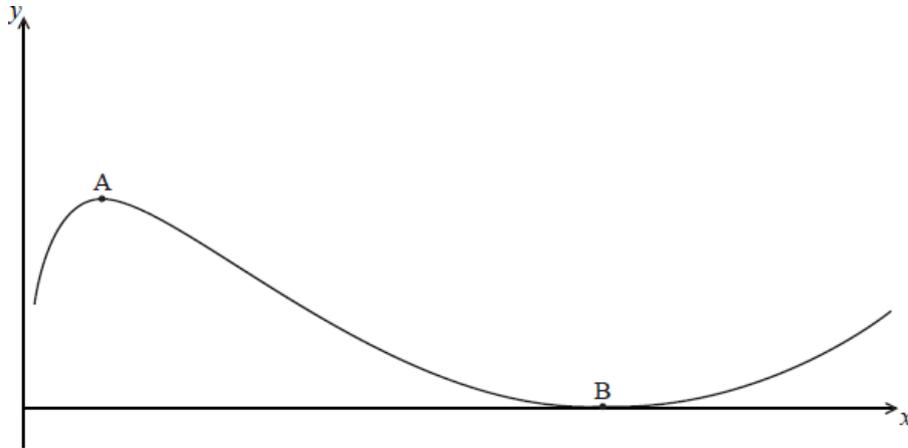


Topic 6 Part 1 [429 marks]

The diagram shows the graph of the function defined by $y = x(\ln x)^2$ for $x > 0$.



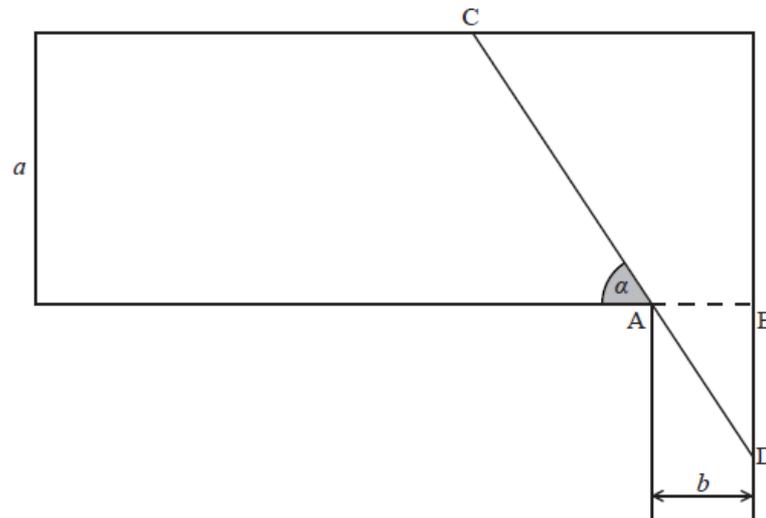
The function has a local maximum at the point A and a local minimum at the point B.

- 1a. Find the coordinates of the points A and B. [5 marks]
- 1b. Given that the graph of the function has exactly one point of inflexion, find its coordinates. [3 marks]

Consider the curve defined by the equation $x^2 + \sin y - xy = 0$.

- 2a. Find the gradient of the tangent to the curve at the point (π, π) . [6 marks]
- 2b. Hence, show that $\tan \theta = \frac{1}{1+2\pi}$, where θ is the acute angle between the tangent to the curve at (π, π) and the line $y = x$. [3 marks]
3. A particle moves along a straight line so that after t seconds its displacement s , in metres, satisfies the equation $s^2 + s - 2t = 0$. Find, in terms of s , expressions for its velocity and its acceleration. [6 marks]
4. By using the substitution $x = \sin t$, find $\int \frac{x^3}{\sqrt{1-x^2}} dx$. [7 marks]
5. Find the area of the region enclosed by the curves $y = x^3$ and $x = y^2 - 3$. [7 marks]

The diagram shows the plan of an art gallery a metres wide. [AB] represents a doorway, leading to an exit corridor b metres wide. In order to remove a painting from the art gallery, CD (denoted by L) is measured for various values of α , as represented in the diagram.



