Topic 6 Part 4 *[317 marks]*

1a. Find $\int (1 + \tan^2 x)\mathrm{d}x$. *[2 marks]*

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1b. Find $\int \sin^2 x\mathrm{d}x$. *[3 marks]*

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A function f is defined by $f(x) = \frac{3x-2}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

2a. Find an expression for $f^{-1}(x)$. *[4 marks]*

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2b. Given that $f(x)$ can be written in the form $f(x) = A + \frac{B}{2x-1}$, find the values of the constants A and B. *[2 marks]*

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2c. Hence, write down $\int \frac{3x-2}{2x-1}\mathrm{d}x$. *[1 mark]*

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3. By using the substitution $u = \mathrm{e}^x + 3$, find $\int \frac{\mathrm{e}^x}{\mathrm{e}^{2x}+6\mathrm{e}^x+13}\mathrm{d}x$. *[7 marks]*

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Let $y(x) = xe^{3x},\ x \in \mathbb{R}$.

4a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[2 marks]*

4b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. *[7 marks]*

4c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. *[5 marks]*

Justify whether any such point is a maximum or a minimum.

4d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. *[5 marks]*

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Consider the function defined by $f(x) = x^3 - 3x^2 + 4$.

5a. Determine the values of x for which $f(x)$ is a decreasing function. *[4 marks]*

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5b. There is a point of inflexion, P, on the curve $y = f(x)$. *[3 marks]*

Find the coordinates of P.

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6. Show that $\int_1^2 x^3 \ln x \mathrm{d}x = 4\ln 2 - \frac{15}{16}$. *[6 marks]*

In triangle ABC, $\mathrm{BC} = \sqrt{3}\ \mathrm{cm}$, $\mathrm{A\hat{B}C} = \theta$ and $\mathrm{B\hat{C}A} = \frac{\pi}{3}$.

7a. Show that length $\mathrm{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$. *[4 marks]*

7b. Given that AB has a minimum value, determine the value of θ for which this occurs. *[4 marks]*

8. By using the substitution $t = \tan x$, find $\int \frac{\mathrm{d}x}{1+\sin^2 x}$. *[8 marks]*

Express your answer in the form $m \arctan(n \tan x) + c$, where m, n are constants to be determined.

Consider the functions $f(x) = \tan x,\ 0 \leq\ x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \neq 1$.

9a. Find an expression for $g \circ f(x)$, stating its domain. *[2 marks]*

9b. Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. *[2 marks]*

9c. Let $y = g \circ f(x)$, find an exact value for *[6 marks]*
$\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$.

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9d. Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. *[6 marks]*

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10. The region R is enclosed by the graph of $y = e^{-x^2}$, the x-axis and the lines $x = -1$ and $x = 1$. *[4 marks]*

Find the volume of the solid of revolution that is formed when R is rotated through 2π about the x-axis.

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11. A bicycle inner tube can be considered as a joined up cylinder of fixed length 200 cm and radius r cm. The radius r increases as the inner tube is pumped up. Air is being pumped into the inner tube so that the volume of air in the tube increases at a constant rate of $30\ \mathrm{cm}^3\mathrm{s}^{-1}$. Find the rate at which the radius of the inner tube is increasing when $r = 2\ \mathrm{cm}$. *[5 marks]*

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12. A function f is defined by $f(x) = x^3 + \mathrm{e}^x + 1,\ x \in \mathbb{R}$. By considering $f'(x)$ determine whether f is a one-to-one or a many-to-one function. *[4 marks]*

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13. Find the equation of the normal to the curve $y = \frac{\mathrm{e}^x\ \cos x \ln(x+\mathrm{e})}{(x^{17}+1)^5}$ at the point where $x = 0$. *[7 marks]*

In your answer give the value of the gradient, of the normal, to three decimal places.

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Richard, a marine soldier, steps out of a stationary helicopter, 1000 m above the ground, at time $t = 0$. Let his height, in metres, above the ground be given by $s(t)$. For the first 10 seconds his velocity, $v(t)\mathrm{ms}^{-1}$, is given by $v(t) = -10t$.

14a. (i) Find his acceleration $a(t)$ for $t < 10$. *[6 marks]*

(ii) Calculate $v(10)$.

(iii) Show that $s(10) = 500$.

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14b. At $t = 10$ his parachute opens and his acceleration $a(t)$ is subsequently given by $a(t) = -10 - 5v,\ t \geq 10$. *[1 mark]*

Given that $\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{1}{\frac{\mathrm{d}v}{\mathrm{d}t}}$, write down $\frac{\mathrm{d}t}{\mathrm{d}v}$ in terms of v.

