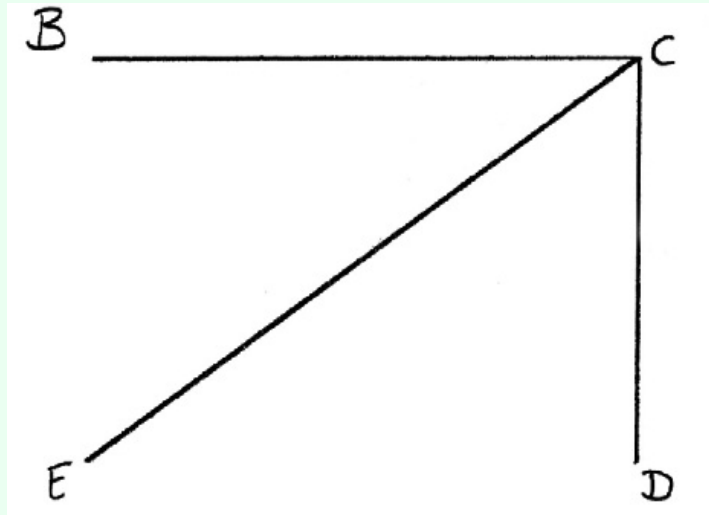# Topic 10 Part 2 *[474 marks]*

1a. *[5 marks]*

Markscheme

using any method, the minimum spanning tree is   *(M1)*

Note: Accept MST = {BC, EC, DC} or {BC, EB, DC}

Note: In graph, line CE may be replaced by BE.

lower bound = weight of minimum spanning tree + 2 smallest weights connected to A   *(M1)*

= 11 + 13 + 14 + 10 + 15 = 63   *A1*

[5 marks]

Examiners report

This question was generally well answered although some candidates failed to realise the significance of the equality of the upper and lower bounds.

1b. *[1 mark]*

Markscheme

weight of ADCBEA = 10 + 14 + 11 + 13 + 15 = 63   *A1*

[1 mark]

Examiners report

This question was generally well answered although some candidates failed to realise the significance of the equality of the upper and lower bounds.

1c. [1 mark]

Examiners report

This question was generally well answered although some candidates failed to realise the significance of the equality of the upper and lower bounds.

2a. [2 marks]

Markscheme

$a = \lambda c + 1$ *M1*

so

$ab = \lambda bc + b \Rightarrow ab \equiv b(\bmod c)$ *A1AG*

[2 marks]

Examiners report

Part (a) was generally well answered. In (b), many candidates tested the result for $n = 1$ instead of $n = 0$. It has been suggested that the reason for this was a misunderstanding of the symbol N with some candidates believing it to denote the positive integers. It is important for candidates to be familiar with IB notation in which N denotes the positive integers and zero. In some scripts the presentation of the proof by induction was poor.

Markscheme

the result is true for $n = 0$ since

$9^0 = 1 \equiv 1 (\bmod\ 4)$ ***A1***

assume the result is true for $n = k$, *i.e.*

$9^k \equiv 1 (\bmod\ 4)$ ***M1***

consider

$9^{k+1} = 9 \times 9^k$ ***M1***

$\equiv 9 \times 1 (\bmod\ 4)$ **or**

$1 \times 9^k (\bmod\ 4)$ ***A1***

$\equiv 1 (\bmod\ 4)$ ***A1***

so true for

$n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 0$ result follows by induction   ***R1***

Note: Do not award the final ***R1*** unless both ***M1*** marks have been awarded.

Note: Award the final ***R1*** if candidates state $n = 1$ rather than $n = 0$

[6 marks]

Examiners report

Part (a) was generally well answered. In (b), many candidates tested the result for $n = 1$ instead of $n = 0$. It has been suggested that the reason for this was a misunderstanding of the symbol N with some candidates believing it to denote the positive integers. It is important for candidates to be familiar with IB notation in which N denotes the positive integers and zero. In some scripts the presentation of the proof by induction was poor.

2c.   [4 marks]

Markscheme

let

$M = (a_n a_{n-1} \ldots a_0)_9$ *(M1)*

$= a_n \times 9^n + a_{n-1} \times 9^{n-1} + \ldots + a_0 \times 9^0$ *A1*

EITHER

$\equiv a_n (\bmod\ 4) + a_{n-1} (\bmod\ 4) + \ldots + a_0 (\bmod\ 4)$ *A1*

$\equiv \sum a_i (\bmod\ 4)$ *A1*

so M is divisible by 4 if

$\sum a_i$ is divisible by 4 *AG*

OR

$= a_n (9^n - 1) + a_{n-1}(9^{n-1} - 1) + \ldots + a_1(9^1 - 1)$

$+ a_n + a_{n-1} + \ldots + a_1 + a_0$ *A1*

Since

$9^n \equiv 1 (\bmod\ 4)$, it follows that

$9^n - 1$ is divisible by 4, *R1*

so M is divisible by 4 if

$\sum a_i$ is divisible by 4 *AG*

[4 marks]

Examiners report

Part (a) was generally well answered. In (b), many candidates tested the result for $n = 1$ instead of $n = 0$. It has been suggested that the reason for this was a misunderstanding of the symbol N with some candidates believing it to denote the positive integers. It is important for candidates to be familiar with IB notation in which N denotes the positive integers and zero. In some scripts the presentation of the proof by induction was poor.

3.   [4 marks]

Markscheme

(i)   any Hamiltonian circuit ACBEFDA *A2*

(ii)   no Eulerian circuit exists because the graph contains vertices of odd degree *A2*

[4 marks]

Examiners report

Parts (a) and (b) were well answered by many candidates. In (c), candidates who tried to prove the result by adding edges to a drawing of G were given no credit. Candidates should be aware that the use of the word 'Prove' indicates that a formal treatment is required Solutions to (d) were often disappointing although a graphical solution was allowed here.

4a. [4 marks]

Markscheme

any clearly indicated method of dividing 1189 by successive numbers   *M1*

find that 1189 has factors 29 and/or 41   *A2*

it follows that 1189 is not a prime number   *A1*

Note: If no method is indicated, award *A1* for the factors and *A1* for the conclusion.

[4 marks]

Examiners report

In (a), some candidates tried to use Fermat's little theorem to determine whether or not 1189 is prime but this method will not always work and in any case the amount of computation involved can be excessive. For this reason, it is strongly recommended that this method should not be used in examinations. In (b), it was clear from the scripts that candidates who had covered this material were generally successful and those who had not previously seen the result were usually unable to proceed.

4b. [6 marks]

Markscheme

(i)   every positive integer, greater than 1, is either prime or can be expressed uniquely as a product of primes   *A1A1*

Note: Award *A1* for "product of primes" and *A1* for "uniquely".

(ii)   **METHOD 1**

let *M* and *N* be expressed as a product of primes as follows

$M = AB$ and $N = AC$ *M1A1*

where *A* denotes the factors which are common and *B*, *C* the disjoint factors which are not common

it follows that $G = A$ *A1*

and $L = GBC$ *A1*

from these equations, it follows that

$GL = A \times ABC = MN$ *AG*

METHOD 2

Let

$M = 2^{x_1} \times 3^{x_2} \times \ldots p_n^{x_n}$ and

$N = 2^{y_1} \times 3^{y_2} \times \ldots p_n^{y_n}$ where

p_n denotes the

n^{th} prime   *M1*

Then

$G = 2^{\min(x_1,y_1)} \times 3^{\min(x_2,y_2)} \times \ldots p_n^{\min(x_n,y_n)}$ *A1*

and

$L = 2^{\max(x_1,y_1)} \times 3^{\max(x_2,y_2)} \times \ldots p_n^{\max(x_n,y_n)}$ *A1*

It follows that

$GL = 2^{x_1} \times 2^{y_1} \times 3^{x_2} \times 3^{y_2} \times \ldots \times p_n^{x_n} \times p_n^{y_n}$ *A1*

$= MN$ *AG*

[6 marks]

In (a), some candidates tried to use Fermat's little theorem to determine whether or not 1189 is prime but this method will not always work and in any case the amount of computation involved can be excessive. For this reason, it is strongly recommended that this method should not be used in examinations. In (b), it was clear from the scripts that candidates who had covered this material were generally successful and those who had not previously seen the result were usually unable to proceed.

5.   *[8 marks]*

Markscheme

(a)   **EITHER**

Prim's algorithm   *A1*

OR

Kruskal's algorithm   *A1*

[1 mark]

(b)   **EITHER**

using Prim's algorithm, starting at A

Edge	Cost	
AC	4	*A1*
CD	3	*A1*
CF	4	*A1*
FE	4	*A1*
AB	5	*A1*

lowest cost road system contains roads AC, CD, CF, FE and AB   *A1*

cost is 20   *A1*

OR

using Kruskal's algorithm

Edge	Cost	
CD	3	*A1*
CF	4	*A1*
FE	4	*A1*
AC	4	*A1*
AB	5	*A1*

lowest cost road system contains roads CD, CF, FE, AC and AB   *A1*

cost is 20   *A1*

Note: Accept alternative correct solutions.

[7 marks]

Total [8 marks]

6.   *[14 marks]*

Markscheme

(a)

$12\,306 = 4 \times 2976 + 402$ *M1*

$2976 = 7 \times 402 + 162$ *M1*

$402 = 2 \times 162 + 78$ *A1*

$162 = 2 \times 78 + 6$ *A1*

$78 = 13 \times 6$

therefore gcd is 6   *R1*

[5 marks]

(b)

$6|996$ means there is a solution

$6 = 162 - 2(78)$ *(M1)(A1)*

$= 162 - 2\,(402 - 2(162))$

$= 5(162) - 2(402)$ *(A1)*

$= 5(2976) - 7(402)\} \right) - 2(402)$

$= 5(2976) - 37(402)$ *(A1)*

$= 5(2976) - 37\,(12\,306 - 4(2976))$

$= 153(2976) - 37(12\,306)$ *(A1)*

$996 = 25\,398(2976) - 6142(12\,306)$

$\Rightarrow x_0 = -6142,\ y_0 = 25\,398$ *(A1)*

$\Rightarrow x = -6142 + \left(\frac{2976}{6}\right) t = -6142 + 496t$

$\Rightarrow y = 25\,398 - \left(\frac{12\,306}{6}\right) t = 25\,398 - 2051t$ *M1A1A1*

[9 marks]

Total [14 marks]

7.