6a. If α is the angle between [CD] and the wall, show that $L = \frac{a}{\sin \alpha} + \frac{b}{\cos \alpha}$, $0 < \alpha < \frac{\pi}{2}$. [3 marks]

6b. If $a = 5$ and $b = 1$, find the maximum length of a painting that can be removed through this doorway. [4 marks]

6c. Let $a = 3k$ and $b = k$. Find $\frac{dL}{d\alpha}$. [3 marks]

6d. Let $a = 3k$ and $b = k$. Find, in terms of k , the maximum length of a painting that can be removed from the gallery through this doorway. [6 marks]

6e. Let $a = 3k$ and $b = k$. Find the minimum value of k if a painting 8 metres long is to be removed through this doorway. [2 marks]

7a. Express $4x^2 - 4x + 5$ in the form $a(x - h)^2 + k$ where $a, h, k \in \mathbb{Q}$. [2 marks]

7b. The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear. [3 marks]

The function f is defined by

$$f(x) = \frac{1}{4x^2 - 4x + 5}$$

7c. Sketch the graph of $y = f(x)$. [2 marks]

7d. Find the range of f . [2 marks]

7e. By using a suitable substitution show that [3 marks]

$$\int f(x)dx = \frac{1}{4} \int \frac{1}{u^2+1} du.$$

7f. Prove that [7 marks]

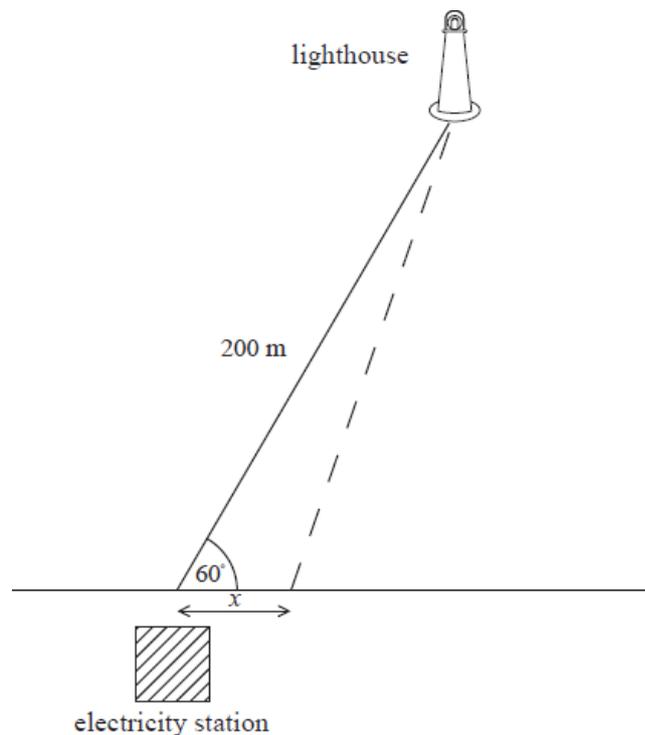
$$\int_1^{3.5} \frac{1}{4x^2-4x+5} dx = \frac{\pi}{16}.$$

8. Find the volume of the solid formed when the region bounded by the graph of [6 marks]

$y = \sin(x - 1)$, and the lines $y = 0$ and $y = 1$ is rotated by

2π about the y -axis.

An electricity station is on the edge of a straight coastline. A lighthouse is located in the sea 200 m from the electricity station. The angle between the coastline and the line joining the lighthouse with the electricity station is 60° . A cable needs to be laid connecting the lighthouse to the electricity station. It is decided to lay the cable in a straight line to the coast and then along the coast to the electricity station. The length of cable laid along the coastline is x metres. This information is illustrated in the diagram below.



The cost of laying the cable along the sea bed is US\$80 per metre, and the cost of laying it on land is US\$20 per metre.

9a. Find, in terms of x , an expression for the cost of laying the cable. [4 marks]

9b. Find the value of x , to the nearest metre, such that this cost is minimized. [2 marks]

A particle, A, is moving along a straight line. The velocity,
 $v_A \text{ ms}^{-1}$, of A t seconds after its motion begins is given by

$$v_A = t^3 - 5t^2 + 6t.$$

10a. Sketch the graph of

[3 marks]

$$v_A = t^3 - 5t^2 + 6t \text{ for}$$

$t \geq 0$, with

v_A on the vertical axis and t on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the t -axis.

10b. Write down the times for which the velocity of the particle is increasing.

[2 marks]

10c. Write down the times for which the magnitude of the velocity of the particle is increasing.

[3 marks]

10d. At $t = 0$ the particle is at point O on the line.

[3 marks]

Find an expression for the particle's displacement,

$x_A \text{ m}$, from O at time t .

10e. A second particle, B, moving along the same line, has position

[4 marks]

$x_B \text{ m}$, velocity

$v_B \text{ ms}^{-1}$ and acceleration,

$a_B \text{ ms}^{-2}$, where

$$a_B = -2v_B \text{ for}$$

$t \geq 0$. At

$t = 0$, $x_B = 20$ and

$$v_B = -20.$$

Find an expression for

v_B in terms of t .

