Topic 1 Part 2 *[399 marks]*

1.   *[5 marks]*

Markscheme

$\frac{\log 3}{\log 2} \times \frac{\log 4}{\log 3} \times \ldots \times \frac{\log 32}{\log 31}$ ***M1A1***

$= \frac{\log 32}{\log 2}$ ***A1***

$= \frac{5 \log 2}{\log 2}$ ***(M1)***

$= 5$ ***A1***

hence $a = 5$

Note: Accept the above if done in a specific base *eg* $\log_2 x$

[5 marks]

Examiners report

[N/A]

2.   *[17 marks]*

Markscheme

(a) $r = 1 + i$ *(A1)*

$u_4 = 3(1 + i)^3$ *M1*

$= -6 + 6i$ *A1*

[3 marks]

(b) $S_{20} = \dfrac{\left((1+i)^{20} - 1\right)}{i}$ *(M1)*

$= \dfrac{3\left((2i)^{10} - 1\right)}{i}$ *(M1)*

Note: Only one of the two *M1*s can be implied. Other algebraic methods may be seen.

$= \dfrac{3\left(-2^{10} - 1\right)}{i}$ *(A1)*

$= 3i\left(2^{10} + 1\right)$ *A1*

[4 marks]

(c) (i) **METHOD 1**

$v_n = \left(3(1 + i)^{n-1}\right)\left(3(1 + i)^{n-1+k}\right)$ *M1*

$9(1 + i)^k (1 + i)^{2n-2}$ *A1*

$= 9(1 + i)^k \left((1 + i)^2\right)^{n-1} \left(= 9(1 + i)^k (2i)^{n-1}\right)$

this is the general term of a geometrical sequence *R1AG*

Notes: Do not accept the statement that the product of terms in a geometric sequence is also geometric unless justified further.

If the final expression for v_n is $9(1 + i)^k (1 + i)^{2n-2}$ award *M1A1R0*.

METHOD 2

$\dfrac{v_{n+1}}{v_n} = \dfrac{u_{n+1} u_{n+k+1}}{u_n u_{n+k}}$ *M1*

$= (1 + i)(1 + i)$ *A1*

this is a constant, hence sequence is geometric *R1AG*

Note: Do not allow methods that do not consider the general term.

(ii) $9(1 + i)^k$ *A1*

(iii) common ratio is $(1 + i)^2 \ (= 2i)$ (which is independent of k) *A1*

[5 marks]

(d) (i) **METHOD 1**

$w_n \ \left|3(1 + i)^{n-1} - 3(1 + i)^n\right|$ *M1*

$= 3|1 + i|^{n-1} |1 - (1 + i)|$ *M1*

$= 3|1 + i|^{n-1}$ *A1*

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence *R1AG*

METHOD 2

$w_n = |u_n - (1 + i)u_n|$ *M1*

$= |u_n| \, |-i|$

$= |u_n|$ *A1*

$= \left|3(1 + i)^{n-1}\right|$

$= 3|(1+i)|^{n-1}$ $\quad$ **A1**

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence $\quad$ **R1AG**

Note: $\quad$ Do not allow methods that do not consider the general term.

(ii) $\quad$ distance between successive points representing $u_{n|}$ in the complex plane forms a geometric sequence $\quad$ **R1**

Note: $\quad$ Various possibilities but must mention distance between successive points.

[5 marks]

Total [17 marks]

Examiners report

[N/A]

3.

Markscheme

METHOD 1

$2^{3(x-1)} = (2 \times 3)^{3x}$ $\quad$ **M1**

Note: $\quad$ Award **M1** for writing in terms of 2 and 3.

$2^{3x} \times 2^{-3} = 2^{3x} \times 3^{3x}$

$2^{-3} = 3^{3x}$ $\quad$ **A1**

$\ln\left(2^{-3}\right) = \ln\left(3^{3x}\right)$ $\quad$ **(M1)**

$-3\ln 2 = 3x\ln 3$ $\quad$ **A1**

$x = -\dfrac{\ln 2}{\ln 3}$ $\quad$ **A1**

METHOD 2

$\ln 8^{x-1} = \ln 6^{3x}$ $\quad$ **(M1)**

$(x-1)\ln 2^3 = 3x\ln(2 \times 3)$ $\quad$ **M1A1**

$3x\ln 2 - 3\ln 2 = 3x\ln 2 + 3x\ln 3$ $\quad$ **A1**

$x = -\dfrac{\ln 2}{\ln 3}$ $\quad$ **A1**

METHOD 3

$\ln 8^{x-1} = \ln 6^{3x}$ $\quad$ **(M1)**

$(x-1)\ln 8 = 3x\ln 6$ $\quad$ **A1**

$x = \dfrac{\ln 8}{\ln 8 - 3\ln 6}$ $\quad$ **A1**

$x = \dfrac{3\ln 2}{\ln\left(\dfrac{2^3}{6^3}\right)}$ $\quad$ **M1**

$x = -\dfrac{\ln 2}{\ln 3}$ $\quad$ **A1**

[5 marks]

Examiners report

[N/A]

4.

Markscheme

(a)   **EITHER**

$$\begin{pmatrix} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textbf{\textit{M1}}$$

row of zeroes implies infinite solutions, (or equivalent).   *R1*

Note: Award *M1* for any attempt at row reduction.

