

Topic 4 Part 2 [222 marks]

Markscheme

(a) (i) **METHOD 1**

$$\overrightarrow{AB} = b - a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (A1)$$

$$\overrightarrow{AC} = c - a = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad (A1)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} \quad M1$$

$$= i(-1 + 1) - j(0 - 2) + k(0 - 2) \quad (A1)$$

$$= 2j - 2k \quad A1$$

Area of triangle ABC } }

$$= \frac{1}{2}|2j - 2k| = \frac{1}{2}\sqrt{8}$$

$$(= \sqrt{2}) \text{ sq. units} \quad M1A1$$

Note: Allow *FT* on final *A1*.

METHOD 2

$$|AB| = \sqrt{2}, |BC| = \sqrt{12}, |AC| = \sqrt{6} \quad A1A1A1$$

Using cosine rule, e.g. on

$$\hat{C} \quad M1$$

$$\cos C = \frac{6+12-2}{2\sqrt{12}} = \frac{2\sqrt{2}}{3} \quad A1$$

$$\therefore \text{Area } \triangle ABC = \frac{1}{2}ab \sin C \quad M1$$

$$= \frac{1}{2}\sqrt{12}\sqrt{6} \sin\left(\arccos \frac{2\sqrt{2}}{3}\right)$$

$$= 3\sqrt{2} \sin\left(\arccos \frac{2\sqrt{2}}{3}\right) (= \sqrt{2}) \quad A1$$

Note: Allow *FT* on final *A1*.

(ii)

$$AB = \sqrt{2} \quad A1$$

$$\sqrt{2} = \frac{1}{2}AB \times h = \frac{1}{2}\sqrt{2} \times h, h \text{ equals the shortest distance} \quad (M1)$$

$$\Rightarrow h = 2 \quad A1$$

(iii) **METHOD 1**

(\pi) has form

$$r \cdot \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = d \quad (M1)$$

Since (1, 1, 2) is on the plane

$$d = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2 - 4 = -2 \quad M1A1$$

Hence

$$r \cdot \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = -2$$

$$2y - 2z = -2 \text{ (or } y - z = -1) \quad A1$$

METHOD 2

$$r = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad (M1)$$

$$x = 1 + 2\mu \quad (i)$$

$$y = 1 + \lambda - \mu \quad (ii)$$

$$z = 2 + \lambda - \mu \quad (iii) \quad AI$$

Note: Award **AI** for all three correct, **A0** otherwise.

From (i)

$$\mu = \frac{x-1}{2}$$

substitute in (ii)

$$y = 1 + \lambda - \left(\frac{x-1}{2}\right)$$

$$\Rightarrow \lambda = y - 1 + \left(\frac{x-1}{2}\right)$$

substitute

λ and

$$\mu \text{ in (iii) } \quad MI$$

$$\Rightarrow z = 2 + y - 1 + \left(\frac{x-1}{2}\right) - \left(\frac{x-1}{2}\right)$$

$$\Rightarrow y - z = -1 \quad AI$$

[14 marks]

(b) (i) The equation of OD is

$$r = \lambda \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix},$$

$$\left(\text{or } r = \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) \quad MI$$

This meets

π where

$$2\lambda + 2\lambda = -1 \quad (M1)$$

$$\lambda = -\frac{1}{4} \quad AI$$

Coordinates of D are

$$\left(0, -\frac{1}{2}, \frac{1}{2}\right) \quad AI$$

(ii)

$$\left| \overrightarrow{OD} \right| = \sqrt{0 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} \quad (M1)AI$$

[6 marks]

Total [20 marks]

Examiners report

It was disappointing to see that a number of candidates did not appear to be well prepared for this question and made no progress at all. There were a number of schools where no candidate made any appreciable progress with the question. A good number of students, however, were successful with part (a) (i). A good number of candidates were also successful with part a (ii) but few realised that the shortest distance was the height of the triangle. Candidates used a variety of methods to answer (a) (iii) but again a reasonable number of correct answers were seen. Candidates also had a reasonable degree of success with part (b), with a respectable number of correct answers seen.

2.

[6 marks]

Markscheme

The normal vector to the plane is

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad (A1)$$

EITHER

θ is the angle between the line and the normal to the plane.

$$\cos \theta = \frac{\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{14}\sqrt{21}} = \frac{3}{\sqrt{14}\sqrt{21}} = \left(\frac{3}{7\sqrt{6}}\right) \quad (M1)(A1)(A1)$$

$$\Rightarrow \theta = 79.9^\circ (= 1.394\dots) \quad A1$$

The required angle is $10.1^\circ (= 0.176)$ **A1**

OR

ϕ is the angle between the line and the plane.

$$\sin \phi = \frac{\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{14}\sqrt{21}} = \frac{3}{\sqrt{14}\sqrt{21}} \quad (M1)(A1)(A1)$$

$$\phi = 10.1^\circ (= 0.176) \quad A2$$

[6 marks]

Examiners report

On the whole this question was well answered. Some candidates failed to find the complementary angle when using the formula with cosine.

