Topic 8 Part 2 *[571 marks]*

1a. The function $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $g(n) = |n| - 1$ for $n \in \mathbb{Z}$. Show that g is neither surjective nor injective. *[2 marks]*

1b. The set S is finite. If the function $f : S \to S$ is injective, show that f is surjective. *[2 marks]*

1c. Using the set $\mathbb{Z}^+$ as both domain and codomain, give an example of an injective function that is not surjective. *[3 marks]*

The group G has a unique element, h , of order 2.

2. (i) Show that ghg^{-1} has order 2 for all $g \in G$. *[5 marks]*

(ii) Deduce that $gh = hg$ for all $g \in G$.

The binary operator multiplication modulo 14, denoted by $*$, is defined on the set $S = \{2, 4, 6, 8, 10, 12\}$.

3a. Copy and complete the following operation table. *[4 marks]*

$*$	**2**	**4**	**6**	**8**	**10**	**12**
2						
4	8	2	10	4	12	6
6						
8						
10	6	12	4	10	2	8
12						

3b. *[11 marks]*

(i) Show that $\{S , *\}$ is a group.

(ii) Find the order of each element of $\{S , *\}$.

(iii) Hence show that $\{S , *\}$ is cyclic and find all the generators.

3c. The set T is defined by $\{x * x : x \in S\}$. Show that $\{T , *\}$ is a subgroup of $\{S , *\}$. *[3 marks]*

The universal set contains all the positive integers less than 30. The set A contains all prime numbers less than 30 and the set B contains all positive integers of the form

$3 + 5n\ (n \in \mathbb{N})$ that are less than 30. Determine the elements of

4a. $A \setminus B$; *[4 marks]*

4b. $A \Delta B$. *[3 marks]*

The relation R is defined for a ,

$b \in \mathbb{Z}^+$ such that aRb if and only if

$a^2 - b^2$ is divisible by 5.

5a. Show that R is an equivalence relation. *[6 marks]*

5b. Identify the three equivalence classes. *[4 marks]*

6. The function *[11 marks]*

$f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ is defined by

$f(x,\, y) = \left(xy^2, \frac{x}{y}\right)$.

Show that f is a bijection.

7a. Given that p , q and r are elements of a group, prove the left-cancellation rule, *i.e.* *[4 marks]*

$pq = pr \Rightarrow q = r$.

Your solution should indicate which group axiom is used at each stage of the proof.

7b. Consider the group G , of order 4, which has distinct elements a , b and c and the identity element e . *[10 marks]*

(i) Giving a reason in each case, explain why ab cannot equal a or b .

(ii) Given that c is self inverse, determine the two possible Cayley tables for G .

(iii) Determine which one of the groups defined by your two Cayley tables is isomorphic to the group defined by the set $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ under multiplication of complex numbers. Your solution should include a correspondence between a, b, c, e and $1, -1, \mathrm{i}, -\mathrm{i}$.

8. The function *[10 marks]*

$f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = 2e^x - e^{-x}.$$

(a) Show that f is a bijection.

(b) Find an expression for

$f^{-1}(x)$.

9. Let R be a relation on the set *[8 marks]*

$\mathbb{Z}$ such that

$aRb \Leftrightarrow ab \geqslant 0$, for a, b

$\in \mathbb{Z}$.

(a) Determine whether R is

(i) reflexive;

(ii) symmetric;

(iii) transitive.

(b) Write down with a reason whether or not R is an equivalence relation.

10a. Let *[4 marks]*

$f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R},\ f(m, x) = (-1)^m x$. Determine whether f is

(i) surjective;

(ii) injective.

10b. P is the set of all polynomials such that *[4 marks]*

$P = \left\{ \sum\limits_{i=0}^{n} a_i x^i | n \in \mathbb{N} \right\}$.

Let

$g : P \to P,\ g(p) = xp$. Determine whether g is

(i) surjective;

(ii) injective.

10c. Let *[7 marks]*

$h : \mathbb{Z} \to \mathbb{Z}^+$,

$h(x) = \left\{ \begin{array}{cc} 2x, & x > 0 \\ 1 - 2x, & x \leqslant 0 \end{array} \right\}$. Determine whether h is

(i) surjective;

(ii) injective.