10f. Find the value of t when the two particles meet.

[6 marks]

The function f has inverse

f^{-1} and derivative

$f'(x)$ for all

$x \in \mathbb{R}$. For all functions with these properties you are given the result that for

$a \in \mathbb{R}$ with

$b = f(a)$ and

$f'(a) \neq 0$

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

11a. Verify that this is true for

[6 marks]

$$f(x) = x^3 + 1 \text{ at } x = 2.$$

11b. Given that

[3 marks]

$$g(x) = xe^{x^2}, \text{ show that}$$

$$g'(x) > 0 \text{ for all values of } x.$$

11c. Using the result given at the start of the question, find the value of the gradient function of

[4 marks]

$$y = g^{-1}(x) \text{ at } x = 2.$$

11d. (i) With f and g as defined in parts (a) and (b), solve

[6 marks]

$$g \circ f(x) = 2.$$

(ii) Let

$$h(x) = (g \circ f)^{-1}(x). \text{ Find}$$

$$h'(2).$$

12. Find the exact value of

[6 marks]

$$\int_1^2 \left((x-2)^2 + \frac{1}{x} + \sin \pi x \right) dx.$$

The curve C is given by

$$y = \frac{x \cos x}{x + \cos x}, \text{ for}$$

$$x \geq 0.$$

13a. Show that

[4 marks]

$$\frac{dy}{dx} = \frac{\cos^2 x - x^2 \sin x}{(x + \cos x)^2}, \quad x \geq 0.$$

13b. Find the equation of the tangent to C at the point

[3 marks]

$$\left(\frac{\pi}{2}, 0 \right).$$

The curve C is given implicitly by the equation

$$\frac{x^2}{y} - 2x = \ln y \text{ for}$$

$$y > 0.$$

14a. Express

[4 marks]

$$\frac{dy}{dx} \text{ in terms of } x \text{ and } y.$$

14b. Find the value of

[2 marks]

$$\frac{dy}{dx} \text{ at the point on } C \text{ where } y = 1 \text{ and}$$

$$x > 0.$$

The function f is defined by

$$f(x) = \frac{2x-1}{x+2}, \text{ with domain}$$

$$D = \{x : -1 \leq x \leq 8\}.$$

15a. Express

[2 marks]

$f(x)$ in the form

$$A + \frac{B}{x+2}, \text{ where}$$

A and

$$B \in \mathbb{Z}.$$

15b. Hence show that

[2 marks]

$$f'(x) > 0 \text{ on } D.$$

15c. State the range of f .

[2 marks]

15d. (i) Find an expression for

[8 marks]

$$f^{-1}(x).$$

(ii) Sketch the graph of

$$y = f(x), \text{ showing the points of intersection with both axes.}$$

(iii) On the same diagram, sketch the graph of

$$y = f'(x).$$

15e. (i) On a different diagram, sketch the graph of

[7 marks]

$$y = f(|x|) \text{ where}$$

$$x \in D.$$

(ii) Find all solutions of the equation

$$f(|x|) = -\frac{1}{4}.$$

16a. Find [4 marks]

$$\int x \sec^2 x dx.$$

16b. Determine the value of m if [2 marks]

$$\int_0^m x \sec^2 x dx = 0.5, \text{ where } m > 0.$$

17. The acceleration of a car is [6 marks]

$$\frac{1}{40}(60 - v) \text{ ms}^{-2}, \text{ when its velocity is}$$

$v \text{ ms}^{-2}$. Given the car starts from rest, find the velocity of the car after 30 seconds.

Consider the differential equation

$$y \frac{dy}{dx} = \cos 2x.$$

18a. (i) Show that the function [10 marks]

$y = \cos x + \sin x$ satisfies the differential equation.

(ii) Find the general solution of the differential equation. Express your solution in the form

$y = f(x)$, involving a constant of integration.

(iii) For which value of the constant of integration does your solution coincide with the function given in part (i)?

18b. A different solution of the differential equation, satisfying $y = 2$ when [12 marks]

$x = \frac{\pi}{4}$, defines a curve C .

(i) Determine the equation of C in the form

$y = g(x)$, and state the range of the function g .

A region R in the xy plane is bounded by C , the x -axis and the vertical lines $x = 0$ and

$x = \frac{\pi}{2}$.

(ii) Find the area of R .