3.

[6 marks]

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{6} & -\frac{1}{12} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$x + \frac{1}{6}\lambda = -\frac{1}{12}$$

$$y - \frac{2}{3}\lambda = -\frac{1}{6}$$

$$r = \left(-\frac{1}{12}i - \frac{1}{6}j\right) + \lambda\left(-\frac{1}{6}i + \frac{2}{3}j + k\right)$$

Examiners report

A large number of candidates did not use their GDC in this question. Some candidates who attempted analytical solutions looked for a point solution although the question specifically states that the planes intersect in a line. Other candidates eliminated one variable and then had no clear strategy for proceeding with the solution.

Some candidates failed to write ' $r =$ ', and others did not give the equation in vector form.

4.

[6 marks]

Markscheme

METHOD 1

Use of

$$|a$$

$$\times b$$

$$| =$$

$$|a$$

$$||b$$

$$|\sin \theta| \quad (M1)$$

$$|a$$

$$\times b$$

$$|^2 =$$

$$|a$$

$$|^2|b$$

$$|^2 \sin^2 \theta \quad (A1)$$

Note: Only one of the first two marks can be implied.

$$=$$

$$|a$$

$$|^2|b$$

$$|^2(1 - \cos^2 \theta) \quad A1$$

$$=$$

$$|a$$

$$|^2|b$$

$$|^2 -$$

$$|a$$

$$|^2|b$$

$$|^2 \cos^2 \theta \quad (A1)$$

$$=$$

$$|a$$

$$|^2|b$$

$$|^2 -$$

$$(|a$$

$$||b$$

$$|\cos \theta|^2 \quad (A1)$$

Note: Only one of the above two *A1* marks can be implied.

=

$|a$

$|^2|b$

$|^2 - (a$

$\cdot b)$

$^2 \quad AI$

Hence LHS = RHS $AG \quad NO$

[6 marks]

METHOD 2

Use of a

$\cdot b =$

$|a$

$||b$

$|\cos \theta \quad (M1)$

$|a$

$|^2|b$

$|^2 - (a$

$\cdot b)$

$^2 =$

$|a$

$|^2|b$

$|^2 -$

$(|a$

$||b$

$|\cos \theta)^2 \quad (A1)$

=

$|a$

$|^2|b$

$|^2 -$

$|a$

$|^2|b$

$|^2$

$\cos^2 \theta \quad (A1)$

Note: Only one of the above two $A1$ marks can be implied.

=

$$|a$$

$$|^2|b$$

$$|^2(1 - \cos^2\theta) \quad AI$$

=

$$|a$$

$$|^2|b$$

$$|^2\sin^2\theta \quad AI$$

=

$$|a$$

$$\times b$$

$$|^2 \quad AI$$

Hence LHS = RHS *AG NO*

Notes: Candidates who independently correctly simplify both sides and show that LHS = RHS should be awarded full marks.

If the candidate starts off with expression that they are trying to prove and concludes that

$\sin^2\theta = (1 - \cos^2\theta)$ award *MIAIAIAIA0A0*.

If the candidate uses two general 3D vectors and explicitly finds the expressions correctly award full marks. Use of 2D vectors gains a maximum of 2 marks.

If two specific vectors are used no marks are gained.

[6 marks]

Examiners report

Those candidates who chose to use the trigonometric version of Pythagoras' Theorem were generally successful, although a minority were unconvincing in their reasoning. Some candidates adopted a full component approach, but often seemed to lose track of what they were trying to prove. A few candidates used 2-dimensional vectors or specific rather than general vectors.

Markscheme

(a) Use of

$$\cos \theta = \frac{\vec{OA} \cdot \vec{AB}}{|\vec{OA}| |\vec{AB}|} \quad (M1)$$

$$\vec{AB} = i - j + k \quad A1$$

$$|\vec{AB}| = \sqrt{3} \text{ and}$$

$$|\vec{OA}| = 3\sqrt{2} \quad A1$$

$$\vec{OA} \cdot \vec{AB} = 6 \quad A1$$

substituting gives

$$\cos \theta = \frac{2}{\sqrt{6}} \left(= \frac{\sqrt{6}}{3} \right) \text{ or equivalent} \quad M1 \quad N1$$

[5 marks]

(b)

$$L_1 : r =$$

$$\vec{OA} + s\vec{AB} \text{ or equivalent} \quad (M1)$$

$$L_1 : r = i - j + 4k + s(i - j + k)$$

or equivalent $A1$

Note: Award (M1)A0 for omitting “ $r =$ ” in the final answer.

[2 marks]

(c) Equating components and forming equations involving s and t $(M1)$

$$1 + s = 2 + 2t, -1 - s = 4 + t, 4 + s = 7 + 3t$$

Having two of the above three equations $A1A1$

Attempting to solve for s or t $(M1)$

Finding either $s = -3$ or $t = -2$ $A1$

Explicitly showing that these values satisfy the third equation $R1$

Point of intersection is $(-2, 2, 1)$ $A1 \quad N1$

Note: Position vector is not acceptable for final $A1$.