11. Set [20 marks]

$S = \{x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5\}$ and a binary operation

$\circ$ on S is defined as

$x_i \circ x_j = x_k$, where

$i + j \equiv k (\bmod 6)$.

(a) (i) Construct the Cayley table for

$\{S,\ \circ\}$ and hence show that it is a group.

(ii) Show that

$\{S,\ \circ\}$ is cyclic.

(b) Let

$\{G,\ *\}$ be an Abelian group of order 6. The element

$a \in$ G has order 2 and the element

$b \in$ G has order 3.

(i) Write down the six elements of

$\{G,\ *\}$.

(ii) Find the order of

a $* b$ and hence show that

$\{G,\ *\}$ is isomorphic to

$\{S,\ \circ\}$.

12. Let [8 marks]

$\{G,\ *\}$ be a finite group that contains an element a (that is not the identity element) and

$H = \{a^n | n \in \mathbb{Z}^+\}$, where

$a^2 = a * a,\ a^3 = a * a * a$ etc.

Show that

$\{H,\ *\}$ is a subgroup of

$\{G,\ *\}$.

13a. Consider the following Cayley table for the set $G = \{1, 3, 5, 7, 9, 11, 13, 15\}$ under the operation *[7 marks]*

$\times_{16}$, where

$\times_{16}$ denotes multiplication modulo 16.

$\times_{16}$	1	3	5	7	9	11	13	15
1	1	3	5	7	9	11	13	15
3	3	a	15	5	11	b	7	c
5	5	15	9	3	13	7	1	11
7	7	d	3	1	e	13	f	9
9	9	11	13	g	1	3	5	7
11	11	h	7	13	3	9	i	5
13	13	7	1	11	5	j	9	3
15	15	13	11	9	7	5	3	1

(i) Find the values of a, b, c, d, e, f, g, h, i and j.

(ii) Given that

$\times_{16}$ is associative, show that the set G, together with the operation

$\times_{16}$, forms a group.

13b. The Cayley table for the set *[8 marks]*

$H = \{e, a_1, a_2, a_3, b_1, b_2, b_3, b_4\}$ under the operation

$*$, is shown below.

$*$	e	a_1	a_2	a_3	b_1	b_2	b_3	b_4
e	e	a_1	a_2	a_3	b_1	b_2	b_3	b_4
a_1	a_1	a_2	a_3	e	b_4	b_3	b_1	b_2
a_2	a_2	a_3	e	a_1	b_2	b_1	b_4	b_3
a_3	a_3	e	a_1	a_2	b_3	b_4	b_2	b_1
b_1	b_1	b_3	b_2	b_4	e	a_2	a_1	a_3
b_2	b_2	b_4	b_1	b_3	a_2	e	a_3	a_1
b_3	b_3	b_2	b_4	b_1	a_3	a_1	e	a_2
b_4	b_4	b_1	b_3	b_2	a_1	a_3	a_2	e

(i) Given that

$*$ is associative, show that H together with the operation

$*$ forms a group.

(ii) Find two subgroups of order 4.

13c. Show that *[2 marks]*

$\{G, \times_{16}\}$ and

$\{H, *\}$ are not isomorphic.

13d. Show that

$\{H, *\}$ is not cyclic.

[3 marks]

14a. Determine, using Venn diagrams, whether the following statements are true.

[6 marks]

(i)

$A' \cup B' = (A \cup B)'$

(ii)

$(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$

14b. Prove, without using a Venn diagram, that

[4 marks]

$A \backslash B$ and

$B \backslash A$ are disjoint sets.

The group G has a subgroup H. The relation R is defined on G by xRy if and only if

$xy^{-1} \in H$, for

$x, y \in G$.

15a. Show that R is an equivalence relation.

[8 marks]

15b. The Cayley table for G is shown below.

[6 marks]

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

The subgroup H is given as

$H = \{e, a^2b\}$.

(i) Find the equivalence class with respect to R which contains ab.