OR

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0 \quad \textbf{\textit{M1}}$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0 \text{ with one valid point} \quad \textbf{\textit{R1}}$$

OR

$x + y + 2z = -2$

$3x - y + 14z = 6$

$x + 2y = -5 \implies x = -5 - 2y$

substitute $x = -5 - 2y$ into the first two equations:

$-5 - 2y + y + 2z = -2$

$3(-5 - 2y) - y + 14z = 6$ *M1*

$-y + 2z = 3$

$-7y + 14z = 21$

the latter two equations are equivalent (by multiplying by 7) therefore an infinite number of solutions.   *R1*

OR

for example, $7 \times R_1 - R_2$ gives $4x + 8y = -20$ *M1*

this equation is a multiple of the third equation, therefore an infinite

number of solutions.   *R1*

[2 marks]

(b)   let $y = t$ *M1*

then $x = -5 - 2t$ *A1*

$z = \frac{t+3}{2}$ *A1*

OR

let $x = t$ *M1*

then $y = \frac{-5-t}{2}$ *A1*

$z = \frac{1-t}{4}$ *A1*

OR

let $z = t$ *M1*

then $x = 1 - 4t$ *A1*

$y = -3 + 2t$ *A1*

OR

attempt to find cross product of two normal vectors:

$$\text{eg: } \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4i + 2j + k \quad \textbf{\textit{M1A1}}$$

$x = 1 - 4t$

$y = -3 + 2t$

$z = t$ *A1*

(or equivalent)

[3 marks]

Total [5 marks]

Examiners report

[N/A]

5.

Markscheme

let $\mathrm{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2,\ n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

$2! = 2$ and $2^1 (1!)^2 = 2$ so $\mathrm{P}(1)$ is true   *R1*

assume $\mathrm{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2,\ n \in \mathbb{Z}^+$ *M1*

Note: Do not award *M1* for statements such as "let $n = k$".

consider $\mathrm{P}(k+1)$:

$(2(k+1))! = (2k+2)(2k+1)(2k)!$ *M1*

$(2(k+1))! \geq (2k+2)(2k+1)(k!)^2 2^k$ *A1*

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2$ since $2k+1 > k+1$ *R1*

$= 2^{k+1}((k+1)!)^2$ *A1*

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ *R1*

Note: To obtain the final *R1*, four of the previous marks must have been awarded.

[7 marks]

Examiners report

An easy question, but many candidates exhibited discomfort and poor reasoning abilities. The difficulty for most was that the proposition was expressed in terms of an inequality. Hopefully, as most publishers of IB textbooks have realised, inequalities in such questions are within the syllabus.

6a.

Markscheme

use of the addition principle with 3 terms   *(M1)*

to obtain $^4C_3 + {}^5C_3 + {}^6C_3 \ (= 4 + 10 + 20)$ *A1*

number of possible selections is 34 *A1*

[3 marks]

Examiners report

As the last question on section A, candidates had to think about the strategy for finding the answers to these two parts. Candidates often had a mark-worthy approach, in terms of considering separate cases, but couldn't implement it correctly.

6b.

Markscheme

EITHER

recognition of three cases: (2 odd and 2 even or 1 odd and 3 even or 0 odd and 4 even)   *(M1)*

$\left({}^5C_2 \times {}^4C_2 \right) + \left({}^5C_1 \times {}^4C_3 \right) + \left({}^5C_0 \times {}^4C_4 \right) \quad (= 60 + 20 + 1)$ *(M1)A1*

OR

recognition to subtract the sum of 4 odd and 3 odd and 1 even from the total   *(M1)*

$^9C_4 - {}^5C_4 - \left({}^5C_3 \times {}^4C_1 \right) \quad (= 126 - 5 - 40)$ *(M1)A1*

THEN

number of possible selections is 81 *A1*

[4 marks]

Total [7 marks]

Examiners report

As the last question on section A, candidates had to think about the strategy for finding the answers to these two parts. Candidates often had a mark-worthy approach, in terms of considering separate cases, but couldn't implement it correctly.

7a.

Markscheme

(i)   METHOD 1

$$\left(1+\mathrm{i}\tan\theta\right)^n + \left(1-\mathrm{i}\tan\theta\right)^n = \left(1+\mathrm{i}\tfrac{\sin\theta}{\cos\theta}\right)^n + \left(1-\mathrm{i}\tfrac{\sin\theta}{\cos\theta}\right)^n \quad \textbf{M1}$$

$$= \left(\tfrac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n + \left(\tfrac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n \quad \textbf{A1}$$

by de Moivre's theorem   **(M1)**

$$\left(\tfrac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n = \tfrac{\cos n\theta + i\sin n\theta}{\cos^n\theta} \quad \textbf{A1}$$

recognition that $\cos\theta - i\sin\theta$ is the complex conjugate of $\cos\theta + i\sin\theta$ **(R1)**

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$$\left(\tfrac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n = \tfrac{\cos n\theta - i\sin n\theta}{\cos^n\theta} \quad \textbf{A1}$$

$$\left(1+\mathrm{i}\tan\theta\right)^n + \left(1-\mathrm{i}\tan\theta\right)^n = \tfrac{2\cos n\theta}{\cos^n\theta} \quad \textbf{AG}$$