[7 marks]

(d) **METHOD 1**

$$r = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} \quad (A1)$$

$$x = 1 + 2\lambda - 3\mu, y = -1 + \lambda + 3\mu \text{ and } z = 4 + 3\lambda - 3\mu \quad M1A1$$

Elimination of the parameters $M1$

$$x + y = 3\lambda \text{ so } 4(x + y) = 12\lambda \text{ and } y + z = 4\lambda + 3 \text{ so } 3(y + z) = 12\lambda + 9$$

$$3(y + z) = 4(x + y) + 9 \quad A1$$

Cartesian equation of plane is $4x + y - 3z = -9$ (or equivalent) $A1 \quad N1$

[6 marks]

METHOD 2

EITHER

The point $(2, 4, 7)$ lies on the plane.

The vector joining $(2, 4, 7)$ and $(1, -1, 4)$ and $2i + j + 3k$ are parallel to the plane. So they are perpendicular to the normal to the plane.

$$(i - j + 4k) - (2i + 4j + 7k) = -i - 5j - 3k \quad (A1)$$

$$n = \begin{vmatrix} i & j & k \\ -1 & -5 & -3 \\ 2 & 1 & 3 \end{vmatrix} \quad M1$$

$$= -12i - 3j + 9k$$

or equivalent parallel vector $A1$

OR

L_1 and

L_2 intersect at $D(-2, 2, 1)$

$$n = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ -3 & 3 & -3 \end{vmatrix} \quad M1$$

$$= -12i - 3j + 9k$$

or equivalent parallel vector AI

THEN

r

$$\cdot n = (i - j + 4k)$$

$$\cdot (-12i - 3j + 9k) \quad M1$$

$$= 27 \quad AI$$

Cartesian equation of plane is $4x + y - 3z = -9$ (or equivalent) $AI \quad NI$

[6 marks]

Total [20 marks]

Examiners report

Most candidates scored reasonably well on this question. The most common errors were: Using OB rather than AB in (a); omitting the $r =$ in (b); failure to check that the values of the two parameters satisfied the third equation in (c); the use of an incorrect vector in (d). Even when (d) was correctly answered, there was usually little evidence of why a specific vector had been used.

6.

[6 marks]

Markscheme

$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$$

$$\overrightarrow{CA} = \mathbf{a} - \mathbf{c}$$

$$\Rightarrow \mathbf{a}$$

$$\cdot (\mathbf{c} - \mathbf{b}) = 0 \quad \text{MI}$$

and \mathbf{b}

$$\cdot (\mathbf{a} - \mathbf{c}) = 0 \quad \text{MI}$$

$$\Rightarrow \mathbf{a}$$

$$\cdot \mathbf{c} = \mathbf{a}$$

$$\cdot \mathbf{b} \quad \text{AI}$$

and \mathbf{a}

$$\cdot \mathbf{b} = \mathbf{b}$$

$$\cdot \mathbf{c} \quad \text{AI}$$

$$\Rightarrow \mathbf{a}$$

$$\cdot \mathbf{c} = \mathbf{b}$$

$$\cdot \mathbf{c} \quad \text{MI}$$

$$\Rightarrow \mathbf{b}$$

$$\cdot \mathbf{c} - \mathbf{a}$$

$$\cdot \mathbf{c} = 0$$

$$\mathbf{c}$$

$$\cdot (\mathbf{b} - \mathbf{a}) = 0 \quad \text{AI}$$

$$\Rightarrow$$

\overrightarrow{OC} is perpendicular to

\overrightarrow{AB} , as \mathbf{b}

$$\neq \mathbf{a} \quad \text{AG}$$

[6 marks]

Examiners report

Only the better candidates were able to make significant progress with this question. Many candidates understood how to begin the question, but did not see how to progress to the last stage. On the whole the candidates' use of notation in this question was poor.

7.

[6 marks]

Markscheme

$$a \cdot b = |a| |b| \cos \theta \quad (M1)$$

$$a \cdot b = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ m \end{pmatrix} = 7 + 3m \quad A1$$

$$|a| = \sqrt{14}$$

$$|b| = \sqrt{13 + m^2} \quad A1$$

$$|a| |b| \cos \theta = \sqrt{14} \sqrt{13 + m^2} \cos 30^\circ$$

$$7 + 3m = \sqrt{14} \sqrt{13 + m^2} \cos 30^\circ \quad A1$$

$$m = 2.27, m = 25.7 \quad A1A1$$

[6 marks]

Examiners report

Many candidates gained the first 4 marks by obtaining the equation, in unsimplified form, satisfied by m but then made mistakes in simplifying and solving this equation.