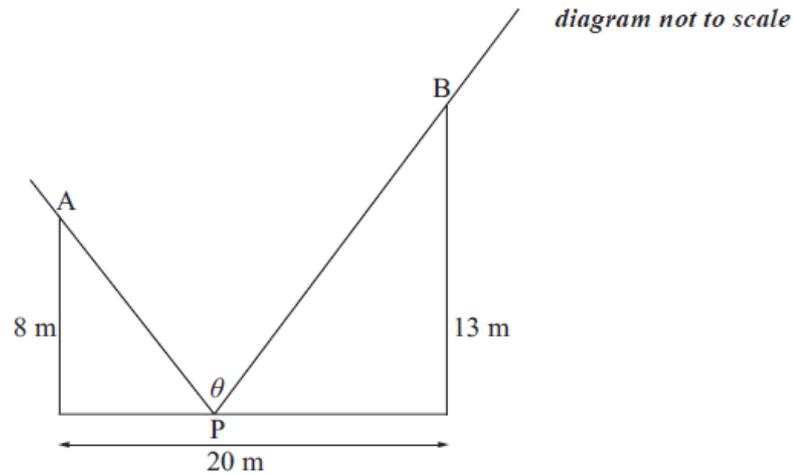
(iii) Find the volume generated when that part of R above the line $y = 1$ is rotated about the x -axis through

2π radians.

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle

θ where

$\theta = \widehat{APB}$, as shown in the diagram.



19a. Find an expression for [2 marks]

θ in terms of x , where x is the distance of P from the base of the wall of height 8 m.

19b. (i) Calculate the value of [2 marks]

θ when $x = 0$.

(ii) Calculate the value of

θ when $x = 20$.

19c. Sketch the graph of [2 marks]

θ , for

$0 \leq x \leq 20$.

19d. Show that [6 marks]

$$\frac{d\theta}{dx} = \frac{5(744 - 64x - x^2)}{(x^2 + 64)(x^2 - 40x + 569)}.$$

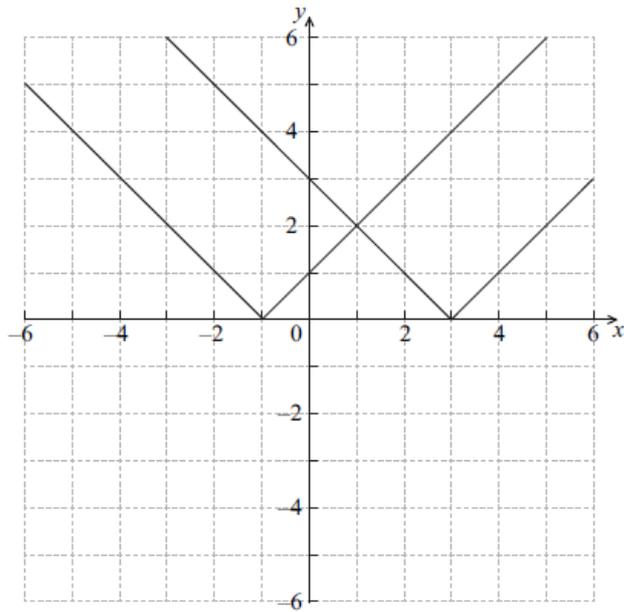
19e. Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give [3 marks]
your answer to four significant figures.

19f. The point P moves across the street with speed [4 marks]

0.5 ms^{-1} . Determine the rate of change of

θ with respect to time when P is at the midpoint of the street.

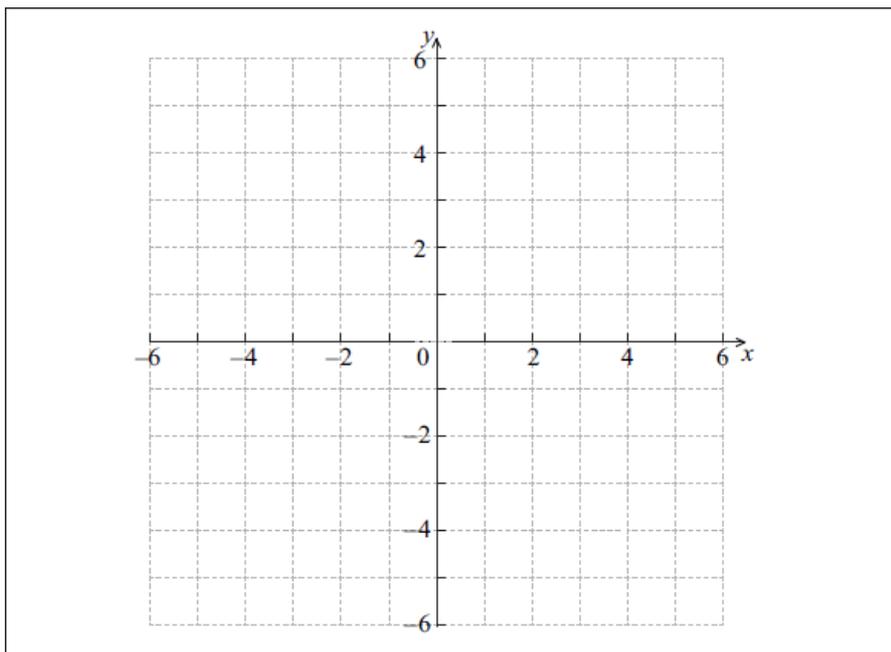
The graphs of
 $y = |x + 1|$ and
 $y = |x - 3|$ are shown below.