METHOD 2

$$\left(1+\mathrm{i}\tan\theta\right)^n + \left(1-\mathrm{i}\tan\theta\right)^n = \left(1+\mathrm{i}\tan\theta\right)^n + \left(1+\mathrm{i}\tan(-\theta)\right)^n \quad \textbf{(M1)}$$

$$= \tfrac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta} + \tfrac{(\cos(-\theta)+i\sin(-\theta))^n}{\cos^n\theta} \quad \textbf{M1A1}$$

Note: Award **M1** for converting to cosine and sine terms.

use of de Moivre's theorem   **(M1)**

$$= \tfrac{1}{\cos^n\theta}\left(\cos n\theta + \mathrm{i}\sin n\theta + \cos(-n\theta) + \mathrm{i}\sin(-n\theta)\right) \quad \textbf{A1}$$

$$= \tfrac{2\cos n\theta}{\cos^2\theta} \quad \text{as} \quad \cos(-n\theta) = \cos n\theta \quad \text{and} \quad \sin(-n\theta) = -\sin n\theta \quad \textbf{R1AG}$$

(ii) $$\left(1+\mathrm{i}\tan\tfrac{3\pi}{8}\right)^4 + \left(1-\mathrm{i}\tan\tfrac{3\pi}{8}\right)^4 = \tfrac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}} \quad \textbf{(A1)}$$

$$= \tfrac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}} \quad \textbf{A1}$$

$$= 0 \quad \text{as} \quad \cos\tfrac{3\pi}{2} = 0 \quad \textbf{R1}$$

Note: The above working could involve theta and the solution of $\cos(4\theta) = 0$.

so $\mathrm{i}\tan\tfrac{3\pi}{8}$ is a root of the equation   **AG**

(iii)   either $-\mathrm{i}\tan\tfrac{3\pi}{8}$ or $-\mathrm{i}\tan\tfrac{\pi}{8}$ or $\mathrm{i}\tan\tfrac{\pi}{8}$ **A1**

Note: Accept $\mathrm{i}\tan\tfrac{5\pi}{8}$ or $\mathrm{i}\tan\tfrac{7\pi}{8}$.

Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$

[10 marks]

Examiners report

Fairly successful.

Markscheme

(i) $\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ *(M1)*

$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$ *A1*

let $t = \tan\frac{\pi}{8}$

attempting to solve $t^2 + 2t - 1 = 0$ for t *M1*

$t = -1 \pm \sqrt{2}$ *A1*

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8} > 0$ *R1*

$\tan\frac{\pi}{8} = \sqrt{2} - 1$ *AG*

(ii) $\cos 4x = 2\cos^2 2x - 1$ *A1*

$= 2\left(2\cos^2 x - 1\right)^2 - 1$ *M1*

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$ *A1*

$= 8\cos^4 x - 8\cos^2 x + 1$ *AG*

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,\mathrm{d}x$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,\mathrm{d}x$ *M1*

Note: The *M1* is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}\left(\cos 2x + 1\right)$ *M1*

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,\mathrm{d}x$ *A1*

$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$ *A1*

$= 4\sqrt{2} - \pi - 2$ (or equivalent)   *A1*

[13 marks]

Total [23 marks]

Examiners report

(i)   Most candidates attempted to use the hint. Those who doubled the angle were usually successful – but many lost the final mark by not giving a convincing reason to reject the negative solution to the intermediate quadratic equation. Those who halved the angle got nowhere.

(ii)   The majority of candidates obtained full marks.

(iii)   This was poorly answered, few candidates realising that part of the integrand could be re-expressed using $\frac{1}{\cos^2 x} = \sec^2 x$, which can be immediately integrated.

Markscheme

using $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$ to form $\frac{a+2d}{a+6d} = \frac{a}{a+2d}$ **(M1)**

$a(a+6d) = (a+2d)^2$ **A1**

$2d(2d - a) = 0$ (or equivalent)   **A1**

since $d \neq 0 \Rightarrow d = \frac{a}{2}$ **AG**

[3 marks]

Examiners report

Part (a) was reasonably well done. A number of candidates used $r = \frac{u_1}{u_2} = \frac{u_2}{u_3}$ rather than $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$. This invariably led to candidates obtaining $r = 2$ in part (b).

8b.

Markscheme

substituting $d = \frac{a}{2}$ into $a + 6d = 3$ and solving for a and d **(M1)**

$a = \frac{3}{4}$ and $d = \frac{3}{8}$ **(A1)**

$r = \frac{1}{2}$ **A1**

$\frac{n}{2}\left(2 \times \frac{3}{4} + (n-1)\frac{3}{8}\right) - \frac{3\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \geq 200$ **(A1)**

attempting to solve for n **(M1)**

$n \geq 31.68\ldots$

so the least value of n is 32   **A1**

[6 marks]

Total [9 marks]

Examiners report

In part (b), most candidates were able to correctly find the first term and the common difference for the arithmetic sequence. However a number of candidates either obtained $r = 2$ via means described in part (a) or confused the two sequences and used $u_1 = \frac{3}{4}$ for the geometric sequence.