Markscheme

(a)

$$x = 3 + 2m$$

$$y = 2 - m$$

$$z = 7 + 2m \quad \text{AI}$$

$$x = 1 + 4n$$

$$y = 4 - n$$

$$z = 2 + n \quad \text{AI}$$

[2 marks]

(b)

$$3 + 2m = 1 + 4n \Rightarrow 2m - 4n = -2 \quad (\text{i})$$

$$2 - m = 4 - n \Rightarrow m - n = -2 \quad (\text{ii}) \quad \text{MI}$$

$$7 + 2m = 2 + n \Rightarrow 2m - n = -5 \quad (\text{iii})$$

$$(\text{iii}) - (\text{ii}) \Rightarrow m = -3 \quad \text{AI}$$

$$\Rightarrow n = -1 \quad \text{AI}$$

Substitute in (i), $-6 + 4 = -2$. Hence lines intersect. **RI**Point of intersection A is $(-3, 5, 1)$ **AI**

[5 marks]

(c)

$$\begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ 4 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} \quad \text{MIAI}$$

$$r \cdot \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} \quad (\text{MI})$$

$$r \cdot \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} = 29$$

$$x + 6y + 2z = 29 \quad \text{AI}$$

Note: Award **MIA0** if answer is not in Cartesian form.

[4 marks]

(d)

$$x = -8 + 3\lambda$$

$$y = -3 + 8\lambda \quad (\text{MI})$$

$$z = 2\lambda$$

Substitute in equation of plane.

$$-8 + 3\lambda - 18 + 48\lambda + 4\lambda = 29 \quad \text{MI}$$

$$55\lambda = 55$$

$$\lambda = 1 \quad \text{AI}$$

Coordinates of B are $(-5, 5, 2)$ **AI**

[4 marks]

(e) Coordinates of C are

$$\left(-4, 5, \frac{3}{2}\right) \quad (\text{AI})$$

$$r = \begin{pmatrix} -4 \\ 5 \\ \frac{3}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} \quad \text{MIAI}$$

Note: Award **MIA0** unless candidate writes $r =$ or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

[3 marks]

Total [18 marks]

Examiners report

Most candidates found this question to their liking and many correct solutions were seen. In (b), some candidates solved two equations for m and n but then failed to show that these values satisfied the third equation. In (e), some candidates used an incorrect formula to determine the coordinates of the mid-point of AB .

Markscheme

(a) identifies a direction vector *e.g.*

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \text{ or}$$

$$\overrightarrow{BA} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \quad \text{AI}$$

identifies the point $(1, -1, 2)$ AI

line

$$l_1 : \frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{1} \quad \text{AG}$$

[2 marks]

(b)

$$r = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$1 + 2\lambda = 1 + \mu, \quad -1 + \lambda = 2 + 2\mu, \quad 2 + \lambda = 3 + \mu \quad (MI)$$

equating two of the three equations gives

$$\lambda = -1 \text{ and}$$

$$\mu = -2 \quad \text{AIAI}$$

check in the third equation

satisfies third equation therefore the lines intersect RI

therefore coordinates of intersection are $(-1, -2, 1)$ AI

[5 marks]

(c) *d*

$$_1 = 2i + j + k, d$$

$$_2 = i + 2j + k \quad \text{AI}$$

d

i

$$\times d$$

j

$$= \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -i$$

$$-j$$

$$+ 3k \quad \text{MIAI}$$

Note: Accept scalar multiples of above vectors.

[3 marks]

(d) equation of plane is

$$-x - y + 3z = k \quad \text{MIAI}$$

contains $(1, 2, 3)$

$$(\text{or } (-1, -2, 1) \text{ or } (1, -1, 2)) \therefore k = -1 - 2 + 3 \times 3 = 6 \quad \text{AI}$$

$$-x - y + 3z = 6 \quad \text{AG}$$

[3 marks]

(e) direction vector of the perpendicular line is

$$\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \quad (M1)$$

$$r = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} + m \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \quad AI$$

Note: Award *A0* if *r* omitted.

[2 marks]

(f) (i) find point where line meets plane

$$-(3-m) - (1-m) + 3(-4+3m) = 6 \quad M1$$

$$m = 2 \quad AI$$

point of intersection is $(1, -1, 2)$ *AI*

(ii) for

$$T', m = 4 \quad (M1)$$

so

$$T' = ($$

$$-1,$$

$$-3,$$

$$8) \quad AI$$

(iii)

$$\overrightarrow{TT'} = \sqrt{(3+1)^2 + (1+3)^2 + (-4-8)^2} \quad (M1)$$

$$= \sqrt{176} \quad (= 4\sqrt{11}) \quad AI$$

[7 marks]

Total [22 marks]

Examiners report

This question was done very well by many students. The common errors were using the same variable for line 2 and in stating the vectors in b) were not parallel and therefore the lines did intersect. Many students did not check the solution in order to establish this.

When required to give the equation of the line in e) many did not state it as an equation, let alone a vector equation.

The difference between position vectors and coordinates was not clear on many papers.

In f) many used inefficient techniques that were time consuming to find the point of reflection.

10.