Markscheme

(a)
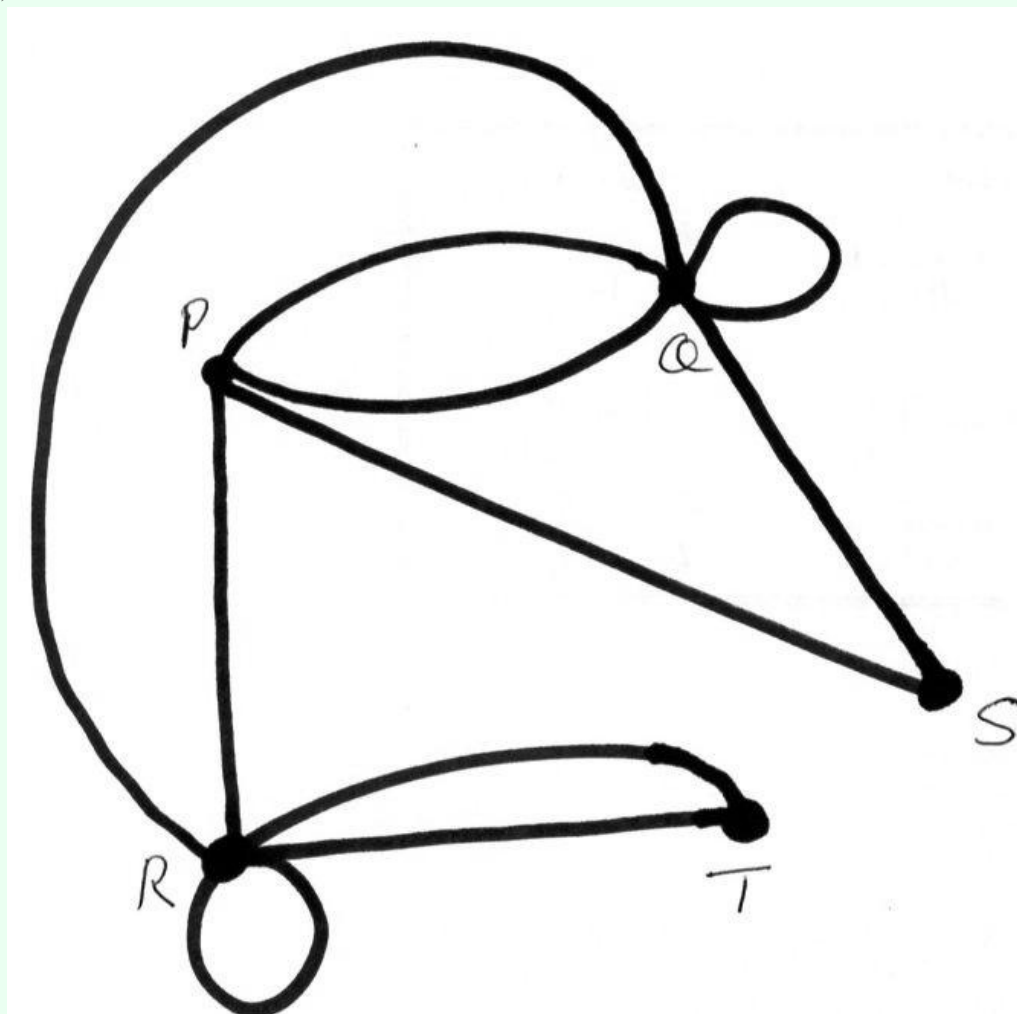
A3

Note: Award *A2* for one missing or misplaced edge,

A1 for two missing or misplaced edges.

[3 marks]

(b) (i)   an Eulerian circuit is one that contains every edge of the graph exactly once   *A1*

(ii)   a possible Eulerian circuit is

$P \rightarrow Q \rightarrow S \rightarrow P \rightarrow Q \rightarrow Q \rightarrow R \rightarrow T \rightarrow R \rightarrow R \rightarrow P$ *A2*

[3 marks]

(c) (i)   a Hamiltonian cycle passes through each vertex of the graph   *A1*

exactly once   *A1*

(ii)   to pass through T, you must have come from R and must return to R.   *R3*

hence there is no Hamiltonian cycle

[5 marks]

8.

Markscheme

the mth term of the first sequence

$= 2 + 4(m-1)$ *(M1)(A1)*

the nth term of the second sequence

$= 7 + 5(n-1)$ *(A1)*

EITHER

equating these,   *M1*

$5n = 4m - 4$

$5n = 4(m-1)$ *(A1)*

4 and 5 are coprime   *(M1)*

$\Rightarrow 4 \mid n$ so

$n = 4s$ or

$5 \mid (m-1)$ so

$m = 5s + 1$,

$s \in \mathbb{Z}^+$ *(A1)A1*

thus the common terms are of the form

$\{2 + 20s;\ s \in \mathbb{Z}^+\}$ *A1*

OR

the numbers of both sequences are

2, 6, 10, 14, 18, 22

7, 12, 17, 22   *A1*

so 22 is common   *A1*

identify the next common number as 42   *(M1)A1*

the general solution is

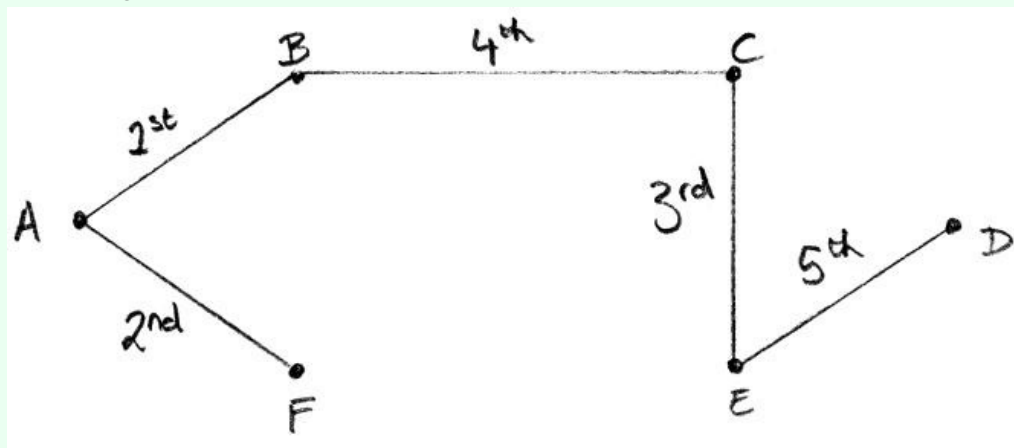$\{2 + 20s;\ s \in \mathbb{Z}^+\}$ *(M1)A1*

[9 marks]

9.

Markscheme

(a) (i)   all the vertices have even degree   *A1*

(ii)   for example ABCDECFBEFA   *A2*

[3 marks]

(c)   the edges are included in the order shown

M1A1A1A1A1A1

Note: Award each *A1* for the edge added in the correct order. Award no further marks after the first error.

[6 marks]

Total [9 marks]

10.

[5 marks]

Markscheme

the complete graph with 6 vertices has 15 edges so

G' has

6 vertices and 5 edges   ***M1A1***

the number of faces in

G',

$f = 2 + e - v = 1$ ***M1A1***

it is therefore a tree because

$f = 1$ ***R1***

Note: Accept it is a tree because

$v = e + 1$

[5 marks]

Examiners report

Part (a) was well answered by many candidates.

11a.

[7 marks]

Markscheme

consider the decimal number

$A = a_n a_{n-1} \ldots a_0$ ***M1***

$A = A_n \times 10^n + a_{n-1} \times 10^{n-1} + \ldots + a_1 \times 10 + a_0$ ***M1***

$= a_n \times (10^n - 1) + a_{n-1} \times (10^{n-1} - 1) + \ldots + a_1 \times (10 - 1) + a_n + a_{n-1} + \ldots + a_0$ ***M1A1***

$= a_n \times 99 \ldots 9(n \text{ digits}) + a_{n-1} \times 99 \ldots 9(n-1 \text{ digits}) + \ldots + 9a_1 + a_n + a_{n-1} + \ldots + a_0$ ***A1***

all the numbers of the form $99 \ldots 9$ are divisible by 9 (to give $11 \ldots 1$),   ***R1***

hence A is divisible by 9 if

$\sum_{i=0}^{n} a_i$ is divisible by 9   ***R1***

Note: A method that uses the fact that

$10^t \equiv 1 (\mathrm{mod}\ 9)$ is equally valid.

[7 marks]

Examiners report

Questions similar to (a) have been asked in the past so it was surprising to see that solutions this time were generally disappointing.

Markscheme

by Fermat's Little Theorem

$5^6 \equiv 1 (\bmod 7)$ *M1A1*

$(126)_7 = (49 + 14 + 6)_{10} = (69)_{10}$ *M1A1*

$5^{(126)_7} \equiv 5^{(11 \times 6 + 3)_{10}} \equiv 5^{(3)_{10}} (\bmod 7)$ *M1A1*

$5^{(3)_{10}} = (125)_{10} = (17 \times 7 + 6)_{10} \equiv 6 (\bmod 7)$ *M1A1*

hence

$a_0 = 6$ *A1*

[9 marks]

Examiners report

In (b), most candidates changed the base 7 number 126 to the base 10 number 69. After that the expectation was that Fermat's little theorem would be used to complete the solution but few candidates actually did that. Many were unable to proceed any further and others used a variety of methods, for example working modulo 7,
$5^{69} = (5^2)^{34}.5 = 4^{34}.5 = (4^2)^{17}.5 = 2^{17}.5$ etc
This is of course a valid method, but somewhat laborious.

Markscheme

Suppose the graph is bipartite so that the vertices belong to one of two disjoint sets M, N.   *M1*

Then consider any vertex V in M. To generate a cycle returning to V, we must go to a vertex in N, then to a vertex in M, then to a vertex in N, then to a vertex in M, *etc.*   *R1*

To return to V, therefore, which belongs to M, an even number of steps will be required.   *R1*

Now suppose the graph contains only cycles of even length.   *M1*

Starting at any vertex V, define the set M as containing those vertices accessible from V in an even number of steps and the set N as containing those vertices accessible from V in an odd number of steps.   *R1*

Suppose that the vertex X belongs to both M and N. Then consider the closed walk from V to X one way and back to V the other way. This closed walk will be of odd length. This closed walk can be contracted to a cycle which will also be of odd length, giving a contradiction to the initial assumption.   *R1*

There can therefore be no vertices common to M and N which shows that the vertices can be divided into two disjoint sets and the graph is bipartite.   *R1*

Consider any edge joining P to vertex Q. Then either

P ∈ M in which case

Q ∈ M or vice versa. In either case an edge always joins a vertex in M to a vertex in N so the graph is bipartite.   *R1*

[8 marks]

Examiners report

Many candidates made a reasonable attempt at showing that bipartite implies cycles of even length but few candidates even attempted the converse.

Markscheme

(i)

$4^8 = 65536 \equiv 7(\bmod\ 9)$ **A1**

not valid because 9 is not a prime number   **R1**

Note: The **R1** is independent of the **A1**.

(ii)   using Fermat's little theorem   **M1**

$5^6 \equiv 1(\bmod\ 7)$ **A1**

therefore

$(5^6)^{10} = 5^{60} \equiv 1(\bmod\ 7)$ **A1**

also,

$5^4 = 625$ **M1**

$\equiv 2(\bmod\ 7)$ **A1**

therefore

$5^{64} \equiv 1 \times 2 \equiv 2(\bmod\ 7)\ \ (\text{so } n = 2)$ **A1**

Note: Accept alternative solutions not using Fermat.