(ii) Another equivalence relation

ρ is defined on G by

$x\rho y$ if and only if

$x^{-1}y \in H$, for

$x, y \in G$. Find the equivalence class with respect to

ρ which contains ab.

Consider the functions

$f : A \to B$ and

$g : B \to C$.

16a. Show that if both f and g are injective, then *[3 marks]*

$g \circ f$ is also injective.

16b. Show that if both f and g are surjective, then *[4 marks]*

$g \circ f$ is also surjective.

16c. Show, using a single counter example, that both of the converses to the results in part (a) and part (b) are false. *[3 marks]*

The binary operation

$*$ is defined for

$a, b \in \mathbb{Z}^+$ by

$$a * b = a + b - 2.$$

17. (a) Determine whether or not *[12 marks]*

$*$ is

(i) closed,

(ii) commutative,

(iii) associative.

(b) (i) Find the identity element.

(ii) Find the set of positive integers having an inverse under

$*$.

The function f is defined by

$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}}, \ x \in \mathbb{R} .$$

18. (a) Find the range of f. *[10 marks]*

(b) Prove that f is an injection.

(c) Taking the codomain of f to be equal to the range of f, find an expression for

$f^{-1}(x)$.

19. (a) Find the six roots of the equation *[17 marks]*

$z^6 - 1 = 0$, giving your answers in the form

$r \operatorname{cis} \theta, r \in \mathbb{R}^+, 0 \leqslant \theta < 2\pi$.

(b) (i) Show that these six roots form a group G under multiplication of complex numbers.

(ii) Show that G is cyclic and find all the generators.

(iii) Give an example of another group that is isomorphic to G, stating clearly the corresponding elements in the two groups.

The relation R is defined on ordered pairs by

$$(a,\ b)R(c,\ d) \text{ if and only if } ad = bc \text{ where } a,\ b,\ c,\ d \in \mathbb{R}^+.$$

20. (a) Show that R is an equivalence relation. *[9 marks]*

(b) Describe, geometrically, the equivalence classes.

21. Let *[12 marks]*

$p = 2^k + 1,\ k \in \mathbb{Z}^+$ be a prime number and let G be the group of integers 1, 2, ..., $p - 1$ under multiplication defined modulo p.

By first considering the elements

$2^1,\ 2^2,\ \ldots,\ 2^k$ and then the elements

$2^{k+1},\ 2^{k+2},\ \ldots,$ show that the order of the element 2 is $2k$.

Deduce that

$k = 2^n$ for $n \in \mathbb{N}$.

22. (a) Draw the Cayley table for the set of integers $G = \{0, 1, 2, 3, 4, 5\}$ under addition modulo 6, *[16 marks]*

$+_6$.

(b) Show that

$\{G,\ +_6\}$ is a group.

(c) Find the order of each element.

(d) Show that

$\{G,\ +_6\}$ is cyclic and state its generators.

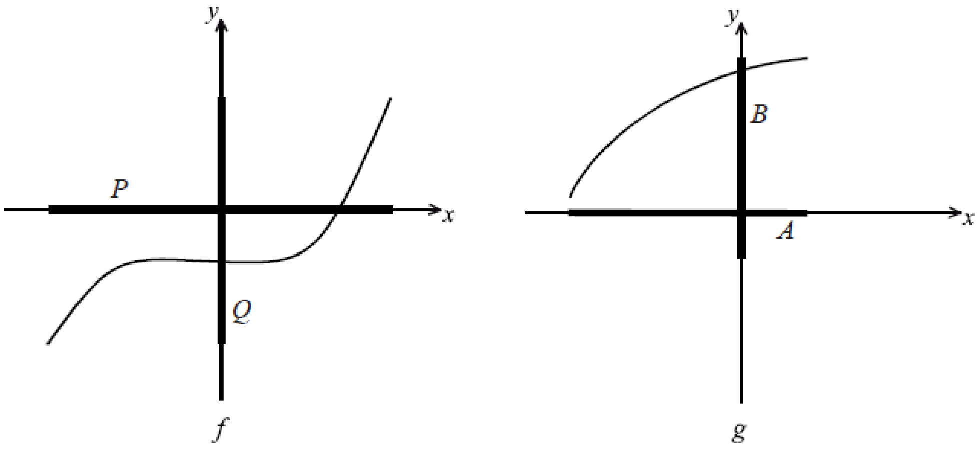
(e) Find a subgroup with three elements.