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14c. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. *[5 marks]*

Hence show that $t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$.

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14d. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. *[2 marks]*

Hence find an expression for the velocity, v, for $t \geq 10$.

14e. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. *[5 marks]*

Find an expression for his height, s, above the ground for $t \geq 10$.

14f. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. *[2 marks]*

Find the value of t when Richard lands on the ground.

The graph of $y = \ln(5x + 10)$ is obtained from the graph of $y = \ln x$ by a translation of a units in the direction of the x-axis followed by a translation of b units in the direction of the y-axis.

15a. Find the value of a and the value of b. *[4 marks]*

15b. The region bounded by the graph of $y = \ln(5x + 10)$, the x-axis and the lines $x = \mathrm{e}$ and $x = 2\mathrm{e}$, is rotated through 2π radians about the x-axis. Find the volume generated. *[2 marks]*

A curve is defined $x^2 - 5xy + y^2 = 7$.

16a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y-2x}{2y-5x}$. *[3 marks]*

16b. Find the equation of the normal to the curve at the point $(6,\ 1)$. *[4 marks]*

16c. Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. *[8 marks]*

A particle moves in a straight line, its velocity $v\ \mathrm{ms}^{-1}$ at time t seconds is given by $v = 9t - 3t^2,\ 0 \leq t \leq 5$.

At time $t = 0$, the displacement s of the particle from an origin O is 3 m.

17a. Find the displacement of the particle when $t = 4$. *[3 marks]*

17b. Sketch a displacement/time graph for the particle, $0 \leq t \leq 5$, showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values. *[5 marks]*

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17c. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \geq 0$. *[3 marks]*

Given further that $s = 16.5$ when $t = 7.5$, find the values of a and b.

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17d. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \geq 0$. *[4 marks]*

Find the times t_1 and $t_2 (0 < t_1 < t_2 < 8)$ when the particle returns to its starting point.

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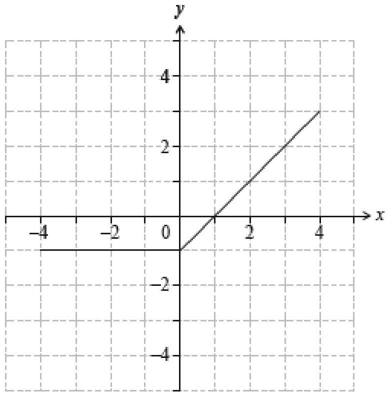
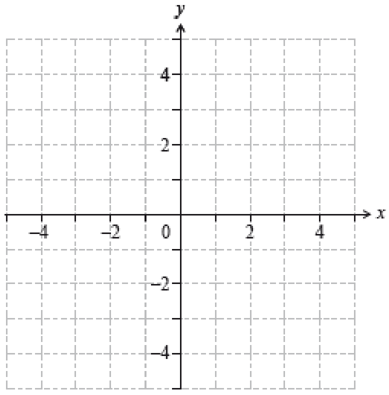
18a. Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to prove that $\cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$, $0 \leqslant x \leqslant \pi$. *[2 marks]*

18b. Find a similar expression for $\sin\frac{1}{2}x,\ 0 \leqslant x \leqslant \pi$. *[2 marks]*

18c. Hence find the value of $\int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x} + \sqrt{1-\cos x}\right) \mathrm{d}x$. *[4 marks]*

19. The first set of axes below shows the graph of $y = f(x)$ for $-4 \leqslant x \leqslant 4$. *[6 marks]*





Let $g(x) = \int_{-4}^{x} f(t)\mathrm{d}t$ for $-4 \leqslant x \leqslant 4$.

(a) State the value of *x* at which $g(x)$ is a minimum.

(b) On the second set of axes, sketch the graph of $y = g(x)$.

20. A body is moving in a straight line. When it is s metres from a fixed point O on the line its velocity, v, is given by $v = -\frac{1}{s^2},\ s > 0$. *[6 marks]*

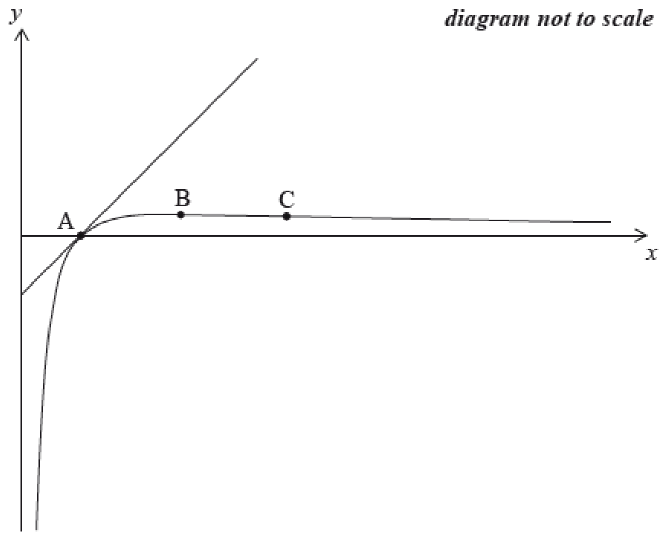
Find the acceleration of the body when it is 50 cm from O.