[8 marks]

Examiners report

Part (a) was generally well answered with a variety of methods seen in (a)(ii). This was set with Fermat's Little Theorem in mind but in the event many candidates started off with many different powers of 5, eg

$5^4 \equiv 2,\ 5^8 \equiv 4$ and

$5^3 \equiv -1(\bmod\ 7)$ were all seen. A variety of methods was also seen in (b), ranging from use of the Chinese Remainder Theorem, finding tables of numbers congruent to

$3(\bmod\ 4)$ and

$4(\bmod\ 5)$ and the use of an appropriate formula.

Markscheme

EITHER

solutions to

$x \equiv 3(\bmod 4)$ are

3, 7, 11, 15, 19, 23, 27, … **A1**

solutions to

$3x \equiv 2(\bmod 5)$. are

4, 9, 14, 19 … **(M1)A1**

so a solution is $x = 19$ **A1**

using the Chinese remainder theorem (or otherwise) **(M1)**

the general solution is

$x = 19 + 20n \ (n \in \mathbb{Z})$ **A1**

(accept 19(mod 20))

OR

$x = 3 + 4t \Rightarrow 9 + 12t \equiv 2(\bmod 5)$ **M1A1**

$\Rightarrow 2t \equiv 3(\bmod 5)$ **A1**

$\Rightarrow 6t \equiv 9(\bmod 5)$

$\Rightarrow t \equiv 4(\bmod 5)$ **A1**

so

$t = 4 + 5n$ and $x = 19 + 20n \ (n \in \mathbb{Z})$ **M1A1**

(accept 19(mod 20))

Note: Also accept solutions done by formula.

[6 marks]

Examiners report

Part (a) was generally well answered with a variety of methods seen in (a)(ii). This was set with Fermat's Little Theorem in mind but in the event many candidates started off with many different powers of 5, eg

$5^4 \equiv 2, \ 5^8 \equiv 4$ and

$5^3 \equiv -1(\bmod 7)$ were all seen. A variety of methods was also seen in (b), ranging from use of the Chinese Remainder Theorem, finding tables of numbers congruent to

3(mod 4)and

4(mod 5) and the use of an appropriate formula.
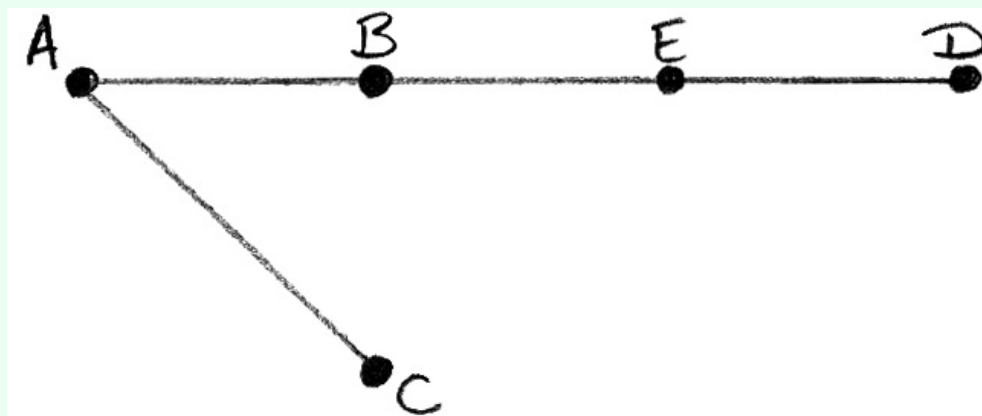
14.

Markscheme

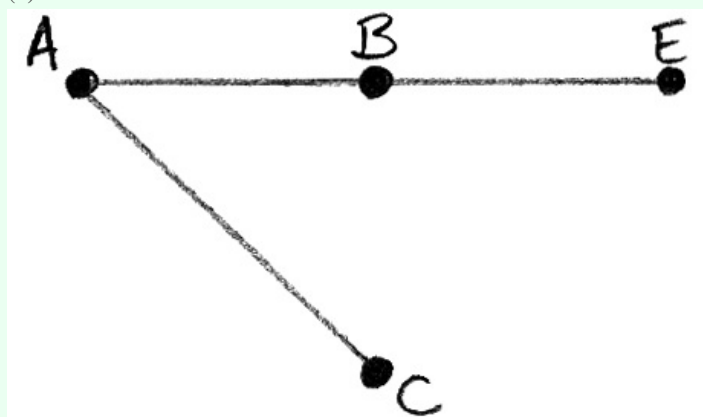(a) (i) the edges are joined in the order

AC

BE

AB

ED   *A2*

Note: Final *A1* independent of the previous *A2*.

(ii)

the weight of this spanning tree is 33   *A1*

to find a lower bound for the travelling salesman problem, we add to that the two smallest weights of edges to D, *i.e.* 15 +16, giving

64   *M1A1*

[7 marks]

(b)   an upper bound is the weight of any Hamiltonian cycle, *e.g.* ABCDEA has weight 75 so 80 is certainly an upper bound

M1A1

[2 marks]

Total [9 marks]

Examiners report

Part (a) was well done by many candidates although some candidates simply drew the minimum spanning tree in (i) without indicating the use of Kruskal's Algorithm. It is important to stress to candidates that, as indicated in the rubric at the top of Page 2, answers must be supported by working and/or explanations. Part (b) caused problems for some candidates who obtained the unhelpful upper bound of 96 by doubling the weight of the minimum spanning tree. It is useful to note that the weight of any Hamiltonian cycle is an upper bound and in this case it was fairly easy to find such a cycle with weight less than or equal to 80.

$$N = a_n a_{n-1} \ldots a_1 a_0 = a_n \times 9^n + a_{n-1} \times 9^{n-1} + \ldots + a_1 \times 9 + a_0$$

a_0

$$N = a_n \times (9^n - 1) + a_{n-1} \times (9^{n-1} - 1) + \ldots + a_1(9 - 1) + \sum_{i=1}^{n} a_i$$

$$\sum_{i=0}^{n} a_i$$

$$9^k \equiv 1(\bmod 2)$$

$$N = a_n a_{n-1} \ldots a_1 a_0 = a_n \times 9^n + a_{n-1} \times 9^{n-1} + \ldots + a_1 \times 9 + a_0 = \sum_{i=0}^{n} a_i(\bmod 2)$$

$$\sum_{i=0}^{n} a_i$$

$$(232430286)_9$$

Examiners report

Parts (a) and (b) were generally well answered. Part (c), however, caused problems for many candidates with some candidates even believing that showing divisibility by 2 and 3 was sufficient to prove divisibility by 12. Some candidates stated that the fact that the sum of the digits was 44 (which itself is divisible by 4) indicated divisibility by 4 but this was only accepted if the candidates extended their proof in (b) to cover divisibility by 4.

16. [18 marks]

Markscheme

(a)   start with a graph consisting of just a single vertex   **M1**

for this graph, $v = 1, f = 1$ and $e = 0$, the relation is satisfied   **A1**

Note: Allow solutions that begin with 2 vertices and 1 edge.

to extend the graph you either join an existing vertex to another existing vertex which increases e by 1 and f by 1 so that $v + f - e$

remains equal to 2   **M1A1**

or add a new vertex and a corresponding edge which increases e by 1 and v by 1 so that $v + f - e$ remains equal to 2   **M1A1**

therefore, however we build up the graph, the relation remains valid   **R1**

[7 marks]

(b)   since every face is bounded by at least 3 edges, the result follows by counting up the edges around each face   **R1**

the factor 2 follows from the fact that every edge bounds (at most) 2 faces   **R1**

hence

$3f \leqslant 2e$ **AG**

from the Euler relation,

$3f = 6 + 3e - 3v$ **M1**

substitute in the inequality,

$6 + 3e - 3v \leqslant 2e$ **A1**

hence

$e \leqslant 3v - 6$ **AG**

[4 marks]

(c)   let G have e edges   **M1**

since G and

G' have a total of

$\binom{12}{2} = 66$ edges   **A1**

it follows that

G' has $66 - e$ edges   **A1**

for planarity we require

$e \leqslant 3 \times 12 - 6 = 30$ **M1A1**

and

$66 - e \leqslant 30 \Rightarrow e \geqslant 36$ **A1**

these two inequalities cannot both be met indicating that both graphs cannot be planar   **R1**

[7 marks]

Total [18 marks]

17. [7 marks]

Markscheme

EITHER

we work modulo 3 throughout

the values of a, b, c, d can only be 0, 1, 2   *R2*

since there are 4 variables but only 3 possible values, at least 2 of the variables must be equal

$(\bmod 3)$ *R2*

therefore at least 1 of the differences must be

$0 (\bmod 3)$ *R2*

the product is therefore

$0 (\bmod 3)$ *R1AG*

OR

we attempt to find values for the differences which do not give

$0 (\bmod 3)$ for the product

we work modulo 3 throughout

we note first that none of the differences can be zero   *R1*

$a - b$ can therefore only be 1 or 2   *R1*

suppose it is 1, then $b - c$ can only be 1

since if it is 2,

$(a - b) + (b - c) \equiv 3 \equiv 0 (\bmod 3)$ *R1*

$c - d$ cannot now be 1 because if it is

$(a - b) + (b - c) + (c - d) = a - d \equiv 3 \equiv 0 (\bmod 3)$ *R1*

$c - d$ cannot now be 2 because if it is

$(b - c) + (c - d) = b - d \equiv 3 \equiv 0 (\bmod 3)$ *R1*

we cannot therefore find values of c and d to give the required result   *R1*

a similar argument holds if we suppose $a - b$ is 2, in which case $b - c$ must be 2 and we cannot find a value of $c - d$ *R1*

the product is therefore

$0 (\bmod 3)$ *AG*

[7 marks]

Examiners report

$e \leqslant 3v - 6$, for $v \geqslant 3$

$v - e + f = 2$

$v - v^2 + 8 = 2$

$v^2 - v - 6 = 0$

$(v + 2)(v - 3) = 0$

$v = 3$

Examiners report

In (b) most candidates gave the required inequality although some just wrote down both inequalities from their formula booklet. The condition

$v \geqslant 3$ was less well known but could be deduced from the next 2 graphs. Euler's relation was used well to obtain the quadratic to solve and many candidates could then draw a correct graph.

19.

[11 marks]

Markscheme

(a)

$N = 3 + 11t$ *M1*

$3 + 11t \equiv 4(\bmod 9)$

$2t \equiv 1(\bmod 9)$ *(A1)*

multiplying by 5,

$10t \equiv 5(\bmod 9)$ *(M1)*

$t \equiv 5(\bmod 9)$ *A1*

$t = 5 + 9s$ *M1*

$N = 3 + 11(5 + 9s)$

$N = 58 + 99s$ *A1*

$58 + 99s \equiv 0(\bmod 7)$

$2 + s \equiv 0(\bmod 7)$

$s \equiv 5(\bmod 7)$ *A1*

$s = 5 + 7u$ *M1*

$N = 58 + 99(5 + 7u)$

$N = 553 + 693u$ *A1*

Note: Allow solutions that are done by formula or an exhaustive, systematic listing of possibilities.

