Topic 1 Part 3 *[483 marks]*

The complex numbers

$z_1 = 2 - 2\text{i}$ and

$z_2 = 1 - \sqrt{3}\text{i}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

1a. *[3 marks]*

Find AB, giving your answer in the form

$a\sqrt{b - \sqrt{3}}$, where a ,

$b \in \mathbb{Z}^+$.

1b. *[3 marks]*

Calculate

$\text{A}\hat{\text{O}}\text{B}$ in terms of

π.

An arithmetic sequence has first term a and common difference d,
$d \neq 0$. The
3^{rd},
4^{th} and
7^{th} terms of the arithmetic sequence are the first three terms of a geometric sequence.

2a. Show that *[3 marks]*

$a = -\frac{3}{2}d$.

2b. Show that the *[5 marks]*

4^{th} term of the geometric sequence is the
16^{th} term of the arithmetic sequence.

3a. Factorize *[2 marks]*

$z^3 + 1$ into a linear and quadratic factor.

3b. Let *[9 marks]*

$\gamma = \frac{1+\text{i}\sqrt{3}}{2}$.

(i) Show that

γ is one of the cube roots of -1.

(ii) Show that

$\gamma^2 = \gamma - 1$.

(iii) Hence find the value of

$(1 - \gamma)^6$.

4. In the arithmetic series with [5 marks]

n^{th} term

u_n , it is given that

$u_4 = 7$ and

$u_9 = 22$.

Find the minimum value of n so that

$u_1 + u_2 + u_3 + \ldots + u_n > 10\,000$.

5a. Prove by mathematical induction that, for [8 marks]

$n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

5b. (a) Using integration by parts, show that [17 marks]

$\int \mathrm{e}^{2x} \sin x \mathrm{d}x = \frac{1}{5}\mathrm{e}^{2x}(2\sin x - \cos x) + C$.

(b) Solve the differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 - y^2}\,\mathrm{e}^{2x} \sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form

$y = f(x)$.

(c) (i) Sketch the graph of

$y = f(x)$, found in part (b), for

$0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the x-axis, and mark it on your sketch.

(ii) The region bounded by the graph of

$y = f(x)$ and the x-axis, between the origin and P, is rotated 360° about the x-axis to form a solid of revolution.

Calculate the volume of this solid.

The integral

I_n is defined by

$I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x} |\sin x| \mathrm{d}x$, for $n \in \mathbb{N}$.

6a. Show that [6 marks]

$I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$.

6b. By letting [4 marks]

$y = x - n\pi$, show that

$I_n = \mathrm{e}^{-n\pi} I_0$.

6c. Hence determine the exact value of *[5 marks]*

$\int_0^\infty \mathrm{e}^{-x}|\sin x|\mathrm{d}x$.

7. Solve the equation *[5 marks]*

$4^{x-1} = 2^x + 8.$

8. (a) Show that *[20 marks]*

$\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x.$

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all

$n \in \mathbb{Z}^+$, $\sin x \neq 0.$

(c) Solve the equation

$\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi.$

9. The system of equations *[5 marks]*

$$2x - y + 3z = 2$$

$$3x + y + 2z = -2$$

$$-x + 2y + az = b$$

is known to have more than one solution. Find the value of a and the value of b.

10. (a) Solve the equation *[7 marks]*

$z^3 = -2 + 2\mathrm{i}$, giving your answers in modulus-argument form.

(b) **Hence** show that one of the solutions is 1 + i when written in Cartesian form.

11. Find the sum of all three-digit natural numbers that are not exactly divisible by 3. *[5 marks]*

12. (a) Consider the following sequence of equations. *[10 marks]*

$1 \times 2 = \frac{1}{3}(1 \times 2 \times 3),$

$1 \times 2 + 2 \times 3 = \frac{1}{3}(2 \times 3 \times 4),$

$1 \times 2 + 2 \times 3 + 3 \times 4 = \frac{1}{3}(3 \times 4 \times 5),$

$\dots .$

(i) Formulate a conjecture for the

n^{th} equation in the sequence.

(ii) Verify your conjecture for $n = 4$.

(b) A sequence of numbers has the

n^{th} term given by

$u_n = 2^n + 3,\ n \in \mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false.

