Topic 8 Part 1 *[382 marks]*

1a.

[3 marks]

Markscheme

$a = 1 \quad b = 8 \quad c = 4$

$d = 8 \quad e = 4 \quad f = 2$

$g = 4 \quad h = 2 \quad i = 1$ ***A3***

Note: Award ***A3*** for 9 correct answers, ***A2*** for 6 or more, and ***A1*** for 3 or more.

[3 marks]

Examiners report

The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

1b.

[3 marks]

Markscheme

Elements	Order
1	1
4, 11, 14	2
2, 7, 8, 13	4

A3

Note: Award ***A3*** for 8 correct answers, ***A2*** for 6 or more, and ***A1*** for 4 or more.

[3 marks]

Examiners report

The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

1c.

[2 marks]

Markscheme

$\{1,\ 4\},\ \{1,\ 11\},\ \{1,\ 14\}$ ***A1A1***

Note: Award ***A1*** for 1 correct answer and ***A2*** for all 3 (and no extras).

[2 marks]

Examiners report

The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

1d. *[4 marks]*

Markscheme

$\{1,\ 2,\ 4,\ 8\},\ \{1,\ 4,\ 7,\ 13\},$ ***A1A1***

$\{1,\ 4,\ 11,\ 14\}$ ***A2***

[4 marks]

Total [12 marks]

Examiners report

The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

2a. *[4 marks]*

Markscheme

METHOD 1

$f(x) = f(y) \Rightarrow \frac{4x+1}{2x-1} = \frac{4y+1}{2y-1}$ ***M1A1***

for attempting to cross multiply and simplify ***M1***

$(4x+1)(2y-1) = (2x-1)(4y+1)$

$\Rightarrow 8xy + 2y - 4x - 1 = 8xy + 2x - 4y - 1 \Rightarrow 6y = 6x$

$\Rightarrow x = y$ ***A1***

hence an injection ***AG***

METHOD 2

$f'(x) = \frac{4(2x-1)-2(4x+1)}{(2x-1)^2} = \frac{-6}{(2x-1)^2}$ ***M1A1***

$< 0 \quad (\text{for all } x \neq 0.5)$ ***R1***

therefore the function is decreasing on either side of the discontinuity

and $f(x) < 2$ and $x < 0.5$ for $f(x) > 0.5$ ***R1***

hence an injection ***AG***

Note: If a correct graph of the function is shown, and the candidate states this is decreasing in each part (or horizontal line test) and hence an injection, award ***M1A1R1***.

[4 marks]

Examiners report

Most students indicated an understanding of the concepts of Injection and Surjection, but many did not give rigorous proofs. Even where graphs were used, it was very common for a sketch to be so imprecise with no asymptotes marked that it was difficult to award even partial credit. Some candidates mistakenly stated that the function was not surjective because 0.5 was not in the domain.

2b.

[4 marks]

Markscheme

METHOD 1

attempt to solve $y = \frac{4x+1}{2x-1}$ ***M1***

$y(2x-1) = 4x+1 \Rightarrow 2xy - y = 4x+1$ ***A1***

$2xy - 4x = 1 + y \Rightarrow x = \frac{1+y}{2y-4}$ ***A1***

no value for $y = 2$ ***R1***

hence not a surjection ***AG***

METHOD 2

consider $y = 2$ ***A1***

attempt to solve $2 = \frac{4x+1}{2x-1}$ ***M1***

$4x - 2 = 4x + 1$ ***A1***

which has no solution ***R1***

hence not a surjection ***AG***

Note: If a correct graph of the function is shown, and the candidate states that because there is a horizontal asymptote at $y = 2$ then the function is not a surjection, award ***M1R1***.

[4 marks]

Total [8 marks]

Examiners report

Most students indicated an understanding of the concepts of Injection and Surjection, but many did not give rigorous proofs. Even where graphs were used, it was very common for a sketch to be so imprecise with no asymptotes marked that it was difficult to award even partial credit. Some candidates mistakenly stated that the function was not surjective because 0.5 was not in the domain.

3a.

[4 marks]

Markscheme

$q \circ p = (1\,3)(2\,5)(1\,2\,5)$ ***(M1)***

$= (1\,5\,3)$ ***M1A1A1***

Note: ***M1*** for an answer consisting of disjoint cycles, ***A1*** for $(1\,5\,3)$,

A1 for either (2) or (2) omitted.

Note: Allow $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$

If done in the wrong order and obtained $(1\,3\,2)$, award ***A2***.

[4 marks]

Examiners report

Many students were unable to start the question, seemingly as they did not understand the cyclic notation. Many of those that did understand found it quite straightforward to obtain good marks on this question.

3b. *[3 marks]*

Markscheme

(i) any cycle with length 4 *eg* (1234) ***A1***

(ii) any permutation with 2 disjoint cycles one of length 2 and one of length 3 *eg* $(1\ 2)\ (3\ 4\ 5)$ ***M1A1***

Note: Award ***M1A0*** for any permutation with 2 non-disjoint cycles one of length 2 and one of length 3.

Accept non cycle notation.

[3 marks]

Examiners report

Many students were unable to start the question, seemingly as they did not understand the cyclic notation. Many of those that did understand found it quite straightforward to obtain good marks on this question.

3c. *[4 marks]*

Markscheme

(i) $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$ ***M1A1***

(ii) $(2\ 3)$, $(2\ 4)$, $(2\ 5)$, $(3\ 4)$, $(3\ 5)$, $(4\ 5)$ ***(M1)***

6 ***A1***

Note: Award ***M1*** for at least one correct cycle.

[4 marks]

Total [11 marks]

Examiners report

Many students were unable to start the question, seemingly as they did not understand the cyclic notation. Many of those that did understand found it quite straightforward to obtain good marks on this question.

4a. *[4 marks]*

Markscheme

$f(e_G) = e_H \Rightarrow f(a * a^{-1}) = e_H$ ***M1***

f is a homomorphism so $f(a * a^{-1}) = f(a) \circ f(a^{-1}) = e_H$ ***M1A1***

by definition $f(a) \circ (f(a))^{-1} = e_H$ so $f(a^{-1}) = (f(a))^{-1}$ (by the left-cancellation law) ***R1***

[4 marks]

Examiners report

Part (a) was well answered by those who understood what a homomorphism is. However many candidates simply did not have this knowledge and consequently could not get into the question.

4b.

[4 marks]

Markscheme

from (a) $f(x^{-1}) = (f(x))^{-1}$

hence $f(x^{-1}) = (p^2)^{-1} = p^5$ ***M1A1***

[2 marks]

Examiners report

Part (b) was well answered, even by those who could not do (a). However, there were many who having not understood what a homomorphism is, made no attempt on this easy question part. Understandably many lost a mark through not simplifying p^{-2} to p^5.

4c.

[4 marks]

Markscheme

$f(x * y) = f(x) \circ f(y)$ (homomorphism) ***(M1)***

$p^2 \circ f(y) = p$ ***A1***

$f(y) = p^5 \circ p$ ***(M1)***

$= p^6$ ***A1***

[4 marks]

Total [10 marks]

Examiners report

Those who knew what a homomorphism is generally obtained good marks in part (c).

5a.

[2 marks]

Markscheme

in a **finite** group the order of any subgroup (exactly) divides the order of the group ***A1A1***

[2 marks]

Examiners report

Many students obtained just half marks in (a) for not stating the requirement of the order to be finite.

5b.