[9 marks]

(b) $u = 3$ or 4

hence N = 553 + 2079 = 2632 or $N = 553 + 2772 = 3325$ *A1A1*

[2 marks]

Total [11 marks]

Examiners report

This was a standard Chinese remainder theorem problem that many candidates gained good marks on. Some candidates employed a formula, which was fine if they remembered it correctly (but not all did), although it did not always show good understanding of the problem.

20.
Markscheme

(a)   (i)   *(Kruskal's: successively take an edge of smallest weight without forming a cycle)*
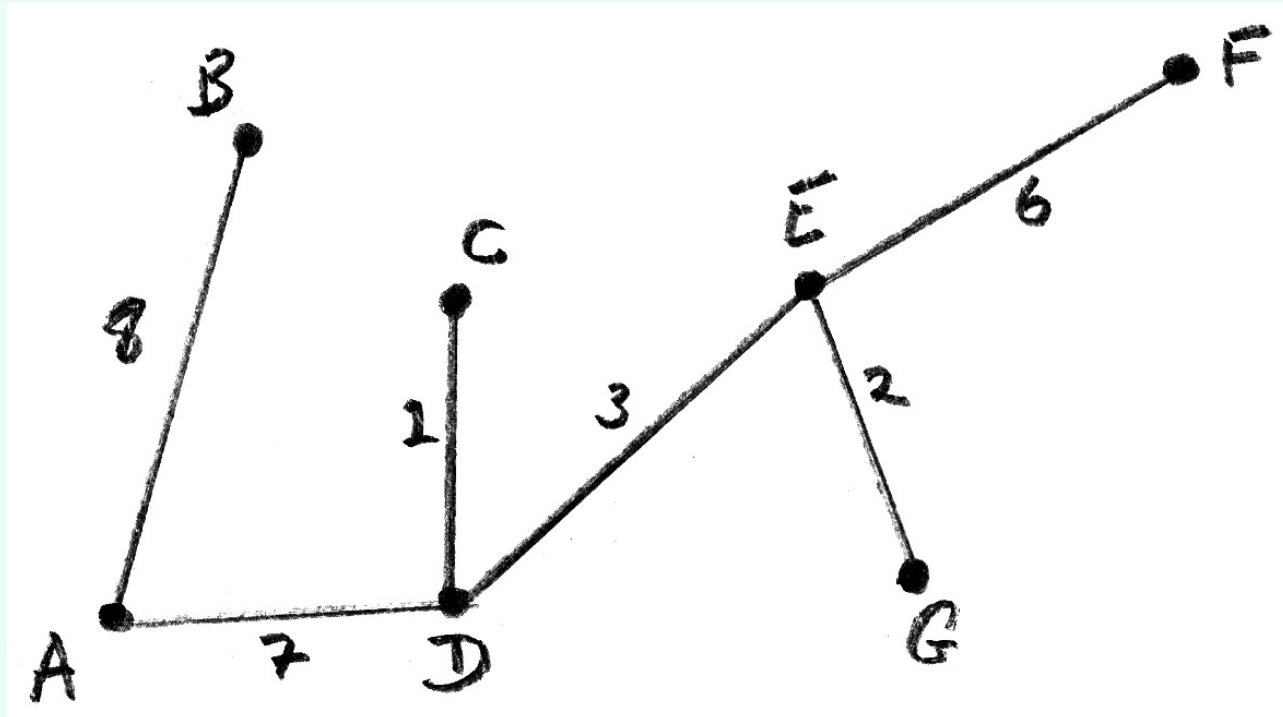
1st edge DC (weight 1)   *A1*

2nd edge EG (weight 2)   *A1*

3rd edge DE (weight 3)   *A1*

4th edge EF (weight 6)   *A1*

5th edge AD (weight 7)   *A1*
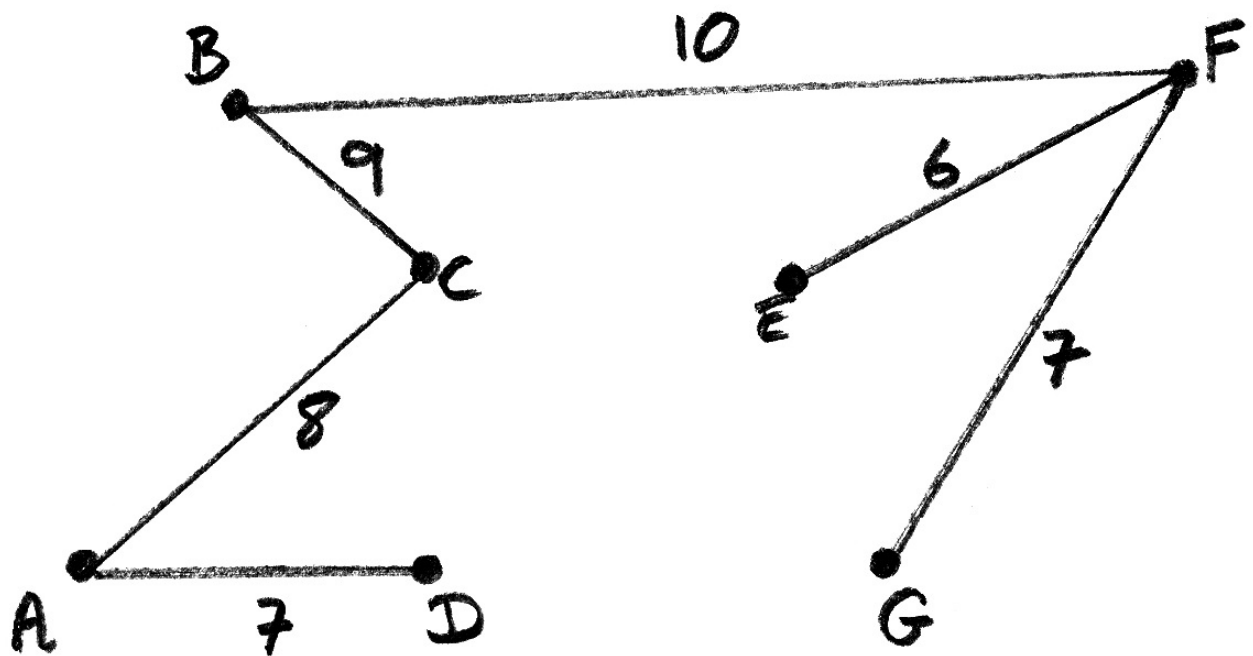
6th edge AB (weight 8)   *A1*

A1

Notes: Weights are not required on the diagram.

Allow *A2(d)* if the (correct) edges are in the wrong order *e.g.* they have used Prim's rather than Kruskal's algorithm.

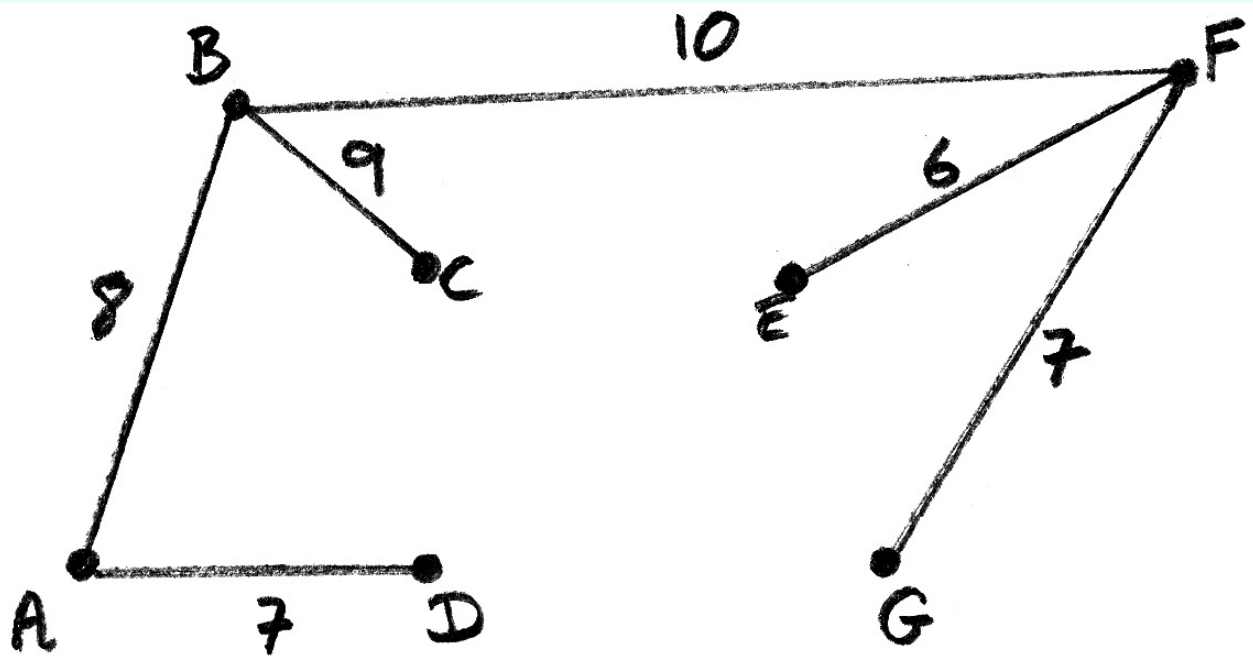(ii)   total weight is $1 + 2 + 3 + 6 + 7 + 8 = 27$ *A1*

[8 marks]

(b)   **EITHER**

A3

OR

A3

Notes: Award *A2* for five or four correct edges,

A1 for three or two correct edges

A0 otherwise.

Weights are not required on the diagram.

THEN

total weight is $6 + 7 + 7 + 8 + 9 + 10 = 47$ *A1*

[4 marks]

Total [12 marks]

Examiners report

Good algorithm work was shown; sometimes there were mistakes in giving the order of the edges chosen by, for example doing

Prim's algorithm instead of Kruskal's.

21.

Markscheme

(a)   **EITHER**

if p is a prime

$a^p \equiv a(\bmod p)$ *A1A1*

OR

if p is a prime and

$a \not\equiv$

$0(\bmod p)$ then

$a^{p-1} \equiv 1(\bmod p)$ *A1A1*

Note: Award *A1* for p being prime and *A1* for the congruence.