Let $f(x) =$
 $|x + 1| - |x - 3|$.

20a. Draw the graph of $y = f(x)$ on the blank grid below.

[4 marks]



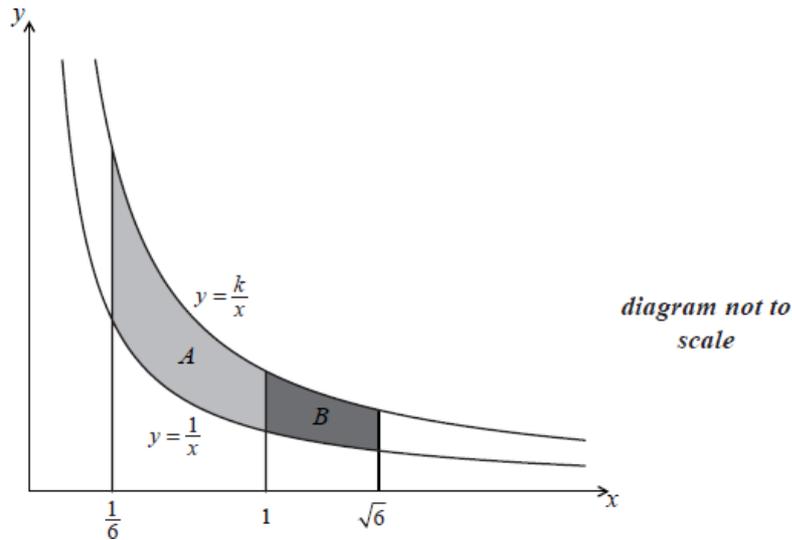
20b. Hence state the value of

[4 marks]

- (i)
 $f'(-3)$;
- (ii)
 $f'(2.7)$;
- (iii)
 $\int_{-3}^{-2} f(x) dx$.

The graph below shows the two curves

$$y = \frac{1}{x} \text{ and } y = \frac{k}{x}, \text{ where } k > 1.$$



21a. Find the area of region A in terms of k . [3 marks]

21b. Find the area of region B in terms of k . [2 marks]

21c. Find the ratio of the area of region A to the area of region B . [3 marks]

22. The curve C has equation $2x^2 + y^2 = 18$. Determine the coordinates of the four points on C at which the normal passes through the point $(1, 0)$. [9 marks]

Let $f(x) = \sqrt{\frac{x}{1-x}}, 0 < x < 1$.

23a. Show that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ and deduce that f is an increasing function. [5 marks]

23b. Show that the curve $y = f(x)$ has one point of inflexion, and find its coordinates. [6 marks]

23c. Use the substitution $x = \sin^2 \theta$ to show that $\int f(x) dx = \arcsin \sqrt{x} - \sqrt{x-x^2} + c$. [11 marks]

24. Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of k . [5 marks]

25. A cone has height h and base radius r . Deduce the formula for the volume of this cone by rotating the triangular region, enclosed by the line $y = h - \frac{h}{r}x$ and the coordinate axes, through 2π about the y -axis. [9 marks]

26. A triangle is formed by the three lines [8 marks]
 $y = 10 - 2x$, $y = mx$ and
 $y = -\frac{1}{m}x$, where
 $m > \frac{1}{2}$.
Find the value of m for which the area of the triangle is a minimum.

The function
 $f(x) = 3 \sin x + 4 \cos x$ is defined for
 $0 < x < 2\pi$.