[6 marks]

Markscheme

taking cross products with \mathbf{a} , *M1*

\mathbf{a}

$$\times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a}$$

$$\times \mathbf{0} = \mathbf{0} \quad \text{A1}$$

using the algebraic properties of vectors and the fact that \mathbf{a}

$$\times \mathbf{a} = \mathbf{0}, \quad \text{M1}$$

\mathbf{a}

$$\times \mathbf{b} + \mathbf{a}$$

$$\times \mathbf{c} = \mathbf{0} \quad \text{A1}$$

\mathbf{a}

$$\times \mathbf{b} = \mathbf{c}$$

$$\times \mathbf{a} \quad \text{AG}$$

taking cross products with \mathbf{b} , *M1*

\mathbf{b}

$$\times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$$

\mathbf{b}

$$\times \mathbf{a} + \mathbf{b}$$

$$\times \mathbf{c} = \mathbf{0} \quad \text{A1}$$

\mathbf{a}

$$\times \mathbf{b} = \mathbf{b}$$

$$\times \mathbf{c} \quad \text{AG}$$

this completes the proof

[6 marks]

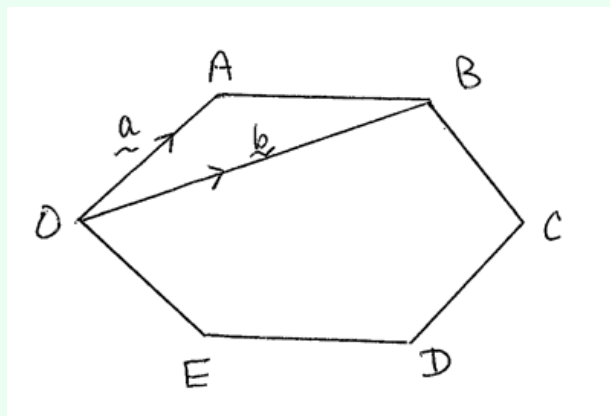
Examiners report

[N/A]

11a.

[2 marks]

Markscheme



$$OC = AB + OA \cos 60 + BC \cos 60 \quad M1$$

$$= AB + AB \times \frac{1}{2} + AB \times \frac{1}{2} \quad A1$$

$$= 2AB \quad AG$$

[2 marks]

Examiners report

[N/A]

11b.

[7 marks]

Markscheme

$$\overrightarrow{OC} = 2\overrightarrow{AB} = 2(b - a) \quad M1A1$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} \quad M1$$

$$= \overrightarrow{OC} + \overrightarrow{AO} \quad A1$$

$$= 2b - 2a - a = 2b - 3a \quad A1$$

$$\overrightarrow{OE} = \overrightarrow{BC} \quad M1$$

$$= 2b - 2a - b = b - 2a \quad A1$$

[7 marks]

Examiners report

[N/A]

12a. [5 marks]

Markscheme

(i)

$$\vec{OA} \times \vec{OB} = i + 7j - 5k \quad AI$$

(ii) area

$$= \frac{1}{2}|i + 7j - 5k|$$

$$= \frac{5\sqrt{3}}{2}(4.33) \quad MIAI$$

(iii) equation of plane is

$$x + 7y - 5z = k \quad MI$$

$$x + 7y - 5z = 0 \quad AI$$

[5 marks]

Examiners report

[N/A]

12b. [7 marks]

Markscheme

(i) direction of line = $(3i + j + 2k) - (i + 2j + 3k) = 2i - j - k \quad MIAI$

equation of line is

$$r = (i + 2j + 3k) + \lambda(2i - j - k) \quad AI$$

(ii) at a point of intersection,

$$1 + 2\lambda = 2 + \mu$$

$$2 - \lambda = 4 + 3\mu \quad MIAI$$

$$3 - \lambda = 3 + 2\mu$$

solving the

2nd and

3rd equations,

$$\lambda = 4, \mu = -2 \quad AI$$

these values do not satisfy the

1st equation so the lines are skew RI

[7 marks]

Examiners report

[N/A]

13a. [3 marks]

Markscheme

$$f'(x) = \frac{1}{x+1} \sin(\pi x) + \pi \ln(x+1) \cos(\pi x) \quad M1A1A1$$

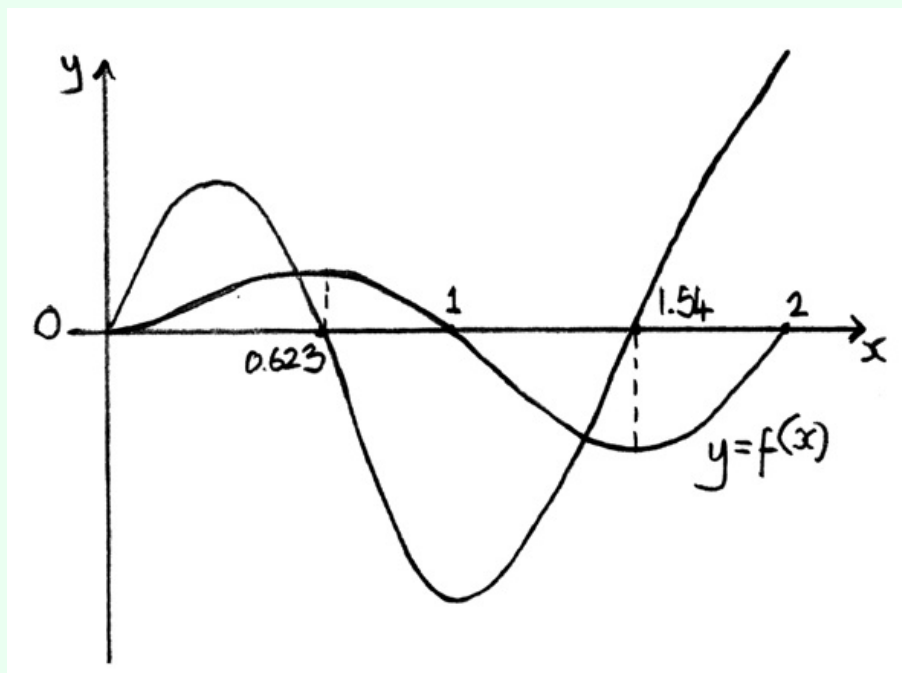
[3 marks]