[3 marks]

Markscheme

METHOD 1

$(a * b^{-1}) * (b * a^{-1}) = a * b^{-1} * b * a^{-1} = a * e * a^{-1} = a * a^{-1} = e$ ***M1A1A1***

Note: ***M1*** for multiplying, ***A1*** for at least one of the next 3 expressions,

A1 for e.

Allow $(b * a^{-1}) * (a * b^{-1}) = b * a^{-1} * a * b^{-1} = b * e * b^{-1} = b * b^{-1} = e$.

METHOD 2

$(a * b^{-1})^{-1} = (b^{-1})^{-1} * a^{-1}$ ***M1A1***

$= b * a^{-1}$***A1***

[3 marks]

Examiners report

Part (b) should have been more straightforward than many found.

5c.

[8 marks]

Markscheme

$a * a^{-1} = e \in H$ (as
H is a subgroup) ***M1***

so aRa and hence R is reflexive

$aRb \Leftrightarrow a * b^{-1} \in H$. H is a subgroup so every element has an inverse in H so

$(a * b^{-1})^{-1} \in H$ ***R1***

$\Leftrightarrow b * a^{-1} \in H \Leftrightarrow bRa$ ***M1***

so R is symmetric

$aRb,\ bRc \Leftrightarrow a * b^{-1} \in H,\ b * c^{-1} \in H$ ***M1***

as H is closed $(a * b^{-1}) * (b * c^{-1}) \in H$ ***R1***

and using associativity ***R1***

$(a * b^{-1}) * (b * c^{-1}) = a * (b^{-1} * b) * c^{-1} = a * c^{-1} \in H \Leftrightarrow aRc$ ***A1***

therefore R is transitive

R is reflexive, symmetric and transitive

Note: Can be said separately at the end of each part.

hence it is an equivalence relation ***AG***

[8 marks]

Examiners report

In part (c) it was evident that most candidates knew what to do, but being a more difficult question fell down on a lack of rigour. Nonetheless, many candidates obtained full or partial marks on this question part.

5d.

[3 marks]

Markscheme

$aRb \Leftrightarrow a * b^{-1} \in H \Leftrightarrow a * b^{-1} = h \in H$ ***A1***

$\Leftrightarrow a = h * b \Leftrightarrow a \in Hb$ ***M1R1***

[3 marks]

Examiners report

Part (d) enabled many candidates to obtain, at least partial marks, but there were few students with the insight to be able to answer part (e) satisfactorily.

5e.

[3 marks]

Markscheme

(d) implies that the right cosets of H are equal to the equivalence classes of the relation in (c) ***R1***

hence the cosets partition G ***R1***

all the cosets are of the same size as the subgroup H so the order of G must be a multiple of $|H|$ ***R1***

[3 marks]

Total [19 marks]

Examiners report

Part (d) enabled many candidates to obtain, at least partial marks, but there were few students with the insight to be able to answer part (e) satisfactorily.

6a.

[4 marks]

Markscheme

$\circ$	p	q	r	s	t	u
p	p	q	r	s	t	u
q	q	p	t	u	r	s
r	r	u	p	t	s	q
s	s	t	u	p	q	r
t	t	s	q	r	u	p
u	u	r	s	q	p	t

(M1)A4

Note: Award ***M1*** for use of Latin square property and/or attempted multiplication, ***A1*** for the first row or column, ***A1*** for the squares of q, r and s, then ***A2*** for all correct.

Examiners report

The majority of candidates were able to complete the Cayley table correctly. Unfortunately, many wasted time and space, laboriously working out the missing entries in the table - the identity is p and the elements q, r and s are clearly of order two, so 14 entries can be filled in without any calculation. A few candidates thought t and u had order two.

6b. *[6 marks]*

Markscheme

(i) $p^{-1} = p,\ q^{-1} = q,\ r^{-1} = r,\ s^{-1} = s$ ***A1***

$t^{-1} = u,\ u^{-1} = t$ ***A1***

Note: Allow FT from part (a) unless the working becomes simpler.

(ii) using the table or direct multiplication ***(M1)***

the orders of $\{p,\ q,\ r,\ s,\ t,\ u\}$ are $\{1,\ 2,\ 2,\ 2,\ 3,\ 3\}$ ***A3***

Note: Award ***A1*** for two, three or four correct, ***A2*** for five correct.

[6 marks]

Examiners report

Generally well done. A few candidates were unaware of the definition of the order of an element.

6c. *[2 marks]*

Markscheme

(i) $\{p,\ r\}$ $(\text{and } (S_3,\ \circ))$ ***A1***

(ii) $\{p,\ u,\ t\}$ $(\text{and } (S_3,\ \circ))$ ***A1***

Note: Award ***A0A1*** if the identity has been omitted.

Award ***A0*** in (i) or (ii) if an extra incorrect “subgroup” has been included.

[2 marks]

Total [13 marks]

Examiners report

Often well done. A few candidates stated extra, and therefore incorrect subgroups.

7a. *[2 marks]*

Markscheme

attempt to solve $e^3y - ey \equiv y \bmod 7$ ***(M1)***

the only solution is $e = 5$ ***A1***

[2 marks]

Examiners report

Many candidates were not sufficiently familiar with modular arithmetic to complete this question satisfactorily. In particular, some candidates completely ignored the requirement that solutions were required to be found modulo 7, and returned decimal answers to parts (a) and (b). Very few candidates invoked Lagrange's theorem in part (b)(ii). Some candidates were under the misapprehension that a group had to be Abelian, so tested for commutativity in part (b)(ii). It was pleasing that many candidates realised that an identity had to be both a left and right identity.

7b.

[5 marks]

Markscheme

(i) attempt to solve $x^4 - x^2 \equiv 5 \bmod 7$ ***(M1)***

least solution is $x = 2$ ***A1***

(ii) suppose $(S, *)$ is a group with order 7 ***A1***

2 has order 2 ***A1***

since 2 does not divide 7, Lagrange's Theorem is contradicted ***R1***

hence, $(S, *)$ is not a group ***AG***

[5 marks]

Examiners report

Many candidates were not sufficiently familiar with modular arithmetic to complete this question satisfactorily. In particular, some candidates completely ignored the requirement that solutions were required to be found modulo 7, and returned decimal answers to parts (a) and (b). Very few candidates invoked Lagrange's theorem in part (b)(ii). Some candidates were under the misapprehension that a group had to be Abelian, so tested for commutativity in part (b)(ii). It was pleasing that many candidates realised that an identity had to be both a left and right identity.

7c.

[3 marks]

Markscheme

(5 is a left-identity), so need to test if it is a right-identity:

ie, is $y * 5 = y$? ***M1***

$1 * 5 = 0 \neq 1$ ***A1***

so 5 is not an identity ***A1***

[3 marks]

Total [10 marks]

Examiners report

Many candidates were not sufficiently familiar with modular arithmetic to complete this question satisfactorily. In particular, some candidates completely ignored the requirement that solutions were required to be found modulo 7, and returned decimal answers to parts (a) and (b). Very few candidates invoked Lagrange's theorem in part (b)(ii). Some candidates were under the misapprehension that a group had to be Abelian, so tested for commutativity in part (b)(ii). It was pleasing that many candidates realised that an identity had to be both a left and right identity.

8a.

[2 marks]

Markscheme

in a product of three consecutive integers either one or two are even ***R1***

and one is a multiple of 3 ***R1***

so the product is divisible by 6 ***AG***

[2 marks]

Examiners report

A surprising number of candidates thought that an example was sufficient evidence to answer this part.

8b.

[3 marks]

Markscheme

to test reflexivity, put $y = x$ ***M1***

then $x^2x - x = (x-1)x(x+1) \equiv 0 \bmod 6$ ***M1A1***

so xRx ***AG***

[3 marks]

Examiners report

Again, a lack of confidence with modular arithmetic undermined many candidates' attempts at this part.