9a.

Markscheme

$P(\text{Ava wins on her first turn}) = \frac{1}{3}$ **A1**

[1 mark]

Examiners report

Parts (a) and (b) were straightforward and were well done.

9b. [2 marks]

Markscheme

P(Barry wins on his first turn) $= \left(\frac{2}{3}\right)^2$ *(M1)*

$= \frac{4}{9}$ $(= 0.444)$ *A1*

[2 marks]

Examiners report

Parts (a) and (b) were straightforward and were well done.

9c. [4 marks]

Markscheme

P(Ava wins in one of her first three turns)

$= \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3}$ *M1A1A1*

Note: Award *M1* for adding probabilities, award *A1* for a correct second term and award *A1* for a correct third term.

Accept a correctly labelled tree diagram, awarding marks as above.

$= \frac{103}{243}$ $(= 0.424)$ *A1*

[4 marks]

Examiners report

Parts (c) and (d) were also reasonably well done.

9d. [4 marks]

Markscheme

P(Ava eventually wins) $= \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \dots$ *(A1)*

using $S_\infty = \frac{a}{1-r}$ with $a = \frac{1}{3}$ and $r = \frac{2}{9}$ *(M1)(A1)*

Note: Award *(M1)* for using $S_\infty = \frac{a}{1-r}$ and award *(A1)* for $a = \frac{1}{3}$ and $r = \frac{2}{9}$.

$= \frac{3}{7}$ $(= 0.429)$ *A1*

[4 marks]

Total [11 marks]

Examiners report

Parts (c) and (d) were also reasonably well done. A pleasingly large number of candidates recognized that an infinite geometric series was required in part (d).

10a.

Markscheme

$r = -x^2, \quad S = \frac{1}{1+x^2}$ **A1AG**

[1 mark]

Examiners report

Most candidates picked up this mark for realizing the common ratio was $-x^2$.

10b.

Markscheme

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

EITHER

$\int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + \ldots dx$ **M1**

$\arctan x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ **A1**

Note: Do not penalize the absence of c at this stage.

when $x = 0$ we have $\arctan 0 = c$ hence $c = 0$ **M1A1**

OR

$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x 1 - t^2 + t^4 - t^6 + \ldots dt$ **M1A1A1**

Note: Allow x as the variable as well as the limit.

M1 for knowing to integrate, **A1** for each of the limits.

$[\arctan t]_0^x = \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right]_0^x$ **A1**

hence $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ **AG**

[4 marks]

Examiners report

Quite a few candidates did not recognize the importance of 'hence' in this question, losing a lot of time by trying to work out the terms from first principles.

Of those who integrated the formula from part (a) only a handful remembered to include the '$+c$' term, and to verify that this must be equal to zero.

10c.

Markscheme

applying the MVT to the function f on the interval $[x, \ y]$ **M1**

$\frac{f(y)-f(x)}{y-x} = f'(c)$ $(\text{for some } c \in]x, \ y[)$ **A1**

$\frac{f(y)-f(x)}{y-x} > 0$ $(\text{as } f'(c) > 0)$ **R1**

$f(y) - f(x) > 0$ as $y > x$ **R1**

$\Rightarrow f(y) > f(x)$ **AG**

Note: If they use x rather than c they should be awarded **M1A0R0**, but could get the next **R1**.

[4 marks]

Examiners report

Most candidates were able to achieve some marks on this question. The most commonly lost mark was through not stating that the inequality was unchanged when multiplying by $y - x$ as $y > x$.

10d. <div align="right">*[4 marks]*</div>

Markscheme

(i) $g(x) = x - \arctan x \Rightarrow g'(x) = 1 - \frac{1}{1+x^2}$ **A1**

this is greater than zero because $\frac{1}{1+x^2} < 1$ **R1**

so $g'(x) > 0$ **AG**

(ii)   (g is a continuous function defined on $[0, \ b]$ and differentiable on $]0, \ b[$ with $g'(x) > 0$ on $]0, \ b[$ for all $b \in \mathbb{R}$)

(If $x \in [0, \ b]$ then) from part (c) $g(x) > g(0)$ **M1**

$x - \arctan x > 0 \Rightarrow \arctan x < x$ **M1**

(as b can take any positive value it is true for all $x > 0$)   **AG**

[4 marks]

Examiners report

The first part of this question proved to be very straightforward for the majority of candidates.

In (ii) very few realized that they had to replace the lower variable in the formula from part (c) by zero.

10e. <div align="right">*[5 marks]*</div>

Markscheme

let $h(x) = \arctan x - \left(x - \frac{x^3}{3}\right)$ *M1*

(h is a continuous function defined on $[0, \ b]$ and differentiable on $]0, \ b[$ with $h'(x) > 0$ on $]0, \ b[$)

$h'(x) = \frac{1}{1+x^2} - (1 - x^2)$ *A1*

$= \frac{1-(1-x^2)(1+x^2)}{1+x^2} = \frac{x^4}{1+x^2}$ *M1A1*

$h'(x) > 0$ hence (for $x \in [0, \ b]$) $h(x) > h(0)(= 0)$ *R1*

$\Rightarrow \arctan x > x - \frac{x^3}{3}$ *AG*

Note: Allow correct working with $h(x) = x - \frac{x^3}{3} - \arctan x$.