21. A curve has equation $\arctan x^2 + \arctan y^2 = \frac{\pi}{4}$. *[9 marks]*

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *x* and *y*.

(b) Find the gradient of the curve at the point where $x = \frac{1}{\sqrt{2}}$ and $y < 0$.

Consider the function
$f(x) = \frac{\ln x}{x},\ x > 0.$
The sketch below shows the graph of
$y = f(x)$ and its tangent at a point A.



22a. Show that [2 marks]
$f'(x) = \frac{1 - \ln x}{x^2}.$

22b. Find the coordinates of B, at which the curve reaches its maximum value. [3 marks]

22c. Find the coordinates of C, the point of inflexion on the curve. [5 marks]

22d. The graph of [4 marks]
$y = f(x)$ crosses the
x-axis at the point A.
Find the equation of the tangent to the graph of
f at the point A.

22e. The graph of [7 marks]
$y = f(x)$ crosses the
x-axis at the point A.
Find the area enclosed by the curve
$y = f(x)$, the tangent at A, and the line
$x = \mathrm{e}$.

The function f is defined by

$$f(x) = \begin{cases} 1 - 2x, & x \le 2 \\ \frac{3}{4}(x-2)^2 - 3, & x > 2 \end{cases}$$

23a. Determine whether or not [2 marks]
f is continuous.

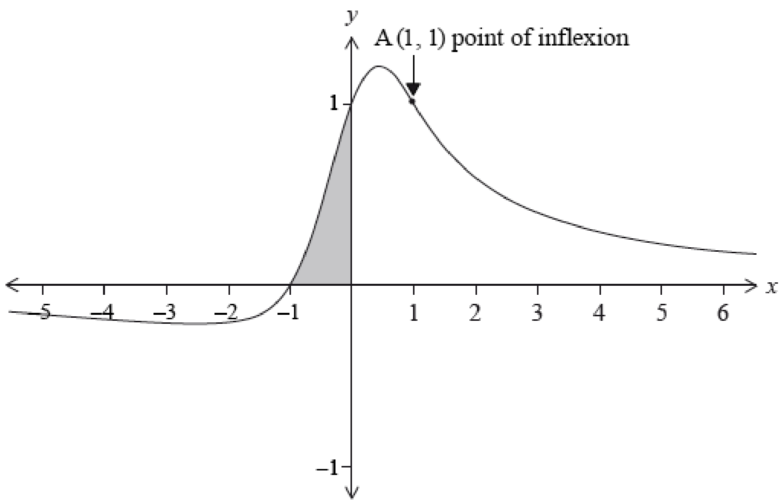
23b. The graph of the function g is obtained by applying the following transformations to the graph of f: [4 marks]

a reflection in the y–axis followed by a translation by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Find $g(x)$.

24. Use the substitution $x = a\sec\theta$ to show that $\int_{a\sqrt{2}}^{2a} \frac{\mathrm{d}x}{x^3\sqrt{x^2-a^2}} = \frac{1}{24a^3}\left(3\sqrt{3}+\pi-6\right)$. [7 marks]

The graph of the function $f(x) = \frac{x+1}{x^2+1}$ is shown below.



25a. Find $f'(x)$. [2 marks]

25b. Hence find the x-coordinates of the points where the gradient of the graph of f is zero. [1 mark]

25c. Find $f''(x)$ expressing your answer in the form $\frac{p(x)}{(x^2+1)^3}$, where $p(x)$ is a polynomial of degree 3. [3 marks]

The point (1, 1) is a point of inflexion. There are two other points of inflexion.

25d. Find the x-coordinates of the other two points of inflexion. [4 marks]

25e. Find the area of the shaded region. Express your answer in the form $\frac{\pi}{a} - \ln\sqrt{b}$, where a and b are integers. *[6 marks]*

26. A tranquilizer is injected into a muscle from which it enters the bloodstream. *[6 marks]*

The concentration C in mgl^{-1}, of tranquilizer in the bloodstream can be modelled by the function $C(t) = \frac{2t}{3+t^2},\ t \geq 0$ where t is the number of minutes after the injection.

Find the maximum concentration of tranquilizer in the bloodstream.