(c) Use mathematical induction to prove that

$5 \times 7^n + 1$ is divisible by 6 for all

$n \in \mathbb{Z}^+$.

13. Consider *[16 marks]*

$\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right).$

(a) Show that

(i)

$\omega^3 = 1;$

(ii)

$1 + \omega + \omega^2 = 0$

(b) (i) Deduce that

$\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0.$

(ii) Illustrate this result for

$\theta = \frac{\pi}{2}$ on an Argand diagram.

(c) (i) Expand and simplify

$F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve

$F(z) = 7$, giving your answers in terms of

ω.

14. Expand and simplify [4 marks]

$\left(x^2 - \frac{2}{x}\right)^4$.

15. The mean of the first ten terms of an arithmetic sequence is 6. The mean of the first twenty terms of the arithmetic sequence is 16. Find the value of the [6 marks]

15^{th} term of the sequence.

16. The sum, [8 marks]

S_n, of the first n terms of a geometric sequence, whose

n^{th} term is

u_n, is given by

$$S_n = \frac{7^n - a^n}{7^n}, \text{ where } a > 0.$$

(a) Find an expression for

u_n.

(b) Find the first term and common ratio of the sequence.

(c) Consider the sum to infinity of the sequence.

(i) Determine the values of a such that the sum to infinity exists.

(ii) Find the sum to infinity when it exists.

17. Consider the complex number [19 marks]

$\omega = \frac{z+\mathrm{i}}{z+2}$, where

$z = x + \mathrm{i}y$ and

$\mathrm{i} = \sqrt{-1}$.

(a) If

$\omega = \mathrm{i}$, determine z in the form

$z = r\,\mathrm{cis}\,\theta$.

(b) Prove that

$\omega = \frac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$.

(c) **Hence** show that when

$\mathrm{Re}(\omega) = 1$ the points

$(x,\, y)$ lie on a straight line,

l_1, and write down its gradient.

(d) Given

$\arg(z) = \arg(\omega) = \frac{\pi}{4}$, find

$|z|$.

18. Consider the polynomial [7 marks]

$p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

Given that 1 + i and 1 − 2i are zeros of $p(x)$, find the values of a, b, c and d.

A geometric sequence u_1, u_2, u_3, ... has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$.

19a. Find the common ratio of the geometric sequence. [2 marks]

19b. An arithmetic sequence v_1, v_2, v_3, ... is such that $v_2 = u_2$ and $v_4 = u_4$. [5 marks]

Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$.

20a. Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. [2 marks]

20b. Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3 marks]

20c. Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. [3 marks]

20d. **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. [6 marks]

20e. By considering the solutions of the equation $\cos 5\theta = 0$, show that $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$. [8 marks]

21a. Write down the quadratic expression $2x^2 + x - 3$ as the product of two linear factors. [1 mark]

21b. Hence, or otherwise, find the coefficient of x in the expansion of $\left(2x^2+x-3\right)^8$. [4 marks]

22. Solve the following system of equations. [6 marks]

$$\log_{x+1} y = 2$$

$$\log_{y+1} x = \frac{1}{4}$$

The equations of three planes, are given by

$$ax + 2y + z = 3$$

$$-x + (a+1)y + 3z = 1$$

$$-2x + y + (a+2)z = k$$

where $a \in \mathbb{R}$.

23a. Given that $a = 0$, show that the three planes intersect at a point. [3 marks]

23b. Find the value of a such that the three planes do not meet at a point. [5 marks]

23c. Given a such that the three planes do not meet at a point, find the value of k such that the planes meet in one line and find an equation of this line in the form [6 marks]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

24. Express [5 marks]

$\frac{1}{(1-\mathrm{i}\sqrt{3})^3}$ in the form $\frac{a}{b}$ where $a,\ b \in \mathbb{Z}$.

25. The common ratio of the terms in a geometric series is [6 marks]

2^x.

(a) State the set of values of x for which the sum to infinity of the series exists.

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40.

26a. Find the sum of the infinite geometric sequence 27, −9, 3, −1, ... [3 marks]

26b. Use mathematical induction to prove that for [7 marks]

$n \in \mathbb{Z}^+$,

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

27. Let [12 marks]

$w = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) Show that *w* is a root of the equation

$z^5 - 1 = 0$.