(f) Find the other proper subgroups of

$\{G,\ +_6\}$.

23a. Below are the graphs of the two functions *[4 marks]*

$F : P \to Q$ and $g : A \to B$.



Determine, with reference to features of the graphs, whether the functions are injective and/or surjective.

23b. Given two functions [9 marks]

$h : X \to Y$ and $k : Y \to Z$.

Show that

(i) if both h and k are injective then so is the composite function

$k \circ h$;

(ii) if both h and k are surjective then so is the composite function

$k \circ h$.

24. Prove that [6 marks]

$(A \cap B) \backslash (A \cap C) = A \cap (B \backslash C)$ where A, B and C are three subsets of the universal set U.

25a. The relation aRb is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ if and only if ab is the square of a positive integer. [10 marks]

(i) Show that R is an equivalence relation.

(ii) Find the equivalence classes of R that contain more than one element.

25b. Given the group [9 marks]

$(G, *)$, a subgroup

$(H, *)$ and

$a, b \in G$, we define

$a \sim b$ if and only if

$ab^{-1} \in H$. Show that

$\sim$ is an equivalence relation.

26. (a) Write down why the table below is a Latin square. [6 marks]

$$\begin{array}{c c} & \begin{array}{ccccc} d & e & b & a & c \end{array} \\ \begin{array}{c} d \\ e \\ b \\ a \\ c \end{array} & \begin{bmatrix} c & d & e & b & a \\ d & e & b & a & c \\ a & b & d & c & e \\ b & a & c & e & d \\ e & c & a & d & b \end{bmatrix} \end{array}$$

(b) Use Lagrange's theorem to show that the table is not a group table.

27. A,

B,

C and

D are subsets of

$\mathbb{Z}$.

[12 marks]

$A = \{m | m \text{ is a prime number less than } 15\}$

$B = \{m | m^4 = 8m\}$

$C = \{m | (m+1)(m-2) < 0\}$

$D = \{m | m^2 < 2m + 4\}$

(a) List the elements of each of these sets.

(b) Determine, giving reasons, which of the following statements are true and which are false.

(i)

$n(D) = n(B) + n(B \cup C)$

(ii)

$D \backslash B \subset A$

(iii)

$B \cap A' = \emptyset$

(iv)

$n(B \Delta C) = 2$

28.

[10 marks]

A binary operation is defined on $\{-1, 0, 1\}$ by

$$A \odot B = \begin{cases} -1, & \text{if } |A| < |B| \\ 0, & \text{if } |A| = |B| \\ 1, & \text{if } |A| > |B|. \end{cases}$$

(a) Construct the Cayley table for this operation.

(b) Giving reasons, determine whether the operation is

(i) closed;

(ii) commutative;

(iii) associative.

29.

[10 marks]

Two functions, F and G , are defined on

$A = \mathbb{R} \backslash \{0, 1\}$ by

$$F(x) = \frac{1}{x},\ G(x) = 1 - x, \text{ for all } x \in A.$$

(a) Show that under the operation of composition of functions each function is its own inverse.

(b) F and G together with four other functions form a closed set under the operation of composition of functions.

Find these four functions.

30. Determine, giving reasons, which of the following sets form groups under the operations given below. Where appropriate you may assume that multiplication is associative. [13 marks]

(a)

$\mathbb{Z}$ under subtraction.

(b) The set of complex numbers of modulus 1 under multiplication.

(c) The set $\{1, 2, 4, 6, 8\}$ under multiplication modulo 10.

(d) The set of rational numbers of the form

$$\frac{3m+1}{3n+1}, \text{ where } m,\ n \in \mathbb{Z}$$

under multiplication.