[5 marks]

Examiners report

Candidates found this part difficult, failing to spot which function was required.

[4 marks]

Markscheme

use of $x - \frac{x^3}{3} < \arctan x < x$ *M1*

choice of $x = \frac{1}{\sqrt{3}}$ *A1*

$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ *M1*

$\frac{8}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ *A1*

Note: Award final *A1* for a correct inequality with a single fraction on each side that leads to the final answer.

$\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$ *AG*

[4 marks]

Total [22 marks]

Examiners report

Many candidates, even those who did not successfully complete (d) (ii) or (e), realized that these parts gave them the necessary inequality.

[2 marks]

Markscheme

$(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ *(M1)A1*

[2 marks]

Examiners report

Well done although some did not use the binomial expansion.

12a.

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = \left(\mathrm{e}^{3x} + 3x\mathrm{e}^{3x}\right)$ **M1A1**

[2 marks]

Examiners report

Well done.

12b.

Markscheme

let $P(n)$ be the statement $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$

prove for $n = 1$ **M1**

LHS of $P(1)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ which is $1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x}$ and RHS is $3^0\mathrm{e}^{3x} + x3^1\mathrm{e}^{3x}$ **R1**

as $\mathrm{LHS} = \mathrm{RHS}$, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true   **M1**

assuming $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = k3^{k-1}\mathrm{e}^{3x} + x3^k\mathrm{e}^{3x}$

$\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)$ **(M1)**

$= k3^{k-1} \times 3\mathrm{e}^{3x} + 1 \times 3^k\mathrm{e}^{3x} + x3^k \times 3\mathrm{e}^{3x}$ **A1**

$= (k+1)3^k\mathrm{e}^{3x} + x3^{k+1}\mathrm{e}^{3x}$ (as required)   **A1**

Note: Can award the **A** marks independent of the **M** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true $(\forall n \in \mathbb{Z}^+)$ **R1**

Note: To gain last **R1** at least four of the above marks must have been gained.

[7 marks]

Examiners report

The logic of an induction proof was not known well enough. Many candidates used what they had to prove rather than differentiating what they had assumed. They did not have enough experience in doing Induction proofs.

12c.

Markscheme

$e^{3x} + x \times 3e^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ **M1A1**

point is $\left(-\frac{1}{3}, \ -\frac{1}{3e} \right)$ **A1**

EITHER

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$

when $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} > 0$ therefore the point is a minimum   **M1A1**

OR

x	$-\dfrac{1}{3}$		
$\dfrac{dy}{dx}$	$-ve$	0	$+ve$

nature table shows point is a minimum   **M1A1**

[5 marks]

Examiners report

Good, some forgot to test for min/max, some forgot to give the y value.

12d.

Markscheme

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$ **A1**

$2 \times 3e^{3x} + x \times 3^2 e^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ **M1A1**

point is $\left(-\frac{2}{3}, \ -\frac{2}{3e^2} \right)$ **A1**

x	$-\dfrac{2}{3}$		
$\dfrac{d^2y}{dx^2}$	$-ve$	0	$+ve$

since the curvature does change (concave down to concave up) it is a point of inflection   **R1**

Note: Allow 3rd derivative is not zero at $-\frac{2}{3}$

[5 marks]

Examiners report

Again quite good, some forgot to check for change in curvature and some forgot the y value.

13a.

Markscheme

(i)   METHOD 1

$\frac{v_{n+1}}{v_n} = \frac{2^{u_{n+1}}}{2^{u_n}}$ **M1**

$= 2^{u_{n+1}-u_n} = 2^d$ **A1**

METHOD 2

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}}$ **M1**

$= 2^d$ **A1**

(ii) $= 2^a$ **A1**

Note: Accept $= 2^{u_1}$.

(iii)   EITHER

v_n is a GP with first term 2^a and common ratio 2^d

$v_n = 2^a (2^d)^{(n-1)}$

OR

$u_n = a + (n-1)d$ as it is an AP

THEN

$v_n = 2^{a+(n-1)d}$ **A1**

[4 marks]

Examiners report

Method of first part was fine but then some algebra mistakes often happened. The next two parts were generally good.

13b.   *[8 marks]*

Markscheme

(i) $S_n = \frac{2^a \left((2^d)^n - 1 \right)}{2^d - 1} = \frac{2^a(2^{dn}-1)}{2^d-1}$ **M1A1**

Note: Accept either expression.

(ii)   for sum to infinity to exist need $-1 < 2^d < 1$ **R1**

$\Rightarrow \log 2^d < 0 \Rightarrow d \log 2 < 0 \Rightarrow d < 0$ **(M1)A1**

Note: Also allow graph of 2^d.