8c.

[3 marks]

Markscheme

if $5Ry$ then $25y \equiv y \bmod 6$ ***(M1)***

$24y \equiv 0 \bmod 6$ ***(M1)***

the set of solutions is $\mathbb{Z}$ ***A1***

Note: Only one of the method marks may be implied.

[3 marks]

Examiners report

(c) and (d) Most candidates started these parts, but some found solutions as fractions rather than integers or omitted zero and/or negative integers.

8d.

[2 marks]

Markscheme

if $3Ry$ then $9y \equiv y \bmod 6$

$8y \equiv 0 \bmod 6 \Rightarrow 4y \equiv 0 \bmod 3$ ***(M1)***

the set of solutions is $3\mathbb{Z}$ (*ie* multiples of 3) ***A1***

[2 marks]

Examiners report

(c) and (d) Most candidates started these parts, but some found solutions as fractions rather than integers or omitted zero and/or negative integers.

8e.

[2 marks]

Markscheme

from part (c) $5R3$ ***A1***

from part (d) $3R5$ is false ***A1***

R is not symmetric ***AG***

Note: Accept other counterexamples.

[2 marks]

Total [12 marks]

Examiners report

Some candidates regarded R as an operation, rather than a relation, so returned answers of the form $aRb \neq bRa$.

9a. *[6 marks]*

Markscheme

(i) to test injectivity, suppose $f(x_1) = f(x_2)$ ***M1***

apply g to both sides $g\left(f(x_1)\right) = g\left(f(x_2)\right)$ ***M1***

$\Rightarrow x_1 = x_2$ ***A1***

so f is injective ***AG***

Note: Do not accept arguments based on "f has an inverse".

(ii) to test surjectivity, suppose $x \in X$ ***M1***

define $y = f(x)$ ***M1***

then $g(y) = g\left(f(x)\right) = x$ A1

so g is surjective ***AG***

[6 marks]

Examiners report

Those candidates who formulated the questions in terms of the basic definitions of injectivity and surjectivity were usually sucessful. Otherwise, verbal attempts such as '$f \text{ is one - to - one} \Rightarrow f \text{ is injective}$' or '$g$ is surjective because its range equals its codomain', received no credit. Some candidates made the false assumption that f and g were mutual inverses.

9b. *[3 marks]*

Markscheme

choose, for example, $f(x) = \sqrt{x}$ and $g(y) = y^2$ ***A1***

then $g \circ f(x) = \left(\sqrt{x}\right)^2 = x$ ***A1***

the function g is not injective as $g(x) = g(-x)$ ***R1***

[3 marks]

Total [9 marks]

Examiners report

Few candidates gave completely satisfactory answers. Some gave functions satisfying the mutual identity but either not defined on the given sets or for which g was actually a bijection.

10a. *[5 marks]*

Markscheme

closure: $\frac{n_1}{6^{i_1}} + \frac{n_2}{6^{i_2}} = \frac{6^{i_2} n_1 + 6^{i_1} n_2}{6^{i_1+i_2}} \in G$ ***A1R1***

Note: Award ***A1*** for RHS of equation. ***R1*** is for the use of two different, but not necessarily most general elements, and the result $\in G$ or equivalent.

identity: 0 ***A1***

inverse: $\frac{-n}{6^i}$ ***A1***

since associativity is given, $(G, +)$ forms a group ***R1AG***

Note: The ***R1*** is for considering closure, the identity, inverses and associativity.

[5 marks]

Examiners report

This part was generally well done. Where marks were lost, it was usually because a candidate failed to choose two different elements in the proof of closure.

10b. *[4 marks]*

Markscheme

it is required to show that H is a proper subset of G ***(M1)***

let $\frac{n}{3^i} \in H$ ***M1***

then $\frac{n}{3^i} = \frac{2^i n}{6^i} \in G$ hence H is a subgroup of G ***A1***

$H \neq G$ since $\frac{1}{6} \in G$ but $\frac{1}{6} \notin H$ ***A1***

Note: The final ***A1*** is only dependent on the first ***M1***.

hence, H is a proper subgroup of G ***AG***

[4 marks]

Examiners report

Only a few candidates realised that they did not have to prove that H is a group - that was stated in the question. Some candidates tried to invoke Lagrange's theorem, even though G is an infinite group.

10c.

[7 marks]

Markscheme

consider $\phi(g_1 + g_2) = (g_1 + g_2) + (g_1 + g_2)$ ***M1***

$= (g_1 + g_1) + (g_2 + g_2) = \phi(g_1) + \phi(g_2)$ ***A1***

(hence ϕ is a homomorphism)

injectivity: let $\phi(g_1) = \phi(g_2)$ ***M1***

working within $\mathbb{Q}$ we have $2g_1 = 2g_2 \Rightarrow g_1 = g_2$ ***A1***

surjectivity: considering even and odd numerators ***M1***

$\phi\left(\frac{n}{6^i}\right) = \frac{2n}{6^i}$ and $\phi\left(\frac{3(2n+1)}{6^{i+1}}\right) = \frac{2n+1}{6^i}$ ***A1A1***

hence ϕ is an isomorphism ***AG***

[7 marks]

Total [16 marks]

Examiners report

Many candidates showed that the mapping is injective. Most attempts at proving surjectivity were unconvincing. Those candidates who attempted to establish the homomorphism property sometimes failed to use two different elements.

11a.

[2 marks]

Markscheme

$*$ is closed ***A1***

because

$1 + ab \in \mathbb{N}$ (when

$a,b \in \mathbb{N}$) ***R1***

[2 marks]

Examiners report

For the commutative property some candidates began by setting
$a * b = b * a$. For the identity element some candidates confused
$e * a$ and
ea stating
$ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

11b.

[2 marks]

Markscheme

consider

$a * b = 1 + ab = 1 + ba = b * a$ ***M1A1***

therefore

$*$ is commutative

[2 marks]

Examiners report

For the commutative property some candidates began by setting $a * b = b * a$. For the identity element some candidates confused $e * a$ and ea stating $ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

11c. *[3 marks]*

Markscheme

EITHER

$a * (b * c) = a * (1 + bc) = 1 + a(1 + bc)$ $(= 1 + a + abc)$ ***A1***

$(a * b) * c = (1 + ab) * c = 1 + c(1 + ab)$ $(= 1 + c + abc)$ ***A1***

(these two expressions are unequal when $a \neq c$) so

$*$ is not associative ***R1***

OR

proof by counter example, for example

$1 * (2 * 3) = 1 * 7 = 8$ ***A1***

$(1 * 2) * 3 = 3 * 3 = 10$ ***A1***

(these two numbers are unequal) so

$*$ is not associative ***R1***

[3 marks]

Examiners report

For the commutative property some candidates began by setting $a * b = b * a$. For the identity element some candidates confused $e * a$ and ea stating $ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

11d. *[3 marks]*

Markscheme

let e denote the identity element; so that

$a * e = 1 + ae = a$ gives

$e = \frac{a-1}{a}$ (where

$a \neq 0$) ***M1***

then any valid statement such as:

$\frac{a-1}{a} \notin \mathbb{N}$ or e is not unique ***R1***

there is therefore no identity element ***A1***

Note: Award the final ***A1*** only if the previous ***R1*** is awarded.