Examiners report

[N/A]

13b. [4 marks]

Markscheme



A4

Note: Award *A1A1* for graphs, *A1A1* for intercepts.

[4 marks]

Examiners report

[N/A]

13c. [2 marks]

Markscheme

0.310, 1.12 *A1A1*

[2 marks]

Examiners report

[N/A]

13d. [3 marks]

Markscheme

$$f'(0.75) = -0.839092 \quad \text{AI}$$

so equation of normal is

$$y - 0.39570812 = \frac{1}{0.839092}(x - 0.75) \quad \text{MI}$$

$$y = 1.19x - 0.498 \quad \text{AI}$$

[3 marks]

Examiners report

[N/A]

13e. [6 marks]

Markscheme

$$A(0, 0)$$

$$B(\overbrace{0.548\dots}^c, \overbrace{0.432\dots}^d) \quad \text{AI}$$

$$C(\overbrace{1.44\dots}^e, \overbrace{-0.881\dots}^f) \quad \text{AI}$$

Note: Accept coordinates for B and C rounded to 3 significant figures.

area

$$\Delta ABC = \frac{1}{2}|(ci + dj)$$

$$\times (ei + fj)$$

$$| \quad \text{MIAI}$$

$$= \frac{1}{2}(de - cf) \quad \text{AI}$$

$$= 0.554 \quad \text{AI}$$

[6 marks]

Examiners report

[N/A]

Markscheme

EITHER

l goes through the point $(1, 3, 6)$, and the plane contains $A(4, -2, 5)$

the vector containing these two points is on the plane, *i.e.*

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} \quad (MI)AI$$

$$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ -1 & 2 & -1 \\ -3 & 5 & 1 \end{vmatrix} = 7i + 4j + k \quad MIAI$$

$$\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} = 25 \quad (MI)$$

hence, Cartesian equation of the plane is

$$7x + 4y + z = 25 \quad AI$$

OR

finding a third point MI

e.g. $(0, 5, 5)$ AI

three points are $(1, 3, 6)$, $(4, -2, 5)$, $(0, 5, 5)$

equation is

$$ax + by + cz = 1$$

system of equations MI

$$a + 3b + 6c = 1$$

$$4a - 2b + 5c = 1$$

$$5b + 5c = 1$$

$$a = \frac{7}{25},$$

$$b = \frac{4}{25},$$

$$c = \frac{1}{25}, \text{ from GDC} \quad MIAI$$

so

$$\frac{7}{25}x + \frac{4}{25}y + \frac{1}{25}z = 1 \quad AI$$

or

$$7x + 4y + z = 25$$

[6 marks]

Examiners report

There were many successful answers to this question, as would be expected. There seemed to be some students, however, that had not been taught the vector geometry section

Markscheme

(a) on

$$l_1: A \begin{pmatrix} -3 + 3\lambda, \\ -4 + 2\lambda, \\ 6 - 2\lambda \end{pmatrix} \quad \text{AI}$$

on

$$l_2: r = \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} \quad (MI)$$

$\Rightarrow B$ (

$$4 - 3\mu, \\ -7 + 4\mu, \\ -3 - \mu) \quad \text{AI}$$

$$\overrightarrow{BA} = a - b = \begin{pmatrix} 3\lambda + 3\mu - 7 \\ 2\lambda - 4\mu + 3 \\ -2\lambda + \mu + 9 \end{pmatrix} \quad (MI) \text{AI}$$