(iii) $S_\infty = \frac{2^a}{1-2^d}$ **A1**

(iv) $\frac{2^a}{1-2^d} = 2^{a+1} \Rightarrow \frac{1}{1-2^d} = 2$ **M1**

$\Rightarrow 1 = 2 - 2^{d+1} \Rightarrow 2^{d+1} = 1$

$\Rightarrow d = -1$ **A1**

[8 marks]

13c.   *[6 marks]*

Markscheme

METHOD 1

$w_n = pq^{n-1}, \ z_n = \ln pq^{n-1}$ *(A1)*

$z_n = \ln p + (n-1)\ln q$ *M1A1*

$z_{n+1} - z_n = (\ln p + n\ln q) - (\ln p + (n-1)\ln q) = \ln q$

which is a constant so this is an AP

(with first term $\ln p$ and common difference $\ln q$)

$\sum_{i=1}^{n} z_i = \frac{n}{2}\left(2\ln p + (n-1)\ln q\right)$ *M1*

$= n\left(\ln p + \ln q^{\left(\frac{n-1}{2}\right)}\right) = n\ln\left(pq^{\left(\frac{n-1}{2}\right)}\right)$ *(M1)*

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ *A1*

METHOD 2

$\sum_{i=1}^{n} z_i = \ln p + \ln pq + \ln pq^2 + \ldots + \ln pq^{n-1}$ *(M1)A1*

$= \ln\left(p^n q^{(1+2+3+\ldots+(n-1))}\right)$ *(M1)A1*

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ *(M1)A1*

[6 marks]

Total [18 marks]

Examiners report

14.   *[4 marks]*

Markscheme

$(3-x)^4 = 1.3^4 + 4.3^3(-x) + 6.3^2(-x)^2 + 4.3(-x)^3 + 1(-x)^4$ or equivalent   *(M1)(A1)*

$= 81 - 108x + 54x^2 - 12x^3 + x^4$ *A1A1*

Note: *A1* for ascending powers, *A1* for correct coefficients including signs.

[4 marks]

Examiners report

[N/A]

15a.   *[6 marks]*

Markscheme

METHOD 1

$z^3 = -\frac{27}{8} = \frac{27}{8}\left(\cos\pi + i\sin\pi\right)$ *M1(A1)*

$= \frac{27}{8}\left(\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)\right)$ *(A1)*

$z = \frac{3}{2}\left(\cos\left(\frac{\pi+2n\pi}{3}\right) + i\sin\left(\frac{\pi+2n\pi}{3}\right)\right)$ *M1*

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}\left(\cos\pi + i\sin\pi\right)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.   *A2*

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award *A1* for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{i\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

METHOD 2

$8z^3 + 27 = 0$

$\Rightarrow z = -\frac{3}{2}$ so $(2z + 3)$ is a factor

Attempt to use long division or factor theorem: *M1*

$\Rightarrow 8z^3 + 27 = (2z + 3)(4z^2 - 6z + 9)$

$\Rightarrow 4z^2 - 6z + 9 = 0$ *A1*

Attempt to solve quadratic: *M1*

$z = \frac{3 \pm 3\sqrt{3}i}{4}$ *A1*

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}\left(\cos\pi + i\sin\pi\right)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.   *A2*

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award *A1* for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{i\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

METHOD 3

$8z^3 + 27 = 0$

Substitute $z = x + iy$ ***M1***

$8(x^3 + 3ix^2y - 3xy^2 - iy^3) + 27 = 0$

$\Rightarrow 8x^3 - 24xy^2 + 27 = 0$ and $24x^2y - 8y^3 = 0$ ***A1***

Attempt to solve simultaneously: ***M1***

$8y(3x^2 - y^2) = 0$

$y = 0, \ y = x\sqrt{3}, \ y = -x\sqrt{3}$

$\Rightarrow \left(x = -\frac{3}{2}, \ y = 0 \right), \ x = \frac{3}{4}, \ y = \pm\frac{3\sqrt{3}}{4}$ ***A1***

$z_1 = \frac{3}{2} \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \right)$,

$z_2 = \frac{3}{2} \left(\cos\pi + i\sin\pi \right)$,

$z_3 = \frac{3}{2} \left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} \right)$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{i\theta} \right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

Examiners report

[N/A]

15b.   *[3 marks]*

Markscheme

EITHER

Valid attempt to use area $= 3 \left(\frac{1}{2} ab \sin C \right)$ *M1*

$= 3 \times \frac{1}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2}$ *A1A1*

Note: Award *A1* for correct sides, *A1* for correct $\sin C$.

OR

Valid attempt to use area $= \frac{1}{2}$ base $\times$ height   *M1*

area $= \frac{1}{2} \times \left(\frac{3}{4} + \frac{3}{2} \right) \times \frac{6\sqrt{3}}{4}$ *A1A1*

Note: *A1* for correct height, *A1* for correct base.