[2 marks]

(b)

$a_0 \equiv X(\bmod 7)$ *M1*

$X = k \times 5^6 + 25 + 15 + 5 - k$

by Fermat

$5^6 \equiv 1(\bmod 7)$ *R1*

$X = k + 45 - k(\bmod 7)$ *(M1)*

$X = 3(\bmod 7)$ *A1*

$a_0 = 3$ *A1*

[5 marks]

(c)

$X = 2 \times 5^6 + 25 + 15 + 3 = 31\,293$ *A1*

EITHER

$X - 7^5 = 14\,486$ *(M1)*

$X - 7^5 - 6 \times 7^4 = 80$

$X - 7^5 - 6 \times 7^4 - 7^2 = 31$

$X - 7^5 - 6 \times 7^4 - 7^2 - 4 \times 7 = 3$

$X = 7^5 + 6 \times 7^4 + 7^2 + 4 \times 7 + 3$ *(A1)*

$X = (160\,143)_7$ *A1*

OR

$31\,293 = 7 \times 4470 + 3$ *(M1)*

$$4470 = 7 \times 638 + 4$$

$$638 = 7 \times 91 + 1$$

$$91 = 7 \times 13 + 0$$

$$13 = 7 \times 1 + 6 \quad \textbf{\textit{(A1)}}$$

$$X = (160\,143)_7 \quad \textbf{\textit{A1}}$$

[4 marks]

Total [11 marks]

Examiners report

Fermat's little theorem was reasonably well known. Not all candidates took the hint to use this in the next part and this part was not done well. Part (c) could and was done even if part (b) was not.

22a.

Markscheme

as each edge contributes 1 to each of the vertices that it is incident with, each edge will contribute 2 to the sum of the degrees of all the vertices   *(R1)*

so

$2e = \sum \text{degrees}$ *(A1)*

$2e = \frac{n(n+1)}{2}$ *A1*

$4 \mid n(n+1)$ *A1*

n and $n+1$ are coprime   *R1*

Note: Accept equivalent reasoning *e.g.* only one of n and $n+1$ can be even.

$4 \mid n$ or $4 \mid n+1$ *A1*

$n \equiv 0(\bmod\,4)$ or $n \equiv 3(\bmod\,4)$ *AG*

[6 marks]

Examiners report

Only the top candidates were able to produce logically, well thought-out proofs.

22b.

Markscheme

since G is simple, the highest degree that a vertex can have is $n-1$ **R1**

the degrees of the vertices must belong to the set

$S = \{0, 1, 2, \ldots, n-1\}$ **A1**

proof by contradiction

if no two vertices have the same degree, all n vertices must have different degrees **R1**

as there are only n different degrees in set S, the degrees must be precisely the n numbers 0, 1, 2, ..., $n-1$ **R1**

let the vertex with degree 0 be A, then A is not adjacent to any of the other vertices **R1**

let the vertex with degree $n-1$ be B, then B is adjacent to all of the other vertices including A **R1R1**

this is our desired contradiction, so there must be two vertices of the same degree **R1**

[8 marks]

Examiners report

Only the top candidates were able to produce logically, well thought-out proofs.

23.

Markscheme

$7854 = 2 \times 3315 + 1224$ **M1A1**

$3315 = 2 \times 1224 + 867$ **A1**

$1224 = 1 \times 867 + 357$

$867 = 2 \times 357 + 153$

$357 = 2 \times 153 + 51$

$153 = 3 \times 51$ **A1**

The gcd is 51. **A1**

Since 51 does not divide 41, **R1**

there are no solutions. **A1**

[7 marks]

Examiners report

Most candidates were able to use the Euclidean Algorithm correctly to find the greatest common divisor. Candidates who used the GCD button on their calculators were given no credit. Some candidates seemed unaware of the criterion for the solvability of Diophantine equations.

24a.

Markscheme

$x \equiv y \,(\operatorname{mod} n) \Rightarrow x = y + kn, \ (k \in \mathbb{Z})$ **A1**

[1 mark]

24b.

Markscheme

$x \equiv y(\bmod n)$

$\Rightarrow x = y + kn$ *M1*

$x^2 = y^2 + 2kny + k^2 n^2$ *A1*

$\Rightarrow x^2 = y^2 + (2ky + k^2 n)n$ *M1A1*

$\Rightarrow x^2 \equiv y^2(\bmod n)$ *AG*

[4 marks]

Examiners report

While most candidates gave a correct meaning to

$x \equiv y(\bmod n)$, there were some incorrect statements, the most common being

$x \equiv y(\bmod n)$ means that when x is divided by n, there is a remainder y. The true statement

$8 \equiv 5(\bmod 3)$ shows that this statement is incorrect. Part (b) was solved successfully by many candidates but (c) caused problems for some candidates who thought that the result in (c) followed automatically from the result in (b).

24c.

Markscheme

EITHER

$x^2 \equiv y^2(\bmod n)$

$\Rightarrow x^2 - y^2 = 0(\bmod n)$ *M1*

$\Rightarrow (x - y)(x + y) = 0(\bmod n)$ *A1*

This will be the case if

$x + y = 0(\bmod n)$ or $x = -y(\bmod n)$ *R1*

so

$x \neq y(\bmod n)$ in general   *R1*

[4 marks]

OR

Any counter example, *e.g.*

$n = 5, x = 3, y = 2$, in which case   *R2*

$x^2 \equiv y^2(\bmod n)$ but

$x \neq y(\bmod n)$. (false)   *R1R1*

[4 marks]

25a. [5 marks]

Markscheme

$N = a_n \times 2^n + a_{n-1} \times 2^{n-1}+\ldots+a_1 \times 2 + a_0$ *M1*

If

$a_0 = 0$, then N is even because all the terms are even.   *R1*

Now consider

$a_0 = N - \sum_{r=1}^{n} a_r \times 2^r$ *M1*

If N is even, then

a_0 is the difference of two even numbers and is therefore even.   *R1*

It must be zero since that is the only even digit in binary arithmetic.   *R1*

[5 marks]

Examiners report

The response to this question was disappointing. Many candidates were successful in showing the 'if' parts of (a) and (b) but failed

even to realise that they had to continue to prove the 'only if' parts.

25b. [6 marks]

Markscheme

$N = a_n \times 3^n + a_{n-1} \times 3^{n-1}+\ldots+a_1 \times 3 + a_0$
$= a_n \times (3^n - 1) + a_{n-1} \times (3^{n-1} - 1)+\ldots+a_1 \times (3 - 1) + a_n + a_{n-1}+\ldots+a_1 + a_0$ *M1A1*

Since

3^n is odd for all

$n \in \mathbb{Z}^+$, it follows that

$3^n - 1$ is even.   *R1*

Therefore if the sum of the digits is even, N is the sum of even numbers and is even.   *R1*

Now consider

$a_n + a_{n-1}+\ldots+a_1 + a_0 = N - \sum_{r=1}^{n} a_r(3^r - 1)$ *M1*

If N is even, then the sum of the digits is the difference of even numbers and is therefore even.   *R1*
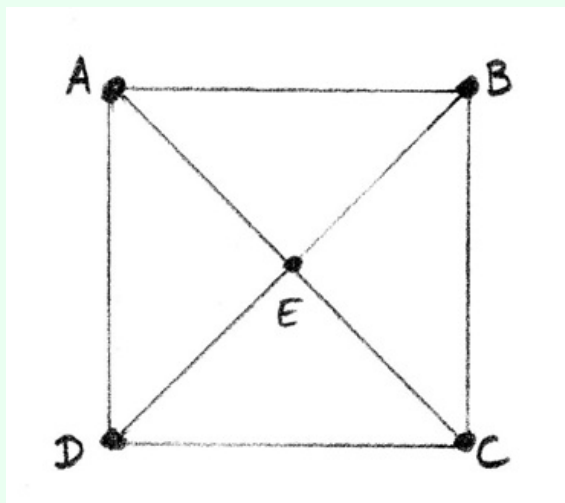
[6 marks]

Examiners report

The response to this question was disappointing. Many candidates were successful in showing the 'if' parts of (a) and (b) but failed

even to realise that they had to continue to prove the 'only if' parts.

26a.

Markscheme

A2

[2 marks]

Examiners report

A fairly common error in (a) was to draw a non-planar version of G, for which no credit was given. In (b), most candidates realised that only one extra edge could be added but a convincing justification was often not provided. Most candidates were reasonably successful in (c) although in some cases not all possible Hamiltonian cycles were stated.

26b.

Markscheme

For a simple planar graph containing triangles,

$e \leqslant 3v - 6$ *M1*

Here

$v = 5$, so $e \leqslant 9$.   *A1*

There are already 8 edges so the maximum number of edges that could be added is 1.   *R1*

This can be done *e.g.* AC or BD   *R1*

[4 marks]

Examiners report

A fairly common error in (a) was to draw a non-planar version of G, for which no credit was given. In (b), most candidates realised that only one extra edge could be added but a convincing justification was often not provided. Most candidates were reasonably successful in (c) although in some cases not all possible Hamiltonian cycles were stated.

26c.

Markscheme

The distinct Hamiltonian cycles are

ABCDEA   *A2*

ABCEDA   *A2*

ABECDA   *A2*

AEBCDA   *A2*

Note: Do not penalise extra cycles.

The weights are 32, 32, 29, 28 respectively.   *A1*

The Hamiltonian cycle of least weight is AEBCDA.   *R1*

[10 marks]

Examiners report

A fairly common error in (a) was to draw a non-planar version of *G*, for which no credit was given. In (b), most candidates realised that only one extra edge could be added but a convincing justification was often not provided. Most candidates were reasonably successful in (c) although in some cases not all possible Hamiltonian cycles were stated.

27a.

Markscheme

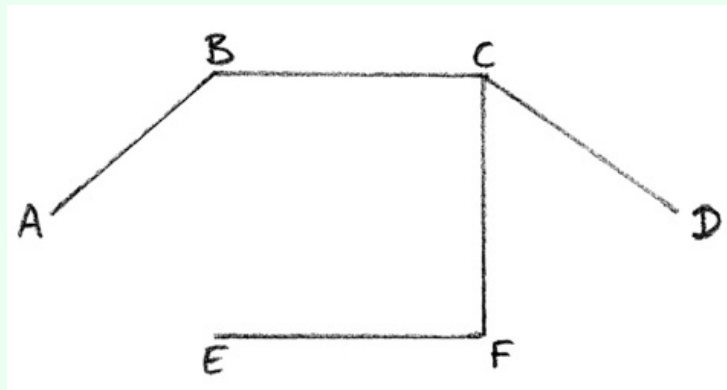The edges are included in the order

CF   *A1*

EF   *A1*

BC   *A1*

CD   *A1*

AB   *A1*

A1

[6 marks]

Examiners report

In (a), many candidates derived the minimum spanning tree although in some cases the method was not clearly indicated as required and some candidates used an incorrect algorithm. Part (b) was reasonably answered by many candidates although some justifications were unsatisfactory. Part (c) caused problems for many candidates who found difficulty in writing down a rigorous proof of the required result.

27b.   [5 marks]

Markscheme

(i)   A tree with v vertices has $v-1$ edges.   *A1*

Using $v + f = e + 2$ with $f = 1$, the result follows.   *R1*

(ii)   Each of the c trees will have one less edge than the number of vertices.   *R1*

Thus the forest will have $v - c$ edges.   *A2*

[5 marks]

Examiners report

In (a), many candidates derived the minimum spanning tree although in some cases the method was not clearly indicated as required and some candidates used an incorrect algorithm. Part (b) was reasonably answered by many candidates although some justifications were unsatisfactory. Part (c) caused problems for many candidates who found difficulty in writing down a rigorous proof of the required result.