31. Three functions mapping [15 marks]

$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ are defined by

$$f_1(m,\ n) = m - n + 4;\ f_2(m,\ n) = |m|;\ f_3(m,\ n) = m^2 - n^2.$$

Two functions mapping

$\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ are defined by

$$g_1(k) = (2k,\ k);\ g_2(k) = (k,\ |k|).$$

(a) Find the range of

(i)

$f_1 \circ g_1$;

(ii)

$f_3 \circ g_2$.

(b) Find all the solutions of

$f_1 \circ g_2(k) = f_2 \circ g_1(k)$.

(c) Find all the solutions of

$f_3(m,\ n) = p$ in each of the cases $p = 1$ and $p = 2$.

32. (a) Show that $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ forms a group of complex numbers G under multiplication. *[17 marks]*

(b) Consider

$S = \{e,\ a,\ b,\ a*b\}$ under an associative operation

$*$ where e is the identity element. If

$a*a = b*b = e$ and

$a*b = b*a$, show that

(i)

$a*b*a = b$,

(ii)

$a*b*a*b = e$.

(c) (i) Write down the Cayley table for

$H = \{S\ , *\}$.

(ii) Show that H is a group.

(iii) Show that H is an Abelian group.

(d) For the above groups, G and H , show that one is cyclic and write down why the other is not. Write down all the generators of the cyclic group.

(e) Give a reason why G and H are not isomorphic.

The binary operation

$*$ is defined on

$\mathbb{R}$ as follows. For any elements a ,

$b \in \mathbb{R}$

$$a*b = a+b+1.$$

33a. (i) Show that *[5 marks]*

$*$ is commutative.

(ii) Find the identity element.

(iii) Find the inverse of the element a .

33b. The binary operation *[6 marks]*

$\cdot$ is defined on

$\mathbb{R}$ as follows. For any elements a ,

$b \in \mathbb{R}$

$a \cdot b = 3ab$. The set S is the set of all ordered pairs

$(x,\ y)$ of real numbers and the binary operation

$\odot$ is defined on the set S as

$(x_1,\ y_1) \odot (x_2,\ y_2) = (x_1 * x_2,\ y_1 \cdot y_2)$.

Determine whether or not

$\odot$ is associative.

34. The relation R is defined on *[14 marks]*

$\mathbb{Z} \times \mathbb{Z}$ such that

$(a,\ b)R(c,\ d)$ if and only if $a - c$ is divisible by 3 and $b - d$ is divisible by 2.

(a) Prove that R is an equivalence relation.

(b) Find the equivalence class for $(2, 1)$.

(c) Write down the five remaining equivalence classes.

35. (a) Show that *[11 marks]*

$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by

$f(x,\ y) = (2x + y,\ x - y)$ is a bijection.

(b) Find the inverse of f .

36. Prove that set difference is not associative. *[7 marks]*

37. The binary operation *[13 marks]*

$*$ is defined on the set $S = \{0, 1, 2, 3\}$ by

$$a * b = a + 2b + ab(\bmod 4) .$$

(a) (i) Construct the Cayley table.

(ii) Write down, with a reason, whether or not your table is a Latin square.

(b) (i) Write down, with a reason, whether or not

$*$ is commutative.

(ii) Determine whether or not

$*$ is associative, justifying your answer.

(c) Find all solutions to the equation

$x * 1 = 2 * x$, for

$x \in S$.

38. The function *[10 marks]*

$f : [0,\ \infty[\to [0,\ \infty[$ is defined by

$f(x) = 2\mathrm{e}^x + \mathrm{e}^{-x} - 3$.

(a) Find

$f'(x)$.

(b) Show that f is a bijection.

(c) Find an expression for

$f^{-1}(x)$.

39. The relations R and S are defined on quadratic polynomials P of the form *[12 marks]*

$$P(z) = z^2 + az + b \text{, where } a, b \in \mathbb{R}, z \in \mathbb{C}.$$

(a) The relation R is defined by

$P_1 R P_2$ if and only if the sum of the two zeros of

P_1 is equal to the sum of the two zeros of

P_2.

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence class containing

$z^2 - 4z + 5$.

(b) The relation S is defined by

$P_1 S P_2$ if and only if

P_1 and

P_2 have at least one zero in common. Determine whether or not S is transitive.