THEN

$= \frac{27\sqrt{3}}{16}$ *AG*

[3 marks]

Total [9 marks]

Examiners report

[N/A]

Markscheme

(i)-(iii) given the three roots $\alpha, \ \beta, \ \gamma$, we have

$x^3 + px^2 + qx + c = (x - \alpha)(x - \beta)(x - \gamma)$ *M1*

$= \left(x^2 - (\alpha + \beta)x + \alpha\beta \right) (x - \gamma)$ *A1*

$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$ *A1*

comparing coefficients:

$p = -(\alpha + \beta + \gamma)$ *AG*

$q = (\alpha\beta + \beta\gamma + \gamma\alpha)$ *AG*

$c = -\alpha\beta\gamma$ *AG*

[3 marks]

Examiners report

[N/A]

Markscheme

METHOD 1

(i)   Given $-\alpha - \beta - \gamma = -6$

And $\alpha\beta + \beta\gamma + \gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$

So $\beta - \alpha = \gamma - \beta$ *M1*

or $2\beta = \alpha + \gamma$

Attempt to solve simultaneous equations: *M1*

$\beta + 2\beta = 6$ *A1*

$\beta = 2$ *AG*

(ii) $\alpha + \gamma = 4$

$2\alpha + 2\gamma + \alpha\gamma = 18$

$\Rightarrow \gamma^2 - 4\gamma + 10 = 0$

$\Rightarrow \gamma = \frac{4 \pm i\sqrt{24}}{2}$ *(A1)*

Therefore $c = -\alpha\beta\gamma = -\left(\frac{4+i\sqrt{24}}{2}\right)\left(\frac{4-i\sqrt{24}}{2}\right)2 = -20$ *A1*

METHOD 2

(i)   let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ *M1*

adding roots   *M1*

to give $3\alpha = 6$ *A1*

$\alpha = 2$ *AG*

(ii) α is a root, so $2^3 - 6 \times 2^2 + 18 \times 2 + c = 0$ *M1*

$8 - 24 + 36 + c = 0$

$c = -20$ *A1*

METHOD 3

(i)   let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ *M1*

adding roots   *M1*

to give $3\alpha = 6$ *A1*

$\alpha = 2$ *AG*

(ii) $q = 18 = 2(2 - d) + (2 - d)(2 + d) + 2(2 + d)$ *M1*

$d^2 = -6 \Rightarrow d = \sqrt{6}i$

$\Rightarrow c = -20$ *A1*

[5 marks]

Examiners report

[N/A]

16c.   *[6 marks]*

Markscheme

METHOD 1

Given $-\alpha - \beta - \gamma = -6$

And $\alpha\beta + \beta\gamma + \gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$.

So $\frac{\beta}{\alpha} = \frac{\gamma}{\beta}$ **M1**

or $\beta^2 = \alpha\gamma$

Attempt to solve simultaneous equations: **M1**

$\alpha\beta + \gamma\beta + \beta^2 = 18$

$\beta(\alpha + \beta + \gamma) = 18$

$6\beta = 18$

$\beta = 3$ **A1**

$\alpha + \gamma = 3,\ \alpha = \frac{9}{\gamma}$

$\Rightarrow \gamma^2 - 3\gamma + 9 = 0$

$\Rightarrow \gamma = \frac{3 \pm i\sqrt{27}}{2}$ **(A1)(A1)**

Therefore $c = -\alpha\beta\gamma = -\left(\frac{3+i\sqrt{27}}{2}\right)\left(\frac{3-i\sqrt{27}}{2}\right)3 = -27$ **A1**

METHOD 2

let the three roots be $a,\ ar,\ ar^2$ **M1**

attempt at substitution of $a,\ ar,\ ar^2$ and p and q into equations from (a)   **M1**

$6 = a + ar + ar^2 \left(= a(1 + r + r^2)\right)$ **A1**

$18 = a^2r + a^2r^3 + a^2r^2 \left(= a^2r(1 + r + r^2)\right)$ **A1**

therefore $3 = ar$ **A1**

therefore $c = -a^3r^3 = -3^3 = -27$ **A1**

[6 marks]

Total [14 marks]

Examiners report

[N/A]

Markscheme

$\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ **M1**

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ **A1**

$= \sqrt{n+1} - \sqrt{n}$ **AG**

[2 marks]

Examiners report

17b.

[2 marks]

Markscheme

$\sqrt{2} - 1 = \frac{1}{\sqrt{2}+\sqrt{1}}$ **A2**

$< \frac{1}{\sqrt{2}}$ **AG**

[2 marks]

Examiners report

17c.

[9 marks]

Markscheme

consider the case $n = 2$: required to prove that $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ **M1**

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2} - 1$

hence $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n = 2$ **A1**

now assume true for $n = k : \sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ **M1**

$\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n = k + 1 : \frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ **(M1)**

from assumption, we have that $\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$ **M1**

so attempt to show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ **(M1)**

EITHER

$\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$ **A1**

$\frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k}+\sqrt{k+1}}$, (from part a), which is true **A1**

OR

$\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}\sqrt{k}+1}{\sqrt{k+1}}$ **A1**

$> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \sqrt{k+1}$ **A1**

THEN

so true for $n = 2$ and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \geq 2$ **R1**

Note: Award **R1** only if all previous **M** marks have been awarded.

[9 marks]

Total [13 marks]

18a.   *[6 marks]*

Markscheme

(i) $\left(\cos\theta + \mathrm{i}\sin\theta\right)^{5}$

$= \cos^{5}\theta + 5\mathrm{i}\cos^{4}\theta\sin\theta + 10\mathrm{i}^{2}\cos^{3}\theta\sin^{2}\theta +$

$10\mathrm{i}^{3}\cos^{2}\theta\sin^{3}\theta + 5\mathrm{i}^{4}\cos\theta\sin^{4}\theta + \mathrm{i}^{5}\sin^{5}\theta$ ***A1A1***

$(= \cos^{5}\theta + 5\mathrm{i}\cos^{4}\theta\sin\theta - 10\cos^{3}\theta\sin^{2}\theta -$

$10\mathrm{i}\cos^{2}\theta\sin^{3}\theta + 5\cos\theta\sin^{4}\theta + \mathrm{i}\sin^{5}\theta)$

Note: Award first ***A1*** for correct binomial coefficients.