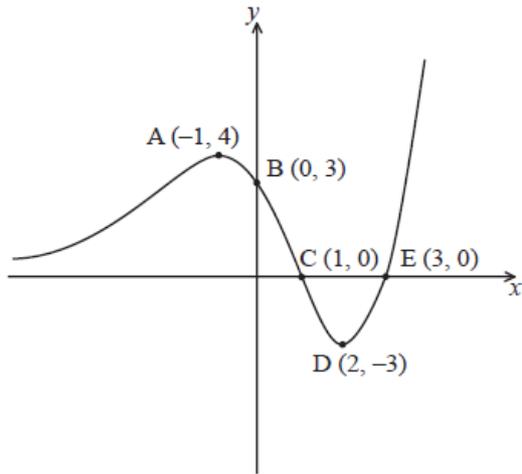
- 27a. Write down the coordinates of the minimum point on the graph of f . [1 mark]

- 27b. The points [2 marks]
 $P(p, 3)$ and
 $Q(q, 3)$, $q > p$, lie on the graph of
 $y = f(x)$.
Find p and q .

- 27c. Find the coordinates of the point, on [4 marks]
 $y = f(x)$, where the gradient of the graph is 3.

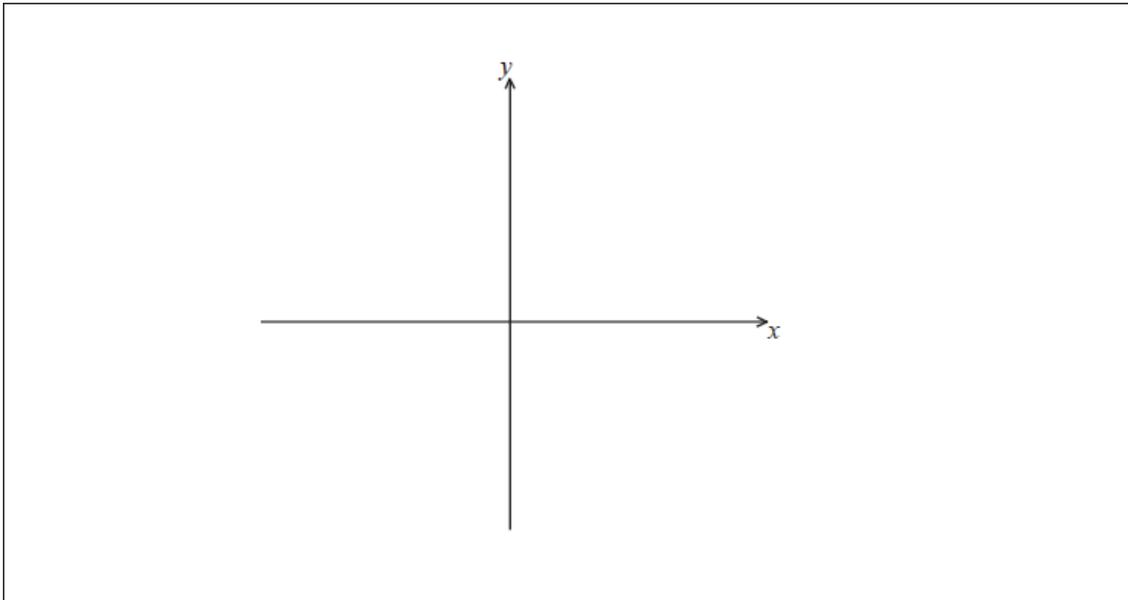
- 27d. Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. [7 marks]

The graph of $y = f(x)$ is shown below, where A is a local maximum point and D is a local minimum point.