28.

Markscheme

(a)

$324 = 2 \times 129 + 66$ ***M1***

$129 = 1 \times 66 + 63$

$66 = 1 \times 63 + 3$ ***A1***

hence gcd $(324, 129) = 3$ ***A1***

[3 marks]

(b)   **METHOD 1**
Since
$3 \mid 12$ the equation has a solution   ***M1***
$3 = 1 \times 66 - 1 \times 63$ ***M1***
$3 = -1 \times 129 + 2 \times 66$
$3 = 2 \times (324 - 2 \times 129) - 129$
$3 = 2 \times 324 - 5 \times 129$ ***A1***
$12 = 8 \times 324 - 20 \times 129$ ***A1***
$(x, y) = (8, -20)$ is a particular solution   ***A1***
Note: A calculator solution may gain ***M1M1A0A0A1***.

A general solution is
$x = 8 + \frac{129}{3}t = 8 + 43t,\ y = -20 - 108t,\ t \in \mathbb{Z}$ ***A1***
METHOD 2
$324x + 129y = 12$
$108x + 43y = 4$ ***A1***
$108x \equiv 4(\bmod\ 43) \Rightarrow 27x \equiv 1(\bmod\ 43)$ ***A1***
$x = 8 + 43t$ ***A1***
$108(8 + 43t) + 43y = 4$ ***M1***
$864 + 4644t + 43y = 4$
$43y = -860 - 4644t$
$y = -20 - 108t$ ***A1***
a particular solution (for example
$t = 0$) is
$(x, y) = (8, -20)$ ***A1***
[6 marks]

(c)   **EITHER**
The left side is even and the right side is odd so there are no solutions   ***M1R1AG***
[2 marks]
OR
gcd$(82, 140) = 2$ ***A1***
2 does not divide 3 therefore no solutions   ***R1AG***
[2 marks]
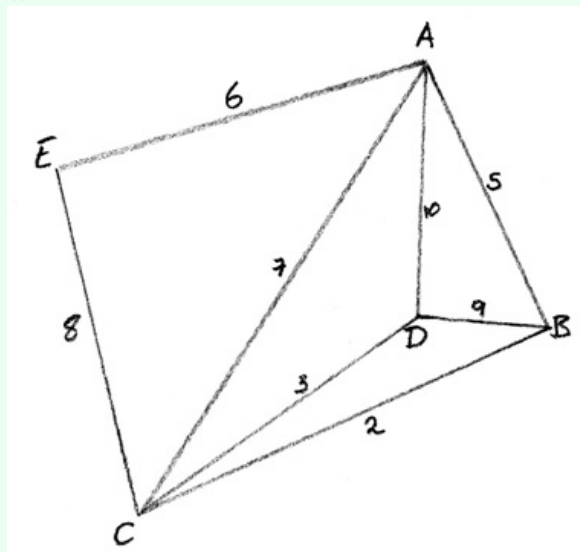
Total [11 marks]

Examiners report

This problem was not difficult but presenting a clear solution and doing part (b) alongside part (a) in two columns was. The simple answer to part (c) was often overlooked.
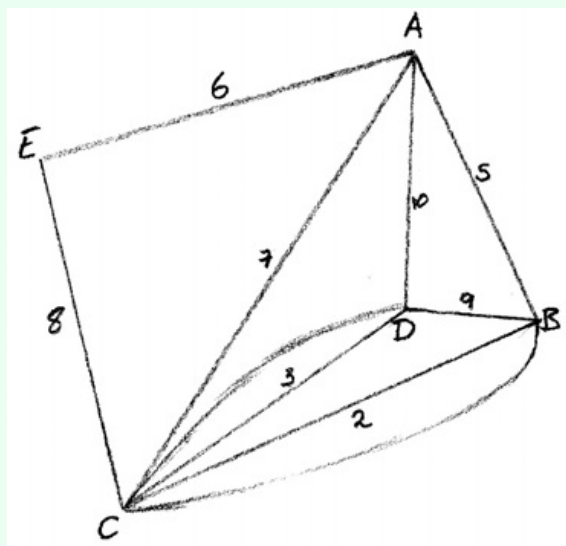
Markscheme

(i)

A1A1A1

Note: Award *A1* for the vertices, *A1* for edges and *A1* for planar form.

(ii)   It is possible to find an Eulerian trail in this graph since exactly two of the vertices have odd degree   *R1*

(iii)   B and D are the odd vertices   *M1*

$BC + CD = 3 + 2 = 5$ and $BD = 9$,   *A1*

since $5 < 9$, BC and CD must be traversed twice   *R1*

A possible walk by inspection is ACBDABCDCEA   *A1*

This gives a total length of

$2(2 + 3) + 8 + 9 + 5 + 7 + 10 + 6 = 55$ for the walk   *A1*
[9 marks]

Examiners report

Drawing the graph usually presented no difficulty. Distinguishing between Eulerian and semi-Eulerian needs attention but this part was usually done successfully.

A simple, clear argument for part (c) was often hidden in mini-essays on graph theory.

29b.

Markscheme

The sum of all the vertex degrees is twice the number of edges, *i.e.* an even number.

Hence a graph cannot have exactly one vertex of odd degree.   *M1R1*

[2 marks]

Examiners report

Drawing the graph usually presented no difficulty. Distinguishing between Eulerian and semi-Eulerian needs attention but this part was usually done successfully.

A simple, clear argument for part (c) was often hidden in mini-essays on graph theory.

30a.

Markscheme

(i)

$a \equiv d(\bmod n)$ and $b \equiv c(\bmod n)$

so

$a - d = pn$ and $b - c = qn$,   *M1A1*

$a - d + b - c = pn + qn$

$(a + b) - (c + d) = n(p + q)$ *A1*

$(a + b) \equiv (c + d)(\bmod n)$ *AG*

(ii)
$$\begin{cases} 2x + 5y \equiv 1(\bmod 6) \\ x + y \equiv 5(\bmod 6) \end{cases}$$
adding

$3x + 6y \equiv 0(\bmod 6)$ *M1*

$6y \equiv 0(\bmod 6)$ so $3x \equiv 0(\bmod 6)$ *R1*

$x \equiv 0$ or $x \equiv 2$ or $x \equiv 4(\bmod 6)$ *A1A1A1*

for

$x \equiv 0,\ 0 + y \equiv 5(\bmod 6)$ so $y \equiv 5(\bmod 6)$ *A1*

for

$x \equiv 2,\ 2 + y \equiv 5(\bmod 6)$ so $y \equiv 3(\bmod 6)$ *A1*

If

$x \equiv 4(\bmod 6),\ 4 + y \equiv 5(\bmod 6)$ so $y \equiv 1(\bmod 6)$ *A1*

[11 marks]

Examiners report

Part (a) (i) was not found difficult but using it in part (a)(ii) resulted in two or three correct lines and then abandonment of the problem.

30b.

Markscheme

Suppose x is a solution

97 is prime so

$x^{97} \equiv x (\bmod\ 97)$ *M1*

$x^{97} - x \equiv 0 (\bmod\ 97)$ *A1*

$x^{97} - x + 1 \equiv 1 \neq 0 (\bmod\ 97)$

Hence there are no solutions   *R1*

[3 marks]

Examiners report

Part (a) (i) was not found difficult but using it in part (a)(ii) resulted in two or three correct lines and then abandonment of the problem.

31a.

Markscheme

At least two of the three vertices in the triangle must lie on one of the two disjoint sets   *M1R1*

These two are joined by an edge so the graph cannot be bipartite   *R1*

[3 marks]

Examiners report

Part (a) was usually done correctly but then clear argument for parts (b) and (c) were rare.

31b.

Markscheme

If there are x vertices in one of the two disjoint sets then there are $(n - x)$ vertices in the other disjoint set   *M1*

The greatest number of edges occurs when all vertices in one set are joined to all vertices in the other to give $x(n - x)$ edges   *A1*

Function $f(x) = x(n - x)$ has a parabolic graph.   *M1*

This graph has a unique maximum at

$\left(\frac{n}{2}, \frac{n^2}{4} \right)$.   *A1*

so

$x(n - x) \leqslant \frac{n^2}{4}$ *R1*

[5 marks]

Examiners report

Part (a) was usually done correctly but then clear argument for parts (b) and (c) were rare.

32.

Markscheme

(a) (i)   A spanning tree with v vertices and $(v-1)$ edges where $f = 1$ **A1A1**

$f - e + v = 1 - (v - 1) + v = 2$ **M1**

So the formula is true for the tree   **AG**

(ii)   Adding one edge connects two different vertices, and hence an extra face is created   **M1R1**

This leaves v unchanged but increases both e and f by 1 leaving $f - e + v$ unchanged. Hence $f - e + v = 2$.   **R1R1**

[7 marks]

(b)   Using

$e \leqslant 3v - 6$,   **M1**

for

K_5, $v = 5$ and

$e = \binom{5}{2} = 10$ **A1A1**

but $3v - 6 = 3(5) - 6 = 9$ **A1**

9 is not greater or equal to 10 so

K_5 is not planar   **R1**

[5 marks]

Total [12 marks]

Examiners report

Part (a)(i) was done successfully but many students did not read part(ii) carefully. It said **'adding an edge'** nothing else. Many candidates assumed it was necessary to add a vertex when this was not the case.

Part (b) was not found to be beyond many candidates if they used the inequality

$e \leqslant 3v - 6$

33a.

Markscheme

the relevant powers of 16 are 16, 256 and 4096

then

$51966 = 12 \times 4096$ remainder 2814   **M1A1**

$2814 = 10 \times 256$ remainder 254

$254 = 15 \times 16$ remainder 14   **A1**

the hexadecimal number is CAFE   **A1**

Note: CAFE is produced using a standard notation, accept explained alternative notations.

[4 marks]

33b.   *[10 marks]*

Markscheme

(i)   using the Euclidean Algorithm   *(M1)*

$901 = 612 + 289$ *(A1)*

$612 = 2 \times 289 + 34$

$289 = 8 \times 34 + 17$

$\gcd(901, 612) = 17$ *A1*

(ii)   working backwards   *(M1)*

$17 = 289 - 8 \times 34$

$= 289 - 8 \times (612 - 2 \times 289)$

$= 17 \times (901 - 612) - 8 \times 612$

$= 27 \times 901 - 25 \times 612$

so $p = 17$, $q = -25$ *A1A1*

(iii)   a particular solution is

$s = 5p = 85$, $t = -5q = 125$ *(A1)*

the general solution is

$s = 85 + 36\lambda$, $t = 125 + 53\lambda$ *M1A1*

by inspection the solution satisfying all conditions is

$(\lambda = -2)$, $s = 13$, $t = 19$ *A1*

[10 marks]

33c.   *[5 marks]*

Markscheme

(i)   the congruence is equivalent to

$9x = 3 + 18\lambda$ *(A1)*

this has no solutions as 9 does not divide the RHS   *R1*

(ii)   the congruence is equivalent to

$3x = 1 + 5\lambda$, $(3x \equiv 1 \,(\bmod 5))$ *A1*

one solution is

$x = 2$, so the general solution is

$x = 2 + 5n$ $(x \equiv 2 \,(\bmod 5))$ *M1A1*

[5 marks]

34a.