[3 marks]

Examiners report

For the commutative property some candidates began by setting $a * b = b * a$. For the identity element some candidates confused $e * a$ and ea stating $ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

12a. *[4 marks]*

Markscheme

$\times_{14}$	**1**	**3**	**5**	**7**	**9**	**11**	**13**
1	1	3	5	7	9	11	13
3	3	9	1	7	13	5	11
5	5	1	11	7	3	13	9
7	7	7	7	7	7	7	7
9	9	13	3	7	11	1	5
11	11	5	13	7	1	9	3
13	13	11	9	7	5	3	1

A4

Note: Award ***A3*** for one error, ***A2*** for two errors, ***A1*** for three errors, ***A0*** for four or more errors.

[4 marks]

Examiners report

There were no problems with parts (a), (b) and (d).

12b. *[1 mark]*

Markscheme

any valid reason, for example ***R1***

not a Latin square

7 has no inverse

[1 mark]

Examiners report

There were no problems with parts (a), (b) and (d).

12c. *[5 marks]*

Markscheme

delete 7 (so that $G = \{1, 3, 5, 9, 11, 13\}$) ***A1***

closure – evident from the table ***A1***

associative because multiplication is associative ***A1***

the identity is 1 ***A1***

13 is self-inverse, 3 and 5 form an inverse

pair and 9 and 11 form an inverse pair ***A1***

the four conditions are satisfied so that

$\{G, \times_{14}\}$ is a group ***AG***

[5 marks]

Examiners report

There were no problems with parts (a), (b) and (d) but in part (c) candidates often failed to state that the set was associative under the operation because multiplication is associative. Likewise they often failed to list the inverses of each element simply stating that the identity was present in each row and column of the Cayley table.

12d. *[4 marks]*

Markscheme

Element	Order
1	1
3	6
5	6
9	3
11	3
13	2

A4

Note: Award ***A3*** for one error, ***A2*** for two errors, ***A1*** for three errors, ***A0*** for four or more errors.

[4 marks]

Examiners report

The majority of candidates did not answer part (d) correctly and often simply listed all subsets of order 2 and 3 as subgroups.

12e. *[2 marks]*

Markscheme

$\{1\}$

$\{1,\ 13\}$

$\{1,\ 9,\ 11\}$ ***A1A1***

[2 marks]

Examiners report

[N/A]

13a. *[5 marks]*

$\mathbb{R}$

Examiners report

For the most part the piecewise function was correctly graphed. Even though the majority of candidates knew that it is required to establish that the function is an injection and a surjection in order to prove it is a bijection, many just quoted the definition of injection or surjection and did not relate their reason to the graph.

13b. *[8 marks]*

Markscheme

considering the linear section, put

$y = 2x + 1$ or

$x = 2y + 1$ ***(M1)***

$x = \frac{y-1}{2}$ or

$y = \frac{x-1}{2}$ ***A1***

so

$f^{-1}(x) = \frac{x-1}{2},\ x \leqslant 5$ ***A1***

EITHER

$y = (x-1)^2 + 4$ ***M1A1***

$(x-1)^2 = y - 4$

$x = 1 \pm \sqrt{y-4}$ ***A1***

$x = 1 + \sqrt{y-4}$

taking the + sign to give the right hand half of the parabola ***R1***

so

$f^{-1}(x) = 1 + \sqrt{x-4},\ x > 5$ ***A1***

OR

considering the quadratic section, put

$y = x^2 - 2x + 5$

$x^2 - 2x + 5 - y = 0$ ***M1***

$x = \frac{2 \pm \sqrt{4 - 4(5-y)}}{2}\ (= 1 \pm \sqrt{y-4})$ ***M1A1***

taking the + sign to give the right hand half of the parabola ***R1***

so

$f^{-1}(x) = \frac{2 + \sqrt{4 - 4(5-x)}}{2},\ x > 5\ (f^{-1}(x) = 1 + \sqrt{x-4},\ x > 5)$ ***A1***

Note: Award ***A0*** for omission of

$f^{-1}(x)$ or omission of the domain. Penalise the omission of the notation

$f^{-1}(x)$ only once. The domain must be seen in both cases.

[8 marks]

14a. *[6 marks]*

Markscheme

reflexive:

$a(a+1) \equiv a(a+1)(\bmod 5)$, therefore aRa ***R1***

symmetric:

$aRb \Rightarrow a(a+1) = b(b+1) + 5N$ ***M1***

$\Rightarrow b(b+1) = a(a+1) - 5N \Rightarrow bRa$ ***A1***

transitive:

EITHER

aRb and $bRc \Rightarrow a(a+1) = b(b+1) + 5M$ and $b(b+1) = c(c+1) + 5N$ ***M1***

it follows that

$a(a+1) = c(c+1) + 5(M+N) \Rightarrow aRc$ ***M1A1***

OR

aRb and $bRc \Rightarrow a(a+1) \equiv b(b+1)(\bmod 5)$ and

$b(b+1) \equiv c(c+1)(\bmod 5)$ ***M1***

$a(a+1) - b(b+1) \equiv 0(\bmod 5)$; $b(b+1) - c(c+1) \equiv 0(\bmod 5)$ ***M1***

$a(a+1) - c(c+1) \equiv 0 \bmod 5 \Rightarrow a(a+1) \equiv c(c+1) \bmod 5 \Rightarrow aRc$ ***A1***

[6 marks]

Examiners report

Candidates knew the properties of equivalence relations but did not show sufficient working out in the transitive case. Others did not do the modular arithmetic correctly, still others omitted the $\bmod (5)$ in part or throughout.

14b. *[3 marks]*

Markscheme

the equivalence can be written as

$a^2 + a - b^2 - b \equiv 0(\bmod 5)$ ***M1***

$(a-b)(a+b) + a - b \equiv 0(\bmod 5)$ ***M1A1***

$(a-b)(a+b+1) \equiv 0(\bmod 5)$ ***AG***

[3 marks]

Examiners report

Candidates knew the properties of equivalence relations but did not show sufficient working out in the transitive case. Others did not do the modular arithmetic correctly, still others omitted the $\bmod (5)$ in part or throughout.

14c. *[4 marks]*

Markscheme

the equivalence classes are

$\{1, 3, 6, 8, 11\}$

$\{2, 7, 12\}$

$\{4, 5, 9, 10\}$ ***A4***

Note: Award ***A3*** for 2 correct classes, ***A2*** for 1 correct class.

[4 marks]

Examiners report

Candidates knew the properties of equivalence relations but did not show sufficient working out in the transitive case. Others did not do the modular arithmetic correctly, still others omitted the $\bmod (5)$ in part or throughout.

15.

[12 marks]

Markscheme

(a) yes ***A1***

because the Cayley table only contains elements of S ***R1***

[2 marks]

(b) yes ***A1***

because the Cayley table is symmetric ***R1***

[2 marks]

(c) no ***A1***

because there is no row (and column) with 1, 2, 3, 4, 5 ***R1***

[2 marks]

(d) attempt to calculate

$(a\Delta b)\Delta c$ and

$a\Delta(b\Delta c)$ for some

$a,\ b,\ c \in S$ ***M1***

counterexample: for example,

$(1\Delta 2)\Delta 3 = 2$

$1\Delta(2\Delta 3) = 1$ ***A1***

Δ is not associative ***A1***

Note: Accept a correct evaluation of

$(a\Delta b)\Delta c$ and

$a\Delta(b\Delta c)$ for some

$a,\ b,\ c \in S$ for the ***M1***.

[3 marks]

(e) for example, attempt to enumerate

$4\Delta b$ for b = 1, 2, 3, 4, 5 and obtain (3, 2, 1, 4, 1) ***(M1)***

find

$(a,\ b) \in \{(2,\ 2),\ (2,\ 3)\}$ for

$a \neq 4$ (or equivalent) ***A1A1***

Note: Award ***M1A1A0*** if extra 'solutions' are listed.

