t \le 5$.

At time $t = 0$, the displacement s of the particle from an origin O is 3 m.

37a. Find the displacement of the particle when $t = 4$. *[3 marks]*

37b. Sketch a displacement/time graph for the particle, $0 \leq t \leq 5$, showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values. *[5 marks]*

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37c. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \geq 0$. *[3 marks]*

Given further that $s = 16.5$ when $t = 7.5$, find the values of a and b.

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37d. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \geq 0$. *[4 marks]*

Find the times t_1 and $t_2 (0 < t_1 < t_2 < 8)$ when the particle returns to its starting point.

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