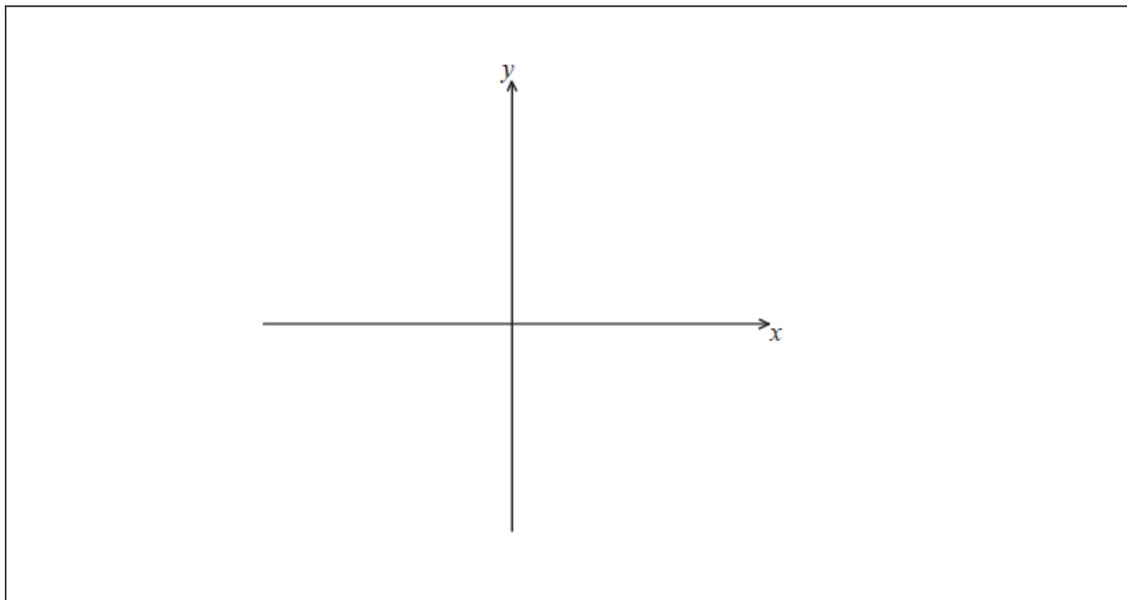
- 28a. On the axes below, sketch the graph of $y = \frac{1}{f(x)}$, clearly showing the coordinates of the images of the points A, B and D, labelling them A' , B' , and D' respectively, and the equations of any vertical asymptotes.

[3 marks]



- 28b. On the axes below, sketch the graph of the derivative $y = f'(x)$, clearly showing the coordinates of the images of the points A and D, labelling them A'' and D'' respectively.

[3 marks]



29. Let $x^3 y = a \sin nx$. Using implicit differentiation, show that

[6 marks]

$$x^3 \frac{d^2 y}{dx^2} + 6x^2 \frac{dy}{dx} + (n^2 x^2 + 6)xy = 0$$

The function f is defined on the domain $[0, \frac{3\pi}{2}]$ by $f(x) = e^{-x} \cos x$.

- 30a. State the two zeros of f .

[1 mark]

- 30b. Sketch the graph of f .

[1 mark]

- 30c. The region bounded by the graph, the x -axis and the y -axis is denoted by A and the region bounded by the graph and the x -axis is denoted by B . Show that the ratio of the area of A to the area of B is

[7 marks]

$$\frac{e^\pi \left(e^{\frac{\pi}{2}} + 1 \right)}{e^\pi + 1}$$

- 31a. Using the definition of a derivative as $f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$, show that the derivative of $\frac{1}{2x+1}$ is $\frac{-2}{(2x+1)^2}$.

[4 marks]

- 31b. Prove by induction that the n^{th} derivative of $(2x+1)^{-1}$ is $(-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$.

[9 marks]

32a. Sketch the curve $y = \frac{\cos x}{\sqrt{x^2+1}}$, $-4 \leq x \leq 4$ showing clearly the coordinates of the x -intercepts, any maximum points and any minimum points. [4 marks]

32b. Write down the gradient of the curve at $x = 1$. [1 mark]

32c. Find the equation of the normal to the curve at $x = 1$. [3 marks]

A particle moves in a straight line with velocity v metres per second. At any time t seconds, $0 \leq t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{dv}{dt} + v^2 + 1 = 0$. It is also given that $v = 1$ when $t = 0$.

33a. Find an expression for v in terms of t . [7 marks]

33b. Sketch the graph of v against t , clearly showing the coordinates of any intercepts, and the equations of any asymptotes. [3 marks]

33c. (i) Write down the time T at which the velocity is zero. [3 marks]
 (ii) Find the distance travelled in the interval $[0, T]$.

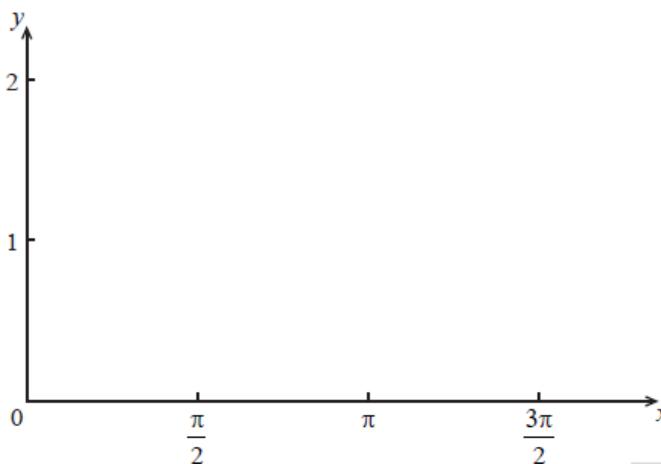
33d. Find an expression for s , the displacement, in terms of t , given that $s = 0$ when $t = 0$. [5 marks]