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27. By using the substitution $u = 1 + \sqrt{x}$, find $\int \frac{\sqrt{x}}{1+\sqrt{x}} \mathrm{d}x$. *[6 marks]*

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Consider two functions f and g and their derivatives f' and g'. The following table shows the values for the two functions and their derivatives at $x = 1, 2$ and 3.

x	1	2	3
$f(x)$	3	1	1
$f'(x)$	1	4	2
$g(x)$	2	1	4
$g'(x)$	4	2	3

Given that $p(x) = f(x)g(x)$ and $h(x) = g \circ f(x)$, find

28a. $p'(3)$; *[2 marks]*

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28b. $h'(2)$. *[4 marks]*

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The function f is defined as $f(x) = \mathrm{e}^{3x+1},\ x \in \mathbb{R}$.

29a. (i) Find $f^{-1}(x)$. *[4 marks]*

(ii) State the domain of f^{-1}.

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29b. The function g is defined as $g(x) = \ln x,\ x \in \mathbb{R}^+$. *[5 marks]*

The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

Find the coordinates of P.

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29c. The graph of $y = g(x)$ intersects the x-axis at the point Q. *[3 marks]*

Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$.

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29d. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. *[5 marks]*

Find the area of the region R.

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29e. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. *[6 marks]*

(i) Show that $g(x) \leq x - 1,\ x \in \mathbb{R}^{+}$.

(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \leq g(x),\ x \in \mathbb{R}^{+}$.

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30. Two cyclists are at the same road intersection. One cyclist travels north at $20\,\mathrm{km\,h^{-1}}$. The other cyclist travels west at $15\,\mathrm{km\,h^{-1}}$. *[5 marks]*

Use calculus to show that the rate at which the distance between the two cyclists changes is independent of time.

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A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time *t* seconds, is given by

$$v(t) = \begin{cases} 5 - (t-2)^2, & 0 \leq t \leq 4 \\ 3 - \frac{t}{2}, & t > 4 \end{cases}.$$

31a. Find the value of t when the particle is instantaneously at rest. *[2 marks]*

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31b. The particle returns to its initial position at $t = T$. *[5 marks]*

Find the value of *T*.

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Consider the triangle PQR where $\mathrm{Q\hat{P}R} = 30^\circ$, $\mathrm{PQ} = (x+2)\ \mathrm{cm}$ and $\mathrm{PR} = (5-x)^2\ \mathrm{cm}$, where $-2 < x < 5$.

32a. Show that the area, $A\ \mathrm{cm}^2$, of the triangle is given by $A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$. *[2 marks]*

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32b. (i) State $\frac{\mathrm{d}A}{\mathrm{d}x}$. *[3 marks]*

(ii) Verify that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$.

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32c. (i) Find $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle PQR. *[7 marks]*

(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum.

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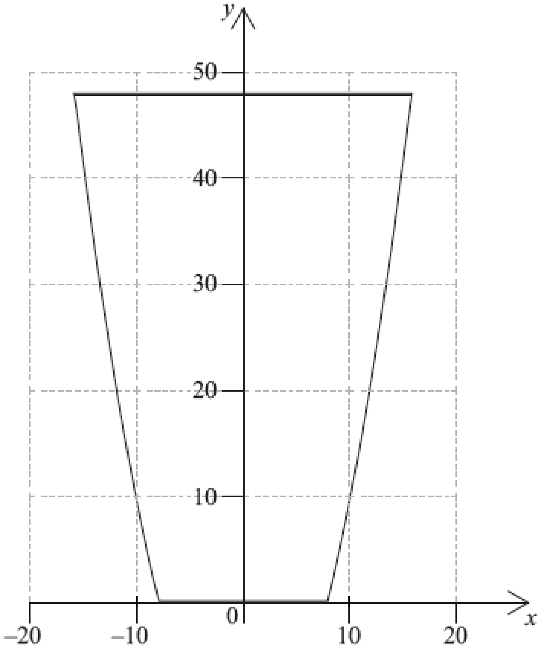
The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is 48 cm.

33a. If the container is filled with water to a depth of $h\,\mathrm{cm}$, show that the volume, $V\,\mathrm{cm}^3$, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. *[3 marks]*

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33b. Once empty, water is pumped back into the container at a rate of $8.5\ \mathrm{cm^3 s^{-1}}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)} \mathrm{cm^3 s^{-1}}$. *[3 marks]*

Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container.

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34. The function f is defined by $f(x) = \mathrm{e}^{-x} \cos x + x - 1$. *[7 marks]*

By finding a suitable number of derivatives of f, determine the first non-zero term in its Maclaurin series.

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