Markscheme

Kruskal's algorithm gives the following edges

CD   (4)   *M1A1*
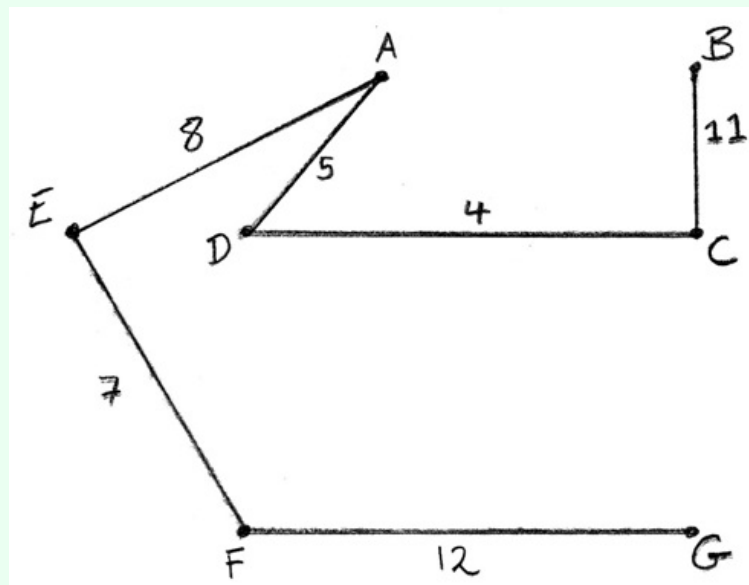
AD   (5)

EF   (7)   *A1*

EA   (8)

BC   (11)

FG   (12)   *A1   N0*

length of the spanning tree is 47   *A1*

[5 marks]

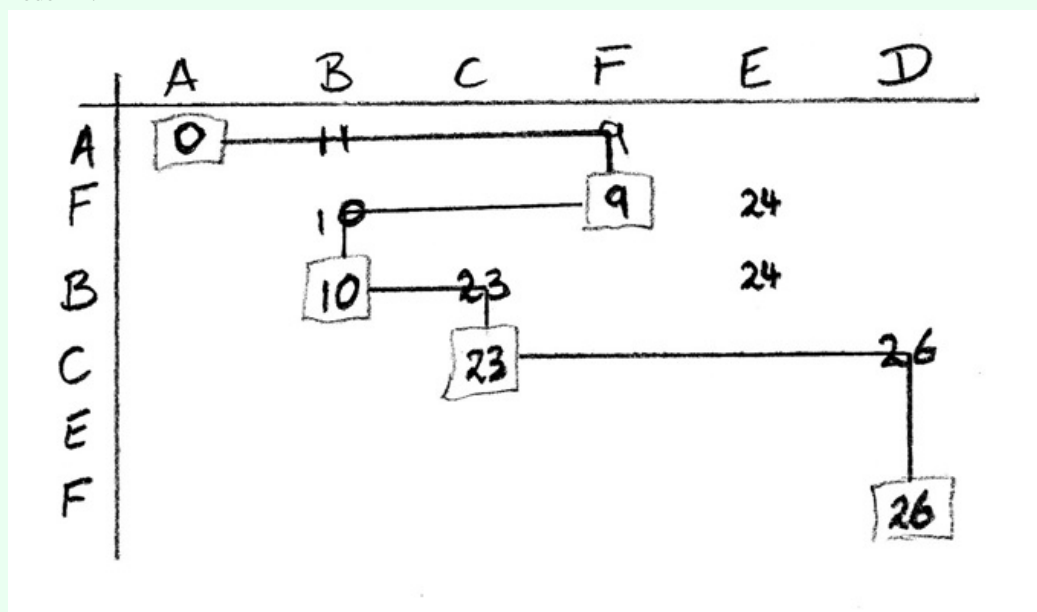34b.   [7 marks]

Markscheme

for Dijkstra's algorithm there are three things associated with a node: order; distance from the initial node as a permanent or temporary node   *M1*

A4

Note: Deduct *A1* for each error or omission.

the shortest path is AFBCD   *A1*

the length is 26   *A1*   *N0*

[7 marks]

Examiners report

Setting out clearly the steps of the algorithms is still a problem for many although getting the correct spanning tree and its length were not.

35a.   [4 marks]

Markscheme

$457\,128 = 2 \times 228\,564$
$228\,564 = 2 \times 114\,282$
$114\,282 = 2 \times 57\,141$
$57\,141 = 3 \times 19\,047$
$19\,047 = 3 \times 6349$
$6349 = 7 \times 907$ *M1A1*
trial division by 11, 13, 17, 19, 23 and 29 shows that 907 is prime   *R1*
therefore
$457\,128 = 2^3 \times 3^2 \times 7 \times 907$ *A1*
[4 marks]

Examiners report

Some candidates were obviously not sure what was meant by 'product of primes' which surprised the examiner.

35b.

Markscheme

by a corollary to Fermat's Last Theorem

$5^{11} \equiv 5(\bmod 11)$ and $17^{11} \equiv 17(\bmod 11)$ *M1A1*

$5^{11} + 17^{11} \equiv 5 + 17 \equiv 0(\bmod 11)$ *A1*

this combined with the evenness of LHS implies

$25 | 5^{11} + 17^{11}$ *R1AG*

[4 marks]

Examiners report

There were some reasonable attempts at part (c) using powers rather than Fermat's little theorem.

by a corollary to Fermat's Last Theorem

$5^{11} \equiv 5(\bmod 11)$ and $17^{11} \equiv 17(\bmod 11)$ *M1A1*

$5^{11} + 17^{11} \equiv 5 + 17 \equiv 0(\bmod 11)$ *A1*

36.
Markscheme

(a)　(i)　Euler's relation is

$e = v - 2 + f \geqslant v - 1$, as $f \geqslant 1$　**M1A1**

(ii)

G is a tree $\Leftrightarrow$ no cycles $\Leftrightarrow$ $f = 1$　**R1R1**

[4 marks]

(b)　the result from (a) (ii) gives

$e = k + 2 + 1 + 1 - 1 = k + 3$　**M1A1**

for a tree we also have

$2e =$ sum of degrees　**M1**

$2k + 6 = k + 4 + 3 + 4 = k + 11$

hence

$k = 5$　**A1**
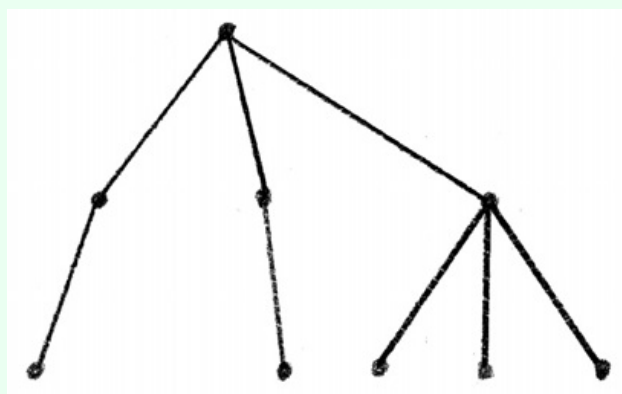
A2

Note: Accept alternative correct solutions.

[6 marks]

(c)　(i)

$v - 1 = 5 < 6 = e$ by (a) (ii)　**M1A1**

G cannot be a tree　**AG**

(ii)

A1
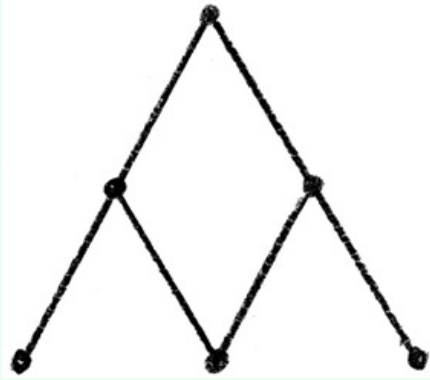
[3 marks]

(d)   take any vertex in the tree and colour it black   **M1**

colour all adjacent vertices white

colour all vertices adjacent to a white vertex black

continue this procedure until all vertices are coloured   **M1**

which must happen since the graph is connected   **R1**

as the tree contains no cycles, no vertex can be both black and white and the graph is proved to be bipartite   **R1**

[4 marks]

Total [17 marks]

Examiners report

Many candidates seem only to have a weak understanding of the requirements for the proof of a mathematical statement.

37.

Markscheme

let x be the number of guests

$x \equiv 1 (\bmod\ 2)$

$x \equiv 1 (\bmod\ 3)$

$x \equiv 1 (\bmod\ 4)$

$x \equiv 1 (\bmod\ 5)$

$x \equiv 1 (\bmod\ 6)$

$x \equiv 0 (\bmod\ 7)$ congruence (i)   *(M1)(A2)*

the equivalent of the first five lines is

$x \equiv 1 (\bmod\ (\text{lcm of } 2, 3, 4, 5, 6)) \equiv 1 (\bmod\ 60)$ *A1*

$\Rightarrow x = 60t + 1$

from congruence (i)

$60t + 1 \equiv 0 (\bmod\ 7)$ *M1A1*

$60t \equiv -1 (\bmod\ 7)$

$60t \equiv 6 (\bmod\ 7)$

$4t \equiv 6 (\bmod\ 7)$

$2t \equiv 3 (\bmod\ 7)$ *A1*

$\Rightarrow t = 7u + 5$ (or equivalent)   *A1*

hence

$x = 420u + 300 + 1$ *A1*

$\Rightarrow x = 420u + 301$

smallest number of guests is 301   *A1*   *N6*

Note: Accept alternative correct solutions including exhaustion or formula from Chinese remainder theorem.

[10 marks]

Examiners report

There were a number of totally correct solutions to this question, but many students were unable to fully justify the result. Some candidates had learnt a formula to apply to the Chinese remainder theorem, but could not apply it well in this situation. Many worked with the conditions for divisibility but did not make much progress with the justification.

Markscheme

(a) using Fermat's little theorem

$n^5 \equiv n(\bmod 5)$ *(M1)*

$n^5 - n \equiv 0(\bmod 5)$ *A1*

now

$n^5 - n = n(n^4 - 1)$ *(M1)*

$= n(n^2 - 1)(n^2 + 1)$

$= n(n - 1)(n + 1)(n^2 + 1)$ *A1*

hence one of the first two factors must be even *R1*

i.e.