EITHER

$$BA \perp l_1 \Rightarrow BA \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 0 \Rightarrow 3(3\lambda + 3\mu - 7) + 2(2\lambda - 4\mu + 3) - 2(-2\lambda + \mu + 9) = 0 \quad \text{MI}$$

$$\Rightarrow 17\lambda - \mu = 33 \quad \text{AI}$$

$$BA \perp l_2 \Rightarrow BA \cdot \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = 0 \Rightarrow -3(3\lambda + 3\mu - 7) + 4(2\lambda - 4\mu + 3) - (-2\lambda + \mu + 9) = 0 \quad \text{MI}$$

$$\Rightarrow \lambda - 26\mu = -24 \quad \text{AI}$$

solving both equations above simultaneously gives

$$\lambda = 2;$$

$$\mu = 1 \Rightarrow A(3, 0, 2), B(1, -3, -4) \quad \text{AIAIAIAI}$$

OR

$$\begin{vmatrix} i & j & k \\ 3 & 2 & -2 \\ -3 & 4 & -1 \end{vmatrix} = 6i + 9j + 18k \quad \text{MIAI}$$

so

$$\overrightarrow{AB} = p \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3\lambda + 3\mu - 7 \\ 2\lambda - 4\mu + 3 \\ -2\lambda + \mu + 9 \end{pmatrix} \quad \text{MIAI}$$

$$3\lambda + 3\mu - 2p = 7$$

$$2\lambda - 4\mu - 3p = -3$$

$$-2\lambda + \mu - 6p = -9$$

$$\lambda = 2,$$

$$\mu = 1,$$

$$p = 1 \quad \text{AIAI}$$

A(

$$-3 + 6,$$

$$-4 + 4,$$

$$6 - 4)$$

= (

$$3,$$

$$0,$$

$$2) \quad \text{AI}$$

B(

$$4 - 3,$$

$$-7 + 4,$$

$$-3 - 1)$$

= (

1,
-3,
-4) *AI*

[13 marks]

(b)

$$\mathbf{AB} = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix} \quad (AI)$$

$$|\mathbf{AB}| = \sqrt{(-2)^2 + (-3)^2 + (-6)^2} = \sqrt{49} = 7 \quad MIAI$$

[3 marks]

(c) from (b)

$2i + 3j + 6k$ is normal to both lines

l_1 goes through $(-3, -4, 6)$

$$\Rightarrow \begin{pmatrix} -3 \\ -4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = 18 \quad MIAI$$

hence, the Cartesian equation of the plane through

l_1 , but not

l_2 , is

$$2x + 3y + 6z = 18 \quad AI$$

[3 marks]

Total [19 marks]

Examiners report

There were a lot of arithmetic errors in the treatment of this question, even though it was apparent that many students did understand the methods involved. In (a) many students failed to realise that

$\vec{}$

\mathbf{AB} should be a multiple of the cross product of the two direction vectors, rather than the cross product itself, and many students failed to give the final answer as coordinates.

16.

[6 marks]

Markscheme

consider a vector parallel to each line,

e.g.

$$u = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \text{ and}$$

$$v = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{A1A1}$$

let

θ be the angle between the lines

$$\cos \theta = \frac{|u \cdot v|}{|u||v|} = \frac{|12 - 6 + 1|}{\sqrt{21}\sqrt{19}} \quad \text{M1A1}$$

$$= \frac{7}{\sqrt{21}\sqrt{19}} = 0.350... \quad (\text{A1})$$

so

$$\theta = 69.5$$

$$\left(\text{or } 1.21 \text{ rad or } \arccos\left(\frac{7}{\sqrt{21}\sqrt{19}}\right) \right) \quad \text{A1} \quad \text{N4}$$

Note: Allow FT from incorrect reasonable vectors.

[6 marks]

Examiners report

Most students knew how to find the angle between two vectors, although many could not find the correct two direction vectors.

Markscheme

(a) let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 4 \end{pmatrix},$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ and}$$

$$B = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} \quad (M1)$$

point of intersection is

$$\left(\frac{11}{12}, \frac{7}{12}, \frac{1}{4}\right) \text{ or}$$

$$(\text{or } (0.917, 0.583, 0.25)) \quad A1$$

(b) **METHOD 1**

(i)

$$\det \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 4 \end{pmatrix} = 0 \quad M1$$

$$-3a + 24 = 0 \quad (A1)$$

$$a = 8 \quad A1 \quad NI$$

(ii) consider the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 4 & 5 \end{array} \right) \quad M1$$

use row reduction to obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{array} \right) \text{ or}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \text{ (or equivalent)} \quad A1$$

any valid reason **R1**(e.g. as the last row is not all zeros, the planes do not meet) **N0**

METHOD 2

use of row reduction (or equivalent manipulation of equations) **M1**

e.g.

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & a-10 & -5 \end{pmatrix} \quad A1A1$$

Note: Award an **A1** for each correctly reduced row.

(i)

$$a - 10 = -2 \Rightarrow a = 8 \quad M1A1 \quad NI$$

(ii) when

$$a = 8, \text{ row 3}$$

$$\neq 2$$

$$\times \text{ row 2} \quad R1 \quad N0$$

[8 marks]

Examiners report

Few students were able to do this question efficiently. Many students were able to do part (a) by manipulating equations, whereas calculator methods would yield the solution quickly and easily. Part (b) was poorly attempted and it was apparent that many students used a lot of time manipulating equations without real understanding of what they were looking for.