40. Let $\{G,$ *[9 marks]*

$*\}$ be a finite group of order n and let H be a non-empty subset of G.

(a) Show that any element

$h \in H$ has order smaller than or equal to n.

(b) If H is closed under

$*$, show that $\{H,$

$*\}$ is a subgroup of $\{G,$

$*\}$.

41. (a) Consider the set $A = \{1, 3, 5, 7\}$ under the binary operation *[14 marks]*

$*$, where

$*$ denotes multiplication modulo 8.

(i) Write down the Cayley table for

$\{A, *\}$.

(ii) Show that

$\{A, *\}$ is a group.

(iii) Find all solutions to the equation

$3 * x * 7 = y$. Give your answers in the form

(x, y).

(b) Now consider the set $B = \{1, 3, 5, 7, 9\}$ under the binary operation

$\otimes$, where

$\otimes$ denotes multiplication modulo 10. Show that

$\{B, \otimes\}$ is not a group.

(c) Another set C can be formed by removing an element from B so that

$\{C, \otimes\}$ is a group.

(i) State which element has to be removed.

(ii) Determine whether or not

$\{A, *\}$ and

$\{C, \otimes\}$ are isomorphic.

42. The permutation *[13 marks]*

p_1 of the set $\{1, 2, 3, 4\}$ is defined by

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

(a) (i) State the inverse of

p_1.

(ii) Find the order of

p_1.

(b) Another permutation

p_2 is defined by

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

(i) Determine whether or not the composition of

p_1 and

p_2 is commutative.

(ii) Find the permutation

p_3 which satisfies

$$p_1 p_3 p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

43. Let G be a finite cyclic group. *[13 marks]*

(a) Prove that G is Abelian.

(b) Given that a is a generator of G, show that

a^{-1} is also a generator.

(c) Show that if the order of G is five, then all elements of G, apart from the identity, are generators of G.

44a. The relation R is defined on *[5 marks]*

$\mathbb{Z}^+$ by aRb if and only if ab is even. Show that only one of the conditions for R to be an equivalence relation is satisfied.

44b. The relation S is defined on *[9 marks]*

$\mathbb{Z}^+$ by aSb if and only if

$a^2 \equiv b^2 (\bmod 6)$.

(i) Show that S is an equivalence relation.

(ii) For each equivalence class, give the four smallest members.

The binary operations

$\odot$ and

$*$ are defined on

$\mathbb{R}^+$ by

$$a \odot b = \sqrt{ab} \text{ and } a * b = a^2 b^2.$$

Determine whether or not

45a. $\odot$ is commutative; *[2 marks]*

45b. $*$ is associative; *[4 marks]*

45c. $*$ is distributive over *[4 marks]*

$\odot$;

45d. $\odot$ has an identity element. *[3 marks]*

The group

$\{G, \times_7\}$ is defined on the set $\{1, 2, 3, 4, 5, 6\}$ where

$\times_7$ denotes multiplication modulo 7.

46a. (i) Write down the Cayley table for *[10 marks]*

$\{G, \times_7\}$.

(ii) Determine whether or not

$\{G, \times_7\}$ is cyclic.

(iii) Find the subgroup of G of order 3, denoting it by H.

(iv) Identify the element of order 2 in G and find its coset with respect to H.

46b. The group *[6 marks]*

$\{K, \circ\}$ is defined on the six permutations of the integers 1, 2, 3 and

$\circ$ denotes composition of permutations.

(i) Show that

$\{K, \circ\}$ is non-Abelian.

(ii) Giving a reason, state whether or not

$\{G, \times_7\}$ and

$\{K, \circ\}$ are isomorphic.

47. The groups *[9 marks]*

$\{K, *\}$ and

$\{H, \odot\}$ are defined by the following Cayley tables.

G

*	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	A	E
C	C	B	E	A

H

$\odot$	e	a
e	e	a
a	a	e

By considering a suitable function from G to H, show that a surjective homomorphism exists between these two groups. State the kernel of this homomorphism.

48. Let *[8 marks]*

$\{G, *\}$ be a finite group and let H be a non-empty subset of G. Prove that

$\{H, *\}$ is a group if H is closed under

$*$.