$n^5 - n \equiv 0(\bmod 2)$

thus

$n^5 - n$ is divisible by 5 and 2

hence it is divisible by 10 *R1*

in base 10, since

$n^5 - n$ is divisible by 10, then

$n^5 - n$ must end in zero and hence

n^5 and n must end with the same digit *R1*

[7 marks]

(b) consider

$n^5 - n = n(n - 1)(n + 1)(n^2 + 1)$

this is divisible by 3 since the first three factors are consecutive integers *R1*

hence

$n^5 - n$ is divisible by 3, 5 and 2 and therefore divisible by 30

in base 30, since

$n^5 - n$ is divisible by 30, then

$n^5 - n$ must end in zero and hence

n^5 and n must end with the same digit *R1*

[2 marks]

Total [9 marks]

Examiners report

There were very few fully correct answers. If Fermat"s little theorem was known, it was not well applied.

Markscheme

$581 = 2 \times 259 + 63$ **M1A1**

$259 = 4 \times 63 + 7$ **A1**

$63 = 9 \times 7$

the GCD is therefore 7   **A1**

[4 marks]

Examiners report

[N/A]

Markscheme

consider

$7 = 259 - 4 \times 63$ **M1**

$= 259 - 4 \times (581 - 2 \times 259)$ **A1**

$= 259 \times 9 + 581 \times (-4)$ **A1**

the general solution is therefore

$x = 9 + 83n;\ y = -4 - 37n$ where $n \in \mathbb{Z}$ **M1A1**

Notes: Accept solutions laid out in tabular form. Dividing the diophantine equation by 7 is an equally valid method.
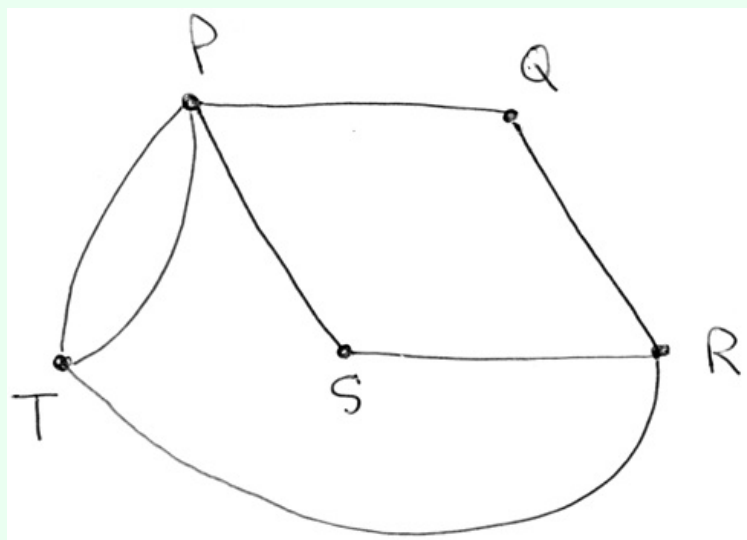
[5 marks]

Examiners report

[N/A]

Markscheme

A2

[2 marks]

40b.   *[4 marks]*

Markscheme

(i)   *G* is not simple because 2 edges join P to T   ***R1***

(ii)   *G* is connected because there is a path joining every pair of vertices   ***R1***

(iii)   (P, R) and (Q, S, T) are disjoint vertices   ***R1***

so *G* is bipartite   ***A1***

Note: Award the ***A1*** only if the ***R1*** is awarded.

[4 marks]

40c.   *[2 marks]*

Markscheme

G has an Eulerian trail because it has two vertices of odd degree (R and T have degree 3), all the other vertices having even degree

R1

the following example is such a trail

TPTRSPQR   ***A1***

[2 marks]

40d.   *[1 mark]*

Markscheme

G has no Eulerian circuit because there are 2 vertices which have odd degree   ***R1***
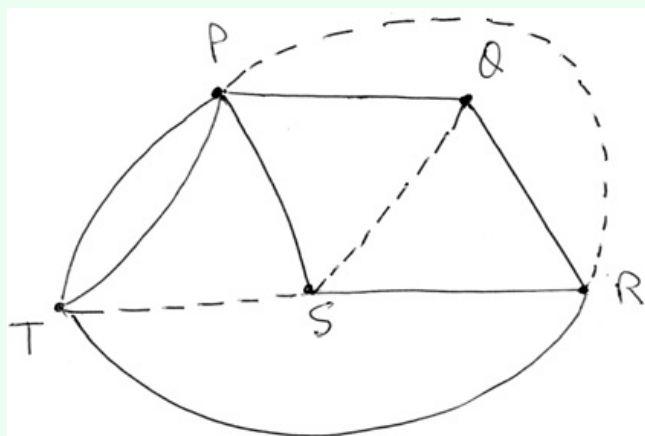
[1 mark]

40e. [4 marks]

Markscheme

consider

so it is possible to add 3 extra edges   *A1*

consider G with one of the edges PT deleted; this is a simple graph with 6 edges; on addition of the new edges, it will still be simple

   M1

$e \leqslant 3v - 6 \Rightarrow e \leqslant 3 \times 5 - 6 = 9$ *R1*

so at most 3 edges can be added   *R1*

[4 marks]

Examiners report

[N/A]

41a. [6 marks]

Markscheme

(i)

$2^8 = 256 \equiv 4(\bmod\ 9)$ (so not true)   *A1*

9 is not prime   *A1*

(ii)   consider various powers of 2, *e.g.* obtaining   *M1*

$2^6 = 64 \equiv 1(\bmod\ 9)$ *A1*

therefore

$2^{45} = (2^6)^7 \times 2^3$ *M1*

$\equiv 8(\bmod\ 9)$ (so $k = 8$)   *A1*

[6 marks]

Examiners report

[N/A]

41b. [6 marks]

Markscheme

EITHER

the solutions to

$3x \equiv 4 (\bmod 5)$ are 3, 8, 13, 18, 23,... *M1A1*

the solutions to

$5x \equiv 6 (\bmod 7)$ are 4, 11, 18,... *A1*

18 is therefore the smallest solution *A1*

the general solution is

$18 + 35n$, $n \in \mathbb{Z}$ *M1*

the required solutions are therefore 123, 158, 193 *A1*

OR

$3x \equiv 4 (\bmod 5) \Rightarrow 2 \times 3x \equiv 2 \times 4 (\bmod 5) \Rightarrow x \equiv 3 (\bmod 5)$ *A1*

$\Rightarrow x = 3 + 5t$ *M1*

$\Rightarrow 15 + 25t \equiv 6 (\bmod 7) \Rightarrow 4t \equiv 5 (\bmod 7) \Rightarrow 2 \times 4t \equiv 2 \times 5 (\bmod 7) \Rightarrow t \equiv 3 (\bmod 7)$ *A1*

$\Rightarrow t = 3 + 7n$ *A1*

$\Rightarrow x = 3 + 5(3 + 7n) = 18 + 35n$ *M1*

the required solutions are therefore 123, 158, 193 *A1*

OR

using the Chinese remainder theorem formula method

first convert the congruences to

$x \equiv 3 (\bmod 5)$ and

$x \equiv 4 (\bmod 7)$ *A1A1*

$M = 35, M_1 = 7, M_2 = 5, m_1 = 5, m_2 = 7, a_1 = 3, a_2 = 4$

x_1 is the solution of

$M_2 x_2 \equiv 1 (\bmod m_1)$, *i.e.*

$7x_1 \equiv 1 (\bmod 5)$ so

$x_1 = 3$

x_2 is the solution of

$M_2 x_2 \equiv 1 (\bmod m_2)$, *i.e.*

$5x_2 \equiv 1 (\bmod 7)$ so

$x_2 = 3$

a solution is therefore

$x = a_1 M_1 x_1 + a_2 M_2 x_2$ *M1*

$= 3 \times 7 \times 3 + 4 \times 5 \times 3 = 123$ *A1*

the general solution is

$123 + 35n$,

$n \in \mathbb{Z}$ *M1*

the required solutions are therefore 123, 158, 193 *A1*

[6 marks]

Examiners report

[N/A]

42a.

Markscheme

using the nearest neighbour algorithm, starting with A,

A → E, E → C   *A1*

C → D, D → B   *A1*

B → A   *A1*

the upper bound is therefore 9 + 10 + 16 + 13 + 11 = 59   *A1*

[4 marks]

Examiners report

[N/A]

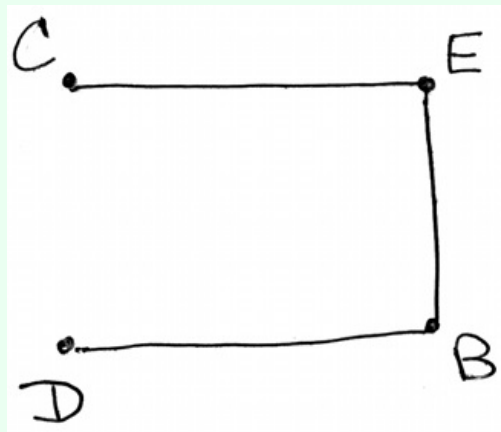42b.

Markscheme

(i)   the edges are added in the order CE   *A1*

BD   *A1*

BE   *A1*

A1

(ii)   the weight of the minimum spanning tree is 37   *(A1)*

we now reconnect A with the 2 edges of least weight   *(M1)*

i.e. AE and AB   *A1*

the lower bound is therefore 37 + 9 + 11 = 57   *A1*

[8 marks]

Examiners report

[N/A]

43a.

Markscheme

the auxiliary equation is

$m^2 - 5m + 6 = 0$ *M1*

giving

$m = 2, 3$ *A1*

the general solution is

$u_n = A \times 2^n + B \times 3^n$ *A1*

substituting $n = 1, 2$

$2A + 3B = 3$ *M1*

$4A + 9B = 3$ *A1*

the solution is $A = 3$, B = –1 giving

$u_n = 3 \times 2^n - 3^n$ *A1*

[6 marks]

Examiners report

[N/A]

43b.

Markscheme

we first prove that

$v_n = 2^n (2n - 1)$ for $n = 1, 2$ *M1*

for $n = 1$, it gives

$2 \times 1 = 2$ which is correct

for $n = 2$, it gives

$4 \times 3 = 12$ which is correct   *A1*

we now assume that the result is true for

$n \leqslant k$ *M1*

consider

$v_{k+1} = 4v_k - 4v_{k-1} \ (k \geqslant 2)$ *M1*

$= 4.2^k(2k - 1) - 4.2^{k-1}(2k - 3)$ *A1*

$= 2^{k+1}(4k - 2 - 2k + 3)$ *A1*

$= 2^{k+1}(2(k+1) - 1)$ *A1*

this proves that if the result is true for

$n \leqslant k$ then it is true for

$n \leqslant k + 1$

since we have also proved it true for

$n \leqslant 2$, the general result is proved by induction   *R1*

Note: A reasonable attempt has to be made to the induction step for the final *R1* to be awarded.

[8 marks]

Examiners report

[N/A]