Markscheme

(a)

$$\overrightarrow{OP} = i + 2j - k \quad (M1)$$

the coordinates of P are (1, 2, -1) *AI*

[2 marks]

(b) **EITHER**

$$x = 1 + t,$$

$$y = 2 - 2t,$$

$$z = 3t - 1 \quad M1$$

$$x - 1 = t,$$

$$\frac{y-2}{-2} = t,$$

$$\frac{z+1}{3} = t \quad AI$$

$$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3} \quad AG \quad N0$$

OR

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \quad M1A1$$

$$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3} \quad AG$$

[2 marks]

(c) (i)

$$2(1+t) + (2-2t) + (3t-1) = 6 \Rightarrow t = 1 \quad M1A1 \quad N1$$

(ii) coordinates are (2, 0, 2) *AI***Note:** Award *A0* for position vector.(iii) distance travelled is the distance between the two points *(M1)*

$$\sqrt{(2-1)^2 + (0-2)^2 + (2+1)^2} = \sqrt{14} \quad ($$

$$= 3.74) \quad (M1)AI$$

[6 marks]

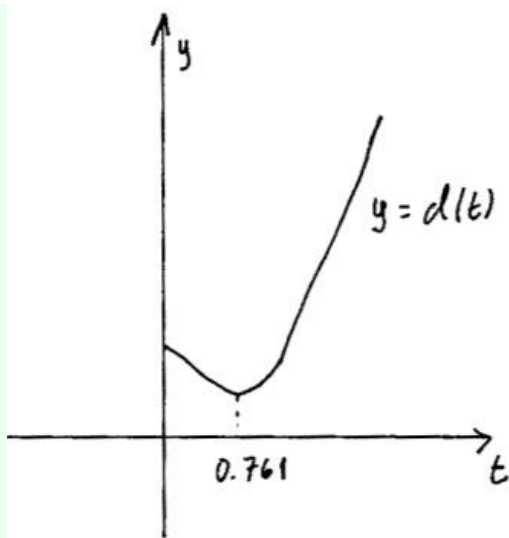
(d) (i) distance from Q to the origin is given by

$$d(t) = \sqrt{t^4 + (1-t)^2 + (1-t^2)^2} \quad (\text{or equivalent}) \quad M1A1$$

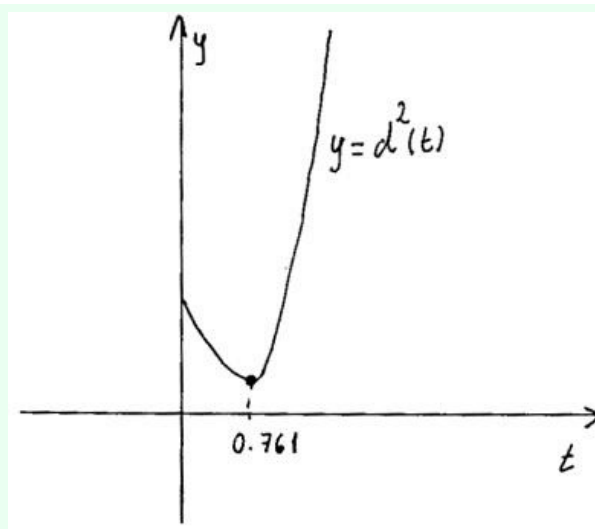
e.g. for labelled sketch of graph of

d or

$$d^2 \quad (M1)(AI)$$



or



the minimum value is obtained for

$t = 0.761$ **AI N3**

(ii) the coordinates are (0.579, 0.239, 0.421) **AI**

Note: Accept answers given as a position vector.

[6 marks]

(e) (i)

$$a = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and}$$

$$c = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \quad \text{(M1)AI}$$

substituting in the equation

$a - b = k(b - c)$, we have **(M1)**

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = k \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \right) \Leftrightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = k \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \quad \text{AI}$$

$\Rightarrow k = 1$ and

$k = \frac{1}{3}$ which is impossible

so there is no solution for

k **R1**

(ii)

BA and

\overrightarrow{CB}

are not parallel **R2**

(hence A, B, and C cannot be collinear)

Note: Only accept answers that follow from part (i).

[7 marks]

Total [23 marks]

Examiners report

Generally this question was answered well by those students who attempted it. It was common to see confusion between coordinates and position vectors. Part (d) was most easily answered with the use of a GDC, but fewer candidates took advantage of this. In part (e) many students had difficulties expressing their reasoning well to obtain the marks.

19.

[5 marks]

Markscheme

direction vector for line

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ or any multiple} \quad \text{AI}$$

$$\begin{pmatrix} 2 \sin \theta \\ 1 - \sin \theta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \quad \text{MI}$$

$$2 \sin \theta - 1 + \sin \theta = 0 \quad \text{AI}$$

Note: Allow **FT** on candidate's direction vector just for line above only.

$$3 \sin \theta = 1$$

$$\sin \theta = \frac{1}{3} \quad \text{AI}$$

$$\theta = 0.340 \text{ or}$$

$$19.5 \quad \text{AI}$$

Note: A coordinate geometry method using perpendicular gradients is acceptable.

[5 marks]

Examiners report

A variety of approaches were seen, either using a scalar product of vectors, or based on the rule for perpendicular gradients of lines. The main problem encountered in the first approach was in the choice of the correct vector direction for the line.