(b) Show that

$(w-1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that

$w^4 + w^3 + w^2 + w + 1 = 0$.

(c) **Hence** show that

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

28. An 81 metre rope is cut into *n* pieces of increasing lengths that form an arithmetic sequence with a common difference of *d* metres. Given that the lengths of the shortest and longest pieces are 1.5 metres and 7.5 metres respectively, find the values of *n* and *d* . [4 marks]

29a. (a) Use de Moivre's theorem to find the roots of the equation [12 marks]

$z^4 = 1 - \mathrm{i}$.

(b) Draw these roots on an Argand diagram.

(c) If

z_1 is the root in the first quadrant and

z_2 is the root in the second quadrant, find

$\frac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

29b. (a) Expand and simplify [13 marks]

$(x-1)(x^4 + x^3 + x^2 + x + 1)$.

(b) Given that *b* is a root of the equation

$z^5 - 1 = 0$ which does not lie on the real axis in the Argand diagram, show that

$1 + b + b^2 + b^3 + b^4 = 0$.

(c) If

$u = b + b^4$ and

$v = b^2 + b^3$ show that

(i) $u + v = uv = -1$;

(ii)

$u - v = \sqrt{5}$, given that

$u - v > 0$.

30. A geometric sequence has a first term of 2 and a common ratio of 1.05. Find the value of the smallest term which is greater than 500. [5 marks]

31. (a) Find the set of values of k for which the following system of equations has no solution. [5 marks]

$$x + 2y - 3z = k$$

$$3x + y + 2z = 4$$

$$5x + 7z = 5$$

(b) Describe the geometrical relationship of the three planes represented by this system of equations.

32. Consider the complex numbers [6 marks]

$z = 1 + 2\mathrm{i}$ and

$w = 2 + a\mathrm{i}$, where

$a \in \mathbb{R}$.

Find a when

(a)

$|w| = 2\,|z|$; ;

(b)

$\mathrm{Re}(zw) = 2\,\mathrm{Im}(zw)$.

33a. If z is a non-zero complex number, we define [9 marks]

$L(z)$ by the equation

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \leqslant \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number,

$L(z) = \ln z$.

(b) Use the equation to calculate

(i)

$L(-1)$;

(ii)

$L(1 - \mathrm{i})$;

(iii)

$L(-1 + \mathrm{i})$.

(c) Hence show that the property

$L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of

z_1 and

z_2 .

33b. Let *f* be a function with domain [14 marks]

$\mathbb{R}$ that satisfies the conditions,

$f(x+y) = f(x)f(y)$, for all *x* and *y* and

$f(0) \neq 0$.

(a) Show that

$f(0) = 1.$

(b) Prove that

$f(x) \neq 0$, for all

$x \in \mathbb{R}$.

(c) Assuming that

$f'(x)$ exists for all

$x \in \mathbb{R}$, use the definition of derivative to show that

$f(x)$ satisfies the differential equation

$f'(x) = k\,f(x)$, where

$k = f'(0)$.

(d) Solve the differential equation to find an expression for

$f(x)$.

34. Given that [8 marks]

$z_1 = 2$ and

$z_2 = 1 + \sqrt{3}\mathrm{i}$ are roots of the cubic equation

$z^3 + bz^2 + cz + d = 0$

where b, c,

$d \in \mathbb{R}$,

(a) write down the third root,

z_3, of the equation;

(b) find the values of b, c and d ;

(c) write

z_2 and

z_3 in the form

$r\mathrm{e}^{\mathrm{i}\theta}$.

35. Prove by mathematical induction [8 marks]

$\sum\limits_{r=1}^{n} r(r!) = (n+1)! - 1,$

$n \in \mathbb{Z}^+$.

36. The complex number z is defined as [22 marks]

$z = \cos\theta + \mathrm{i}\sin\theta$.

(a) State de Moivre's theorem.

(b) Show that

$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$.

(c) Use the binomial theorem to expand

$\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form.

(d) Hence show that

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(e) Check that your result in part (d) is true for

$\theta = \frac{\pi}{4}$.

(f) Find

$\int_0^{\frac{\pi}{2}} \sin^5\theta \mathrm{d}\theta$.