[3 marks]

Total [12 marks]

Examiners report

[N/A]

16a. *[9 marks]*

Markscheme

(i)

$0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ ***A2***

Notes: ***A2*** for all correct, ***A1*** for three to five correct.

(ii) **EITHER**

closure: if

$s_1,\ s_2 \in S$, then

$s_1 = \frac{m}{2}$ and

$s_2 = \frac{n}{2}$ for some

$m,\ n \in$ ¢. ***M1***

Note: Accept two distinct examples (*eg*,

$\frac{1}{2} + \frac{1}{2} = 1;\ \frac{1}{2} + 1 = \frac{3}{2}$) for the ***M1***.

$s_1 + s_2 = \frac{m+n}{2} \in S$ ***A1***

OR

the sum of two half-integers ***A1***

is a half-integer ***R1***

THEN

identity: 0 is the (additive) identity ***A1***

inverse:

$s + (-s) = 0$, where

$-s \in S$ ***A1***

it is associative (since

$S \subset$ §) ***A1***

the group axioms are satisfied ***AG***

(iii) **EITHER**

the set is not closed under multiplication, ***A1***

for example,

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, but

$\frac{1}{4} \notin S$ ***R1***

OR

not every element has an inverse, ***A1***

for example, 3 does not have an inverse ***R1***

[9 marks]

Examiners report

[N/A]

16b. *[10 marks]*

Markscheme

(i) reflexive: consider

$3s + 5s$ ***M1***

$= 8s \in ¢ \Rightarrow$ reflexive ***A1***

symmetric: if

$s_1 R s_2$, consider

$3s_2 + 5s_1$ ***M1***

for example,

$= 3s_1 + 5s_2 + (2s_1 - 2s_2) \in ¢ \Rightarrow$symmetric ***A1***

transitive: if

$s_1 R s_2$ and

$s_2 R s_3$, consider ***(M1)***

$3s_1 + 5s_3 = (3s_1 + 5s_2) + (3s_2 + 5s_3) - 8s_2$ ***M1***

$\in ¢ \Rightarrow$transitive ***A1***

so *R* is an equivalence relation ***AG***

(ii)

$C_1 = ¢$ ***A1***

$C_2 = \left\{\pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{5}{2},\ \ldots\right\}$ ***A1A1***

Note: ***A1*** for half odd integers and ***A1*** for ±.

[10 marks]

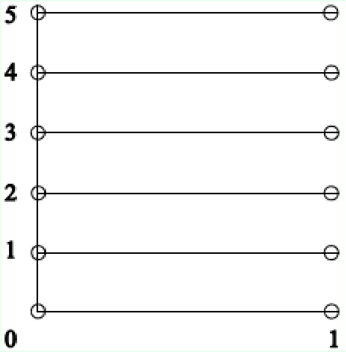
Examiners report

[N/A]

17a. *[5 marks]*

Markscheme

(i)

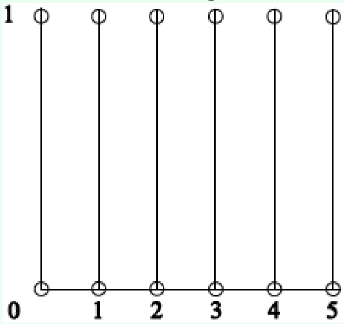


correct horizontal lines *A1*

correctly labelled axes *A1*

clear indication that the endpoints are not included *A1*

(ii)



fully correct diagram *A1*

Note: Do not penalize the inclusion of endpoints twice.

(iii) the intersection is empty *A1*

[5 marks]

Examiners report

[N/A]

17b. *[10 marks]*

Markscheme

(i) range

$(f) =]0,\ 1[\cup]1,\ 2[\cup \mathrm{L} \cup]5,\ 6[$ ***A1A1***

Note: ***A1*** for six intervals and ***A1*** for fully correct notation.

Accept

$0 < x < 6,\ x \neq 0, 1, 2, 3, 4, 5, 6$

(ii) range

$(g) = [0,\ 5[$ ***A1***

(iii) Attempt at solving

$f(x_1,\ y_1) = f(x_2,\ y_2)$ ***M1***

$f(x,\ y) \in]y,\ y+1[\Rightarrow y_1 = y_2$ ***M1***

and then

$x_1 = x_2$ ***A1***

so

f is injective ***AG***

(iv)

$f^{-1}(\pi) = (\pi - 3,\ 3)$ ***A1A1***

(v) solutions: (0.5, 1), (0.25, 2),

$\left(\frac{1}{6},\ 3\right)$, (0.125, 4), (0.1, 5) ***A2***

Note: ***A2*** for all correct, ***A1*** for 2 correct.

[10 marks]

Examiners report

[N/A]

18a. *[2 marks]*

Markscheme

$f(g) = f(e_G g) = f(e_G) f(g)$ for

$g \in G$ ***M1A1***

$\Rightarrow f(e_G) = e_H$ ***AG***

[2 marks]

Examiners report

[N/A]

18b. *[6 marks]*

Markscheme

(i) closure: let k_1 and $k_2 \in K$, then $f(k_1k_2) = f(k_1)f(k_2)$ ***M1A1***

$= e_He_H = e_H$ ***A1***

hence

$k_1k_2 \in K$ ***R1***

(ii) K is non-empty because

e_G belongs to K ***R1***

a closed non-empty subset of a finite group is a subgroup ***R1AG***

[6 marks]

Examiners report

[N/A]

18c. *[6 marks]*

Markscheme

(i)

$f(gkg^{-1}) = f(g)f(k)f(g^{-1})$ ***M1***

$= f(g)e_Hf(g^{-1}) = f(gg^{-1})$ ***A1***

$= f(e_G) = e_H$ ***A1***

$\Rightarrow gkg^{-1} \in K$ ***AG***

(ii) clear definition of both left and right cosets, seen somewhere. ***A1***

use of part (i) to show

$gK \subseteq Kg$ ***M1***

similarly

$Kg \subseteq gK$ ***A1***

hence

$gK = Kg$ ***AG***

[6 marks]

Examiners report

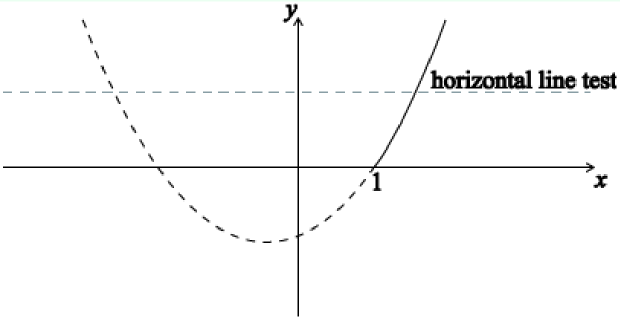
[N/A]

19. [14 marks]

Markscheme

(a) **Method 1**

sketch of the graph of
f ***(M1)***



range of
$f =$ co-domain, therefore
f is surjective ***R1***
graph of
f passes the horizontal line test, therefore
f is injective ***R1***
therefore
f is bijective ***AG***

Note: Other explanations may be given (*eg* use of derivative or description of parabola).