33e. Hence, or otherwise, show that $s = \frac{1}{2} \ln \frac{2}{1+v^2}$. [4 marks]

Given that

$$f(x) = 1 + \sin x, \quad 0 \leq x \leq \frac{3\pi}{2},$$

34a. sketch the graph of f ; [1 mark]



34b. show that $(f(x))^2 = \frac{3}{2} + 2 \sin x - \frac{1}{2} \cos 2x$; [1 mark]

$$(f(x))^2 = \frac{3}{2} + 2 \sin x - \frac{1}{2} \cos 2x;$$

- 34c. find the volume of the solid formed when the graph of f is rotated through 2π radians about the x -axis.

[4 marks]

35. Given that

[7 marks]

$y = \frac{1}{1-x}$, use mathematical induction to prove that

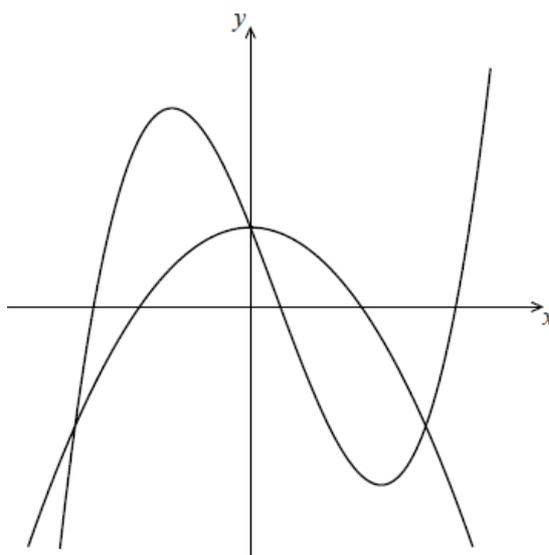
$$\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}, \quad n \in \mathbb{Z}^+.$$

36. The graphs of

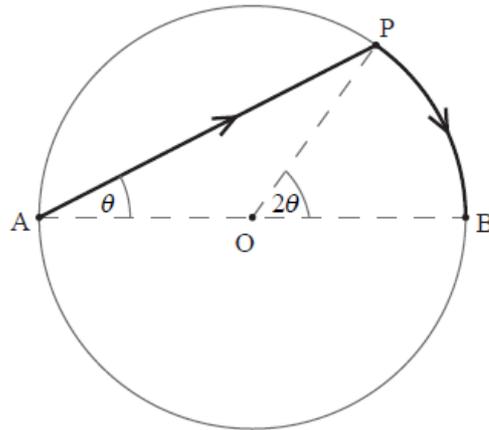
[7 marks]

$$f(x) = -x^2 + 2 \text{ and}$$

$g(x) = x^3 - x^2 - bx + 2$, $b > 0$, intersect and create two closed regions. Show that these two regions have equal areas.



The diagram below shows a circular lake with centre O , diameter AB and radius 2 km.



Jorg needs to get from A to B as quickly as possible. He considers rowing to point P and then walking to point B. He can row at 3 km h^{-1} and walk at 6 km h^{-1} . Let $\widehat{PAB} = \theta$ radians, and t be the time in hours taken by Jorg to travel from A to B.

37a. Show that [3 marks]

$$t = \frac{2}{3}(2 \cos \theta + \theta).$$

37b. Find the value of [2 marks]

θ for which

$$\frac{dt}{d\theta} = 0.$$

37c. What route should Jorg take to travel from A to B in the least amount of time? [3 marks]

Give reasons for your answer.

At 12:00 a boat is 20 km due south of a freighter. The boat is travelling due east at 20 km h^{-1} , and the freighter is travelling due south at 40 km h^{-1} .

38a. Determine the time at which the two ships are closest to one another, and justify your answer. [8 marks]

38b. If the visibility at sea is 9 km, determine whether or not the captains of the two ships can ever see each other's ship. [3 marks]

The curve C with equation

$y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{y}{\ln y}(x + 2), \quad y > 1,$$

and $y = e$ when $x = 2$.

39a. Find the equation of the tangent to C at the point (2, e). [3 marks]

39b. Find [11 marks]
 $f(x)$.

39c. Determine the largest possible domain of f . [6 marks]

39d. Show that the equation [4 marks]
 $f(x) = f'(x)$ has no solution.