(g) Hence, with reference to graphs of circular functions, find

$\int_0^{\frac{\pi}{2}} \cos^5\theta \mathrm{d}\theta$, explaining your reasoning.

37. Find the values of n such that [5 marks]

$\left(1 + \sqrt{3}\mathrm{i}\right)^n$ is a real number.

38a. (a) The sum of the first six terms of an arithmetic series is 81. The sum of its first eleven terms is 231. Find the first term and the common difference. [14 marks]

(b) The sum of the first two terms of a geometric series is 1 and the sum of its first four terms is 5. If all of its terms are positive, find the first term and the common ratio.

(c) The

r^{th} term of a new series is defined as the product of the

r^{th} term of the arithmetic series and the

r^{th} term of the geometric series above. Show that the

r^{th} term of this new series is

$(r+1)2^{r-1}$.

38b. Using mathematical induction, prove that [7 marks]

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n2^n,\ n \in \mathbb{Z}^+.$$

39a. Let [8 marks]

$z = x + \mathrm{i}y$ be any non-zero complex number.

(i) Express

$\frac{1}{z}$ in the form

$u + \mathrm{i}v$.

(ii) If

$z + \frac{1}{z} = k$,

$k \in \mathbb{R}$, show that either $y = 0$ or

$x^2 + y^2 = 1.$

(iii) Show that if

$x^2 + y^2 = 1$ then

$|k| \leqslant 2$.

39b. Let [14 marks]

$w = \cos\theta + \mathrm{i}\sin\theta$.

(i) Show that

$w^n + w^{-n} = 2\cos n\theta$,

$n \in \mathbb{Z}$.

(ii) Solve the equation

$3w^2 - w + 2 - w^{-1} + 3w^{-2} = 0$, giving the roots in the form

$x + \mathrm{i}y$.

40. Three Mathematics books, five English books, four Science books and a dictionary are to be placed on a student's shelf so that the books of each subject remain together. [7 marks]

(a) In how many different ways can the books be arranged?

(b) In how many of these will the dictionary be next to the Mathematics books?

41. Consider the arithmetic sequence 8, 26, 44, [4 marks]

$\ldots$.

(a) Find an expression for the

n^{th} term.

(b) Write down the sum of the first n terms using sigma notation.

(c) Calculate the sum of the first 15 terms.

The three planes

$2x - 2y - z = 3$

$4x + 5y - 2z = -3$

$3x + 4y - 3z = -7$

intersect at the point with coordinates (a, b, c).

42a. Find the value of each of a, b and c. *[2 marks]*

42b. The equations of three planes are *[4 marks]*

$2x - 4y - 3z = 4$

$-x + 3y + 5z = -2$

$3x - 5y - z = 6.$

Find a vector equation of the line of intersection of these three planes.

43. Given that *[7 marks]*

$z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a)

$\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0,\ n \in \mathbb{Z}^+;$

(b)

$\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0,\ z \neq -1.$

44. The interior of a circle of radius 2 cm is divided into an infinite number of sectors. The areas of these sectors form a geometric sequence with common ratio k. The angle of the first sector is *[12 marks]*

θ radians.

(a) Show that

$\theta = 2\pi(1 - k).$

(b) The perimeter of the third sector is half the perimeter of the first sector.

Find the value of k and of

$\theta.$

When
$\left(1+\frac{x}{2}\right)^{2}$,
$n \in \mathbb{N}$, is expanded in ascending powers of
x , the coefficient of
x^3 is
70.

45. (a) Find the value of n . *[6 marks]*

(b) Hence, find the coefficient of x^2 .

Consider the equation
$z^3 + az^2 + bz + c = 0$, where
a ,
b,
$c \in \mathbb{R}$. The points in the Argand diagram representing the three roots of the equation form the vertices of a triangle whose area is 9. Given that one root is
$-1 + 3\mathrm{i}$, find

46. (a) the other two roots; *[7 marks]*

(b) a , b and c .

47. (a) Show that the complex number i is a root of the equation *[6 marks]*

$$x^4 - 5x^3 + 7x^2 - 5x + 6 = 0 .$$

(b) Find the other roots of this equation.

48. Six people are to sit at a circular table. Two of the people are not to sit immediately beside each other. Find the number of ways that the six people can be seated. *[5 marks]*