Method 2

Injective:
$f(a) = f(b) \Rightarrow a = b$ ***M1***
$(a - 1)(a + 2) = (b - 1)(b + 2)$
$a^2 + a = b^2 + b$
solving for
a by completing the square, or the quadratic formula, ***A1***
$a = b$
surjective: for all
$y \in \mathbb{R}^+$ there exists
$x \in]1,\ \infty[$ such that
$f(x) = y$
solving
$y = x^2 + x - 2$ for x,
$x = \frac{\sqrt{4y+9}-1}{2}$. For all positive real
y, the minimum value for
$\sqrt{4y + 9}$ is
3. Hence,
$x \geqslant 1$ ***R1***
since
f is both injective and surjective,
f is bijective. ***AG***

Method 3

f is bijective if and only if
f has an inverse ***(M1)***
solving
$y = x^2 + x - 2$ for
$x,\ x = \frac{\sqrt{4y+9}-1}{2}$. For all positive real

y, the minimum value for

$\sqrt{4y+9}$ is

3. Hence,

$x \geqslant 1$ ***R1***

$f^{-1}(x) = \frac{\sqrt{4x+9}-1}{2}$ ***R1***

f has an inverse, hence

f is bijective ***AG***

[3 marks]

(b) (i) attempt to find counterexample ***(M1)***

eg

$g(x,\,y) = g(y,\,x),\ x \neq y$ ***A1***

g is not injective ***R1***

(ii)

$-1 \leqslant \sin(x+y) \leqslant 1$ ***(M1)***

range of

g is

$[-1,\,1] \times \mathbb{R} \neq \mathbb{R} \times \mathbb{R}$ ***A1***

g is not surjective ***R1***

[6 marks]

(c) let

$h(x,\,y) = (u,\,v)$

then

$u = x + 3y$

$v = 2x + y$ ***(M1)***

solving simultaneous equations ***(M1)***

eg

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$x = \frac{-u+3v}{5} y = \frac{2u-v}{5}$ ***A1***

hence

$h^{-1}(x,\,y) = \left(\frac{-x+3y}{5},\, \frac{2x-y}{5}\right)$ ***A1***

as this expression is defined for any values of

$(x,\,y) \in \mathbb{R} \times \mathbb{R}$ ***R1***

the inverse of

h exists ***AG***

[5 marks]

Examiners report

For part (a), given the command term ‘show that’ and the number of marks for this part, the best approach is a graphical one, i.e., an informal approach. Many candidates chose an algebraic approach and generally made correct statements for injective and surjective. However, they often did not follow through with the necessary algebraic manipulation to make a valid conclusion. In part (b), many candidates were not able to provide valid counter-examples. In part (c) It was obvious that quite a few candidates had not seen this type of function before. Those that were able to find the inverse generally did not justify their result, and hence could not earn the final R mark.

20a. *[9 marks]*

Markscheme

(i) the order of
a is a divisor of the order of
G ***(M1)***
since the order of
G is 12, the order of
a must be 1, 2, 3, 4, 6 or 12 ***A1***
the order cannot be 1, 2, 3 or 6, since
$a^6 \neq e$ ***R1***
the order cannot be 4, since
$a^4 \neq e$ ***R1***
so the order of
a must be 12
therefore,
a is a generator of
G, which must therefore be cyclic ***R1***
another generator is *eg*
$a^{-1},\ a^5,\ \ldots$ ***A1***
[6 marks]

(ii)
$H = \{e,\ a^4,\ a^8\}$ ***(A1)***

	e	a^4	a^8
e	e	a^4	a^8
a^4	a^4	a^8	e
a^8	a^8	e	a^4

M1A1

[3 marks]

Examiners report

In part (a), many candidates could not provide a logical sequence of steps to show that
G is cyclic. In particular, although they correctly quoted Lagrange's theorem, they did not always consider all the orders of a, i.e., all the factors of 12, omitting in particular 1 as a factor. Some candidates did not state the second generator, in particular
a^{-1}. Very few candidates were successful in finding the required subgroup, although they were obviously familiar with setting up a Cayley table.

20b. *[2 marks]*

Markscheme

no ***A1***

eg the group of symmetries of a triangle
S_3 is not cyclic but all its (proper) subgroups are cyclic
eg the Klein four-group is not cyclic but all its (proper) subgroups are cyclic ***R1***
[2 marks]

Examiners report

21.

[9 marks]

Markscheme

(a)

R is reflexive as

$x^{-1} * x = e \in H \Rightarrow xRx$ for any

$x \in G$ ***A1***

if

xRy then

$y^{-1} * x = h \in H$

but

$h \in H \Rightarrow h^{-1} \in H$, *ie*,

$\underbrace{(y^{-1} * x)^{-1}}_{x^{-1}*y} \in H$ ***M1***

therefore

yRx

R is symmetric ***A1***

if

xRy then

$y^{-1} * x = h \in H$ and if

yRz then

$z^{-1} * y = k \in H$ ***M1***

$k * h \in H$, *ie*,

$\underbrace{(z^{-1} * y) * (y^{-1} * x)}_{z^{-1}*x} \in H$ ***A1***

therefore

xRz

R is transitive ***A1***

so

R is an equivalence relation on

G ***AG***

[6 marks]

(b)

$xRe \Leftrightarrow e^{-1} * x \in H$ ***M1***

$x \in H$ ***A1***

$[e] = H$ ***A1*** ***N0***

[3 marks]

Examiners report

Part (a) was fairly well answered by many candidates. They knew how to apply the equivalence relations axioms in this particular example. Part (b) however proved to be very challenging and hardly any correct answers were seen.

22.

[11 marks]

Markscheme

(a)

$(A\backslash B)\cup(B\backslash A)=(A\cap B')\cup(B\cap A')$ ***(M1)***

$=((A\cap B')\cup B)\cap((A\cap B')\cup A')$ ***M1***

$=\left((A\cup B)\cap\underbrace{(B'\cup B)}_{U}\right)\cap\left(\underbrace{(A\cup A')}_{U}\cap(B'\cup A')\right)$ ***A1***

$=(A\cup B')\cap(B'\cup A')=(A\cup B)\cap(A\cap B)'$ ***A1***

$=(A\cup B)\backslash(A\cap B)$ ***AG***

[4 marks]

(b) (i) false ***A1***

counterexample

eg

$D\backslash C=\{2\}\notin S$ ***R1***

(ii) true ***A1***

as

$A\cap D=A,\ B\cap D=B,\ C\cap D=C$ and

$D\cap D=D,$

D is the identity ***R1***

A (or

B or

C) has no inverse as

$A\cap X=D$ is impossible ***R1***

(iii) false ***A1***

when

$Y=D$ the equation has more than one solution (four solutions) ***R1***

[7 marks]

Examiners report

For part (a), candidates who chose to prove the given statement using the properties of Sets were often successful with the proof. Some candidates chose to use the definition of equality of sets, but made little to no progress. In a few cases candidates attempted to use Venn diagrams as a proof. Part (b) was challenging for most candidates, and few correct answers were seen.

23a.

[2 marks]

Markscheme

closure, identity, inverse ***A2***

Note: Award ***A1*** for two correct properties, ***A0*** otherwise.

[2 marks]

Examiners report

This was on the whole a well answered question and it was rare for a candidate not to obtain full marks on part (a). In part (b) the vast majority of candidates were able to show that the set satisfied the properties of a group apart from associativity which they were also familiar with. Virtually all candidates knew the difference between commutativity and associativity and were able to distinguish between the two. Candidates were familiar with Lagrange's Theorem and many were able to see how it did not apply in the case of this problem. Many candidates found a solution method to part (iii) of the problem and obtained full marks.

23b. *[15 marks]*

Markscheme

(i) closure: there are no extra elements in the table ***R1***
identity: s is a (left and right) identity ***R1***
inverses: all elements are self-inverse ***R1***
commutative: no, because the table is not symmetrical about the leading diagonal, or by counterexample ***R1***
associativity: for example,
$(pq)t = rt = p$ ***M1A1***
not associative because
$p(qt) = pr = t \neq p$ ***R1***
Note: Award ***M1A1*** for 1 complete example whether or not it shows non-associativity.

(ii)
$\{s, p\}, \{s, q\}, \{s, r\}, \{s, t\}$ ***A2***
Note: Award ***A1*** for 2 or 3 correct sets.

as 2 does not divide 5, Lagrange's theorem would have been contradicted if T had been a group ***R1***

(iii) any attempt at trying values ***(M1)***
the solutions are q, r, s and t ***A1A1A1A1***
Note: Deduct ***A1*** if p is included.

[15 marks]

Examiners report

This was on the whole a well answered question and it was rare for a candidate not to obtain full marks on part (a). In part (b) the vast majority of candidates were able to show that the set satisfied the properties of a group apart from associativity which they were also familiar with. Virtually all candidates knew the difference between commutativity and associativity and were able to distinguish between the two. Candidates were familiar with Lagrange's Theorem and many were able to see how it did not apply in the case of this problem. Many candidates found a solution method to part (iii) of the problem and obtained full marks.

24a. *[3 marks]*

Markscheme

$P = \{1, 2, 3\}$
$Q' = \{1, 3, 5, 7, 9\}$
so
$P \cap Q' = \{1, 3\}$ ***(M1)(A1)***
so
$(P \cap Q')' = \{2, 4, 5, 6, 7, 8, 9, 10\}$ ***A1***
[3 marks]

Examiners report

This was also a well answered question with many candidates obtaining full marks on both parts of the problem. Some candidates attempted to use a factorial rather than a sum of combinations to solve part (b) (ii) and this led to incorrect answers.

24b. *[5 marks]*

Markscheme

(i)
$P^* = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3), \emptyset\}$ ***A2***
Note: Award ***A1*** if one error, ***A0*** for two or more.

(ii)
R^* contains: the empty set (1 element); sets containing one element (8 elements); sets containing two elements ***(M1)***
$= \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \ldots \binom{8}{8}$ ***(A1)***
$= 2^8 \ (= 256)$ ***A1***
Note: ***FT*** in (ii) applies if no empty set included in (i) and (ii).

[5 marks]

Examiners report

This was also a well answered question with many candidates obtaining full marks on both parts of the problem. Some candidates attempted to use a factorial rather than a sum of combinations to solve part (b) (ii) and this led to incorrect answers.

25a. *[6 marks]*

Markscheme

reflexive:
$a^3 - a^3 = 0$, $\Rightarrow R$ is reflexive ***R1***
symmetric: if
$a^3 \equiv b^3 (\bmod 7)$, then
$b^3 \equiv a^3 (\bmod 7)$ ***M1***
$\Rightarrow R$ is symmetric ***R1***
transitive:
$a^3 = b^3 + 7n$ and
$b^3 = c^3 + 7m$ ***M1***
then
$a^3 = c^3 + 7(n + m)$
$\Rightarrow a^3 \equiv c^3 (\bmod 7)$ ***R1***
$\Rightarrow R$ is transitive ***A1***
and is an equivalence relation ***AG***
Note: Allow arguments that use
$a^3 - b^3 \equiv 0 (\bmod 7)$ *etc.*

[6 marks]

Examiners report

Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

25b. *[2 marks]*

Markscheme

$\{0,\ 7,\ 14,\ 21,\ \ldots\}$ ***A2***
[2 marks]

Examiners report

Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

25c. *[3 marks]*

Markscheme

$\{1,\ 2,\ 4,\ 8,\ 9,\ 11\}$ ***A3***
Note: Deduct 1 mark for each error or omission.

[3 marks]

Examiners report

Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

25d.

[3 marks]

Markscheme

consider

$(n+7)^3 = n^3 + 21n^2 + 147n + 343 = n^3 + 7N$ ***M1A1***

$\Rightarrow n^3 \equiv (n+7)^3 (\bmod 7) \Rightarrow n$ and

$n+7$ are in the same equivalence class ***R1***

[3 marks]

Examiners report

Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

26a.

[2 marks]

Markscheme

non-S: for example –2 does not belong to the range of g ***R1***

non-I: for example

$g(1) = g(-1) = 0$ ***R1***

Note: Graphical arguments have to recognize that we are dealing with sets of integers and not all real numbers

[2 marks]

Examiners report

Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

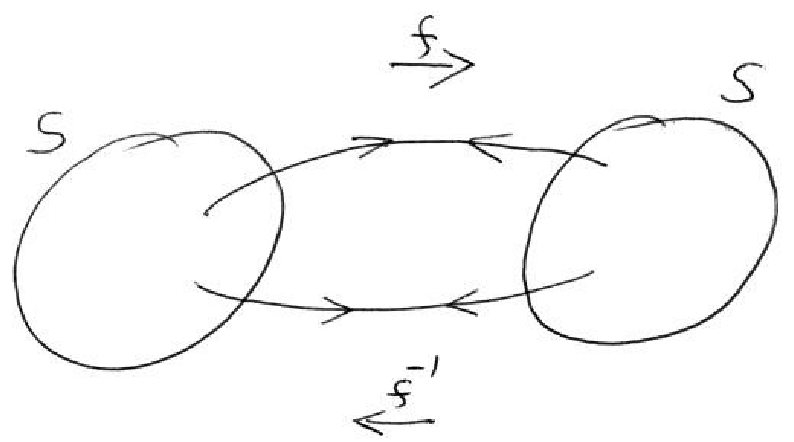
26b.

[2 marks]

Markscheme

as f is injective

$n\left(f(S)\right) = n(S)$ ***A1***



R1

Note: Accept alternative explanations.

f is surjective ***AG***

[2 marks]

Examiners report

Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

26c.

[3 marks]

Markscheme

for example,

$h(n) = n + 1$ ***A1***

Note: Only award the ***A1*** if the function works.

I:

$n + 1 = m + 1 \Rightarrow n = m$ ***R1***

non-S: 1 has no pre-image as

$0 \notin \mathbb{Z}^+$ ***R1***

[3 marks]

Examiners report

Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

27a.

[4 marks]

Markscheme

(i) not reflexive *e.g.* $1 + 1 = 2$ ***R1***

(ii) symmetric since $x + y = y + x$ ***R1***

(iii) *e.g.* $1 + 11 > 7$, $11 + 2 > 7$ but $1 + 2 = 3$, so not transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

Examiners report

Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

27b.

[4 marks]

Markscheme

(i) reflexive since

$x - x = 0$ ***R1***

(ii) symmetric since

$|x - y| = |y - x|$ ***R1***

(iii) *e.g.* 1R2, 2R3 but $1 - 3 = -2$, so not transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

Examiners report

Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

27c. *[4 marks]*

Markscheme

(i) reflexive since

$x^2 > 0$ ***R1***

(ii) symmetric since

$xy = yx$ ***R1***

(iii)

$xy > 0$ and $yz > 0 \Rightarrow xy^2z > 0 \Rightarrow xz > 0$ since $y^2 > 0$, so transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

Examiners report

Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

27d. *[4 marks]*

Markscheme

(i) reflexive since

$\frac{x}{x} = 1$ ***R1***

(ii) not symmetric *e.g.*

$\frac{2}{1} = 2$ but $\frac{1}{2} = 0.5$ ***R1***

(iii)

$\frac{x}{y} \in \mathbb{Z}$ and $\frac{y}{z} \in \mathbb{Z} \Rightarrow \frac{xy}{yz} = \frac{x}{z} \in \mathbb{Z}$, so transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

Examiners report

Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

27e. *[3 marks]*

Markscheme

only (c) is an equivalence relation ***(A1)***

the equivalence classes are

$\{1, 2, 3,\ldots\}$ and $\{-1,-2,-3,\ldots\}$ ***A1A1***

[3 marks]

Examiners report

Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

28a. *[1 mark]*

Markscheme

$\emptyset, \{a\}, \{b\}, \{a, b\}$ ***A1***

[1 mark]

Examiners report

A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

28b.

[3 marks]

Markscheme

Δ	$\varnothing$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\varnothing$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\varnothing$	$\{a, b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{a, b\}$	$\varnothing$	$\{a\}$
$\{a, b\}$	$\{a, b\}$	$\{b\}$	$\{a\}$	$\varnothing$

A3

Note: Award ***A2*** for one error, ***A1*** for two errors, ***A0*** for more than two errors.

[3 marks]

Examiners report

A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

28c.

[3 marks]

Markscheme

closure is seen from the table above ***A1***

$\emptyset$ is the identity ***A1***

each element is self-inverse ***A1***

Note: Showing each element has an inverse is sufficient.

associativity is assumed so we have a group ***AG***

[3 marks]

Examiners report

A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

28d.

[2 marks]

Markscheme

not isomorphic as in the above group all elements are self-inverse whereas in

$(\mathbb{Z}_4, +_4)$ there is an element of order 4 (*e.g.* 1) ***R2***

[2 marks]

Examiners report

A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

28e. *[4 marks]*

Markscheme

(i)

$\emptyset$ is the identity ***A1***

(ii)

$X^{-1} = X$ ***A1***

(iii) if X and Y are subsets of S then

$X \Delta Y$ (the set of elements that belong to X or Y but not both) is also a subset of S, hence closure is proved ***R1***

$\{P(S), \Delta\}$ is a group because it is closed, has an identity, all elements have inverses (and

Δ is associative) ***R1AG***

[4 marks]

Examiners report

A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

28f. *[1 mark]*

Markscheme

not a group because although the identity is

$\emptyset$, if $X \neq \emptyset$ it is impossible to find a set Y such that

$X \cup Y = \emptyset$, so there are elements without an inverse ***R1AG***

[1 mark]

Examiners report

A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

28g. *[1 mark]*

Markscheme

not a group because although the identity is S, if

$X \neq S$ is impossible to find a set Y such that

$X \cap Y = S$, so there are elements without an inverse ***R1AG***

[1 mark]

Examiners report

A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

29a. *[2 marks]*

Markscheme

$\frac{c}{2} * \frac{3c}{4} = \frac{\frac{c}{2}+\frac{3c}{4}}{1+\frac{1}{2}\cdot\frac{3}{4}}$ ***M1***

$= \frac{\frac{5c}{4}}{\frac{11}{8}} = \frac{10c}{11}$ ***A1***

[2 marks]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term "state" in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

29b. *[1 mark]*

Markscheme

identity is 0 ***A1***

[1 mark]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

29c. *[1 mark]*

Markscheme

inverse is $-x$ ***A1***

[1 mark]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

29d. *[2 marks]*

Markscheme

$x * y = \frac{x+y}{1+\frac{xy}{c^2}}$, $y * x = \frac{y+x}{1+\frac{yx}{c^2}}$ ***M1***

(since ordinary addition and multiplication are commutative)

$x * y = y * x$ so $*$ is commutative ***R1***

Note: Accept arguments using symmetry.

[2 marks]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

29e. *[4 marks]*

Markscheme

$$(x*y)*z=\frac{x+y}{1+\frac{xy}{c^2}}*z=\frac{\left(\frac{x+y}{1+\frac{xy}{c^2}}\right)+z}{1+\left(\frac{x+y}{1+\frac{xy}{c^2}}\right)\frac{z}{c^2}} \quad \textbf{\textit{M1}}$$

$$=\frac{\frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\frac{xy}{c^2}\right)}}{\frac{\left(1+\frac{xy}{c^2}+\frac{xz}{c^2}+\frac{yz}{c^2}\right)}{\left(1+\frac{xy}{c^2}\right)}}=\frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\left(\frac{xy+xz+yz}{c^2}\right)\right)} \quad \textbf{\textit{A1}}$$

$$x*(y*z)=x*\left(\frac{y+z}{1+\frac{yz}{c^2}}\right)=\frac{x+\left(\frac{y+z}{1+\frac{yz}{c^2}}\right)}{1+\frac{x}{c^2}\left(\frac{y+z}{1+\frac{yz}{c^2}}\right)}$$

$$=\frac{\frac{\left(x+\frac{xyz}{c^2}+y+z\right)}{\left(1+\frac{yz}{c^2}\right)}}{\frac{\left(1+\frac{yz}{c^2}+\frac{xy}{c^2}+\frac{xz}{c^2}\right)}{\left(1+\frac{yz}{c^2}\right)}}=\frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\left(\frac{xy+xz+yz}{c^2}\right)\right)} \quad \textbf{\textit{A1}}$$

since both expressions are the same

$*$ is associative ***R1***

Note: After the initial ***M1A1***, correct arguments using symmetry also gain full marks.

[4 marks]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

29f.

[2 marks]

Markscheme

(i)

$c > x$ and $c > y \Rightarrow c - x > 0$ and $c - y > 0 \Rightarrow (c-x)(c-y) > 0$ ***R1AG***

(ii)

$c^2 - cx - cy + xy > 0 \Rightarrow c^2 + xy > cx + cy \Rightarrow c + \frac{xy}{c} > x + y$ (as $c > 0$)

so

$x + y < c + \frac{xy}{c}$ ***M1AG***

[2 marks]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

29g.

[2 marks]

Markscheme

if

$x,\ y \in G$ then $-c - \frac{xy}{c} < x + y < c + \frac{xy}{c}$

thus

$-c\left(1 + \frac{xy}{c^2}\right) < x + y < c\left(1 + \frac{xy}{c^2}\right)$ and $-c < \frac{x+y}{1+\frac{xy}{c^2}} < c$ ***M1***

(as $1 + \frac{xy}{c^2} > 0$) so $-c < x * y < c$ ***A1***

proving that G is closed under

$*$ ***AG***

[2 marks]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

29h.

[2 marks]

Markscheme

as

$\{G, *\}$ is closed, is associative, has an identity and all elements have an inverse ***R1***

it is a group ***AG***

as

$*$ is commutative ***R1***

it is an Abelian group ***AG***

[2 marks]

Examiners report

Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

30.

[8 marks]

Markscheme

closure: let

$a,\, b \in H \cap K$, so that

$a,\, b \in H$ and

$a,\, b \in K$ ***M1***

therefore

$ab \in H$ and

$ab \in K$ so that

$ab \in H \cap K$ ***A1***

associativity: this carries over from G ***R1***

identity: the identity

$e \in H$ and

$e \in K$ ***M1***

therefore

$e \in H \cap K$ ***A1***

inverse:

$a \in H \cap K$ implies

$a \in H$ and

$a \in K$ ***M1***

it follows that

$a^{-1} \in H$ and

$a^{-1} \in K$ ***A1***

and therefore that

$a^{-1} \in H \cap K$ ***A1***

the four group axioms are therefore satisfied ***AG***

[8 marks]

Examiners report

This question presented the most difficulty for students. Overall the candidates showed a lack of ability to present a formal proof. Some gained points for the proof of the identity element in the intersection and the statement that the associative property carries over from the group. However, the vast majority gained no points for the proof of closure or the inverse axioms.