(ii) $\left(\mathrm{cis}\theta\right)^{5} = \mathrm{cis}5\theta = \cos 5\theta + \mathrm{i}\sin 5\theta$ ***M1***

$= \cos^{5}\theta + 5\mathrm{i}\cos^{4}\theta\sin\theta - 10\cos^{3}\theta\sin^{2}\theta - 10\mathrm{i}\cos^{2}\theta\sin^{3}\theta +$

$5\cos\theta\sin^{4}\theta + \mathrm{i}\sin^{5}\theta$ ***A1***

Note: Previous line may be seen in (i)

equating imaginary terms   ***M1***

$\sin 5\theta = 5\cos^{4}\theta\sin\theta - 10\cos^{2}\theta\sin^{3}\theta + \sin^{5}\theta$ ***AG***

(iii)   equating real terms

$\cos 5\theta = \cos^{5}\theta - 10\cos^{3}\theta\sin^{2}\theta + 5\cos\theta\sin^{4}\theta$ ***A1***

[6 marks]

Examiners report

In part (i) many candidates tried to multiply it out the binomials rather than using the binomial theorem. In parts (ii) and (iii) many candidates showed poor understanding of complex numbers and made no attempt to equate real and imaginary parts. In a some cases the correct answer to part (iii) was seen although it was unclear how it was obtained.

18b.   *[4 marks]*

Markscheme

$(r\mathrm{cis}\alpha)^5 = 1 \Rightarrow r^5\mathrm{cis}5\alpha = 1\mathrm{cis}0$ ***M1***

$r^5 = 1 \Rightarrow r = 1$ ***A1***

$5\alpha = 0 \pm 360k,\ k \in \mathbb{Z} \Rightarrow a = 72k$ ***(M1)***

$\alpha = 72°$ ***A1***

Note:　Award ***M1A0*** if final answer is given in radians.

[4 marks]

Examiners report

This question was poorly done. Very few candidates made a good attempt to apply De Moivre's theorem and most of them could not even equate the moduli to obtain r.

18c.　

Markscheme

use of $\sin(5 \times 72) = 0$ **OR** the imaginary part of 1 is 0 ***(M1)***

$0 = 5\cos^4\alpha \sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha$ ***A1***

$\sin\alpha \neq 0 \Rightarrow 0 = 5(1 - \sin^2\alpha)^2 - 10(1 - \sin^2\alpha)\sin^2\alpha + \sin^4\alpha$ ***M1***

Note:　Award ***M1*** for replacing $\cos^2\alpha$.

$0 = 5(1 - 2\sin^2\alpha + \sin^4\alpha) - 10\sin^2\alpha + 10\sin^4\alpha + \sin^4\alpha$ ***A1***

Note:　Award ***A1*** for any correct simplification.

so $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$ ***AG***

[4 marks]

Examiners report

This question was poorly done. From the few candidates that attempted it, many candidates started by writing down what they were trying to prove and made no progress.

18d.

Markscheme

$$\sin^2\alpha = \frac{20 \pm \sqrt{400-320}}{32}$$ *M1A1*

$$\sin\alpha = \pm\sqrt{\frac{20 \pm \sqrt{80}}{32}}$$

$$\sin\alpha = \frac{\pm\sqrt{10 \pm 2\sqrt{5}}}{4}$$ *A1*

Note: Award *A1* regardless of signs. Accept equivalent forms with integral denominator, simplification may be seen later.

as $72 > 60$, $\sin 72 > \frac{\sqrt{3}}{2} = 0.866\ldots$ we have to take both positive signs (or equivalent argument)   *R1*

Note: Allow verification of correct signs with calculator if clearly stated

$$\sin 72 = \frac{\sqrt{10+2\sqrt{5}}}{4}$$ *A1*

[5 marks]

Total [19 marks]

Examiners report

Very few made a serious attempt to answer this question. Also very few realised that they could use the answers given in part (c) to attempt this part.

19a.   *[2 marks]*

Markscheme

$$\binom{11}{4} = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$$ *(M1)A1*

[2 marks]

Examiners report

[N/A]

19b.   *[2 marks]*

Markscheme

$$\binom{5}{2} \times \binom{6}{2} = \frac{5\times4}{2\times1} \times \frac{6\times5}{2\times1}$$ *M1*

$$= 150$$ *A1*

[2 marks]

19c. *[2 marks]*

Markscheme

METHOD 1

number of ways all men $= \binom{5}{4} = 5$

$330 - 5 = 325$ ***M1A1***

Note: Allow FT from answer obtained in part (a).

[2 marks]

METHOD 2

$$\binom{6}{1}\binom{5}{3} + \binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{1} + \binom{6}{4}$$ ***M1***

$= 325$ ***A1***

[2 marks]

Total [6 marks]

20a. *[6 marks]*

Markscheme

$2x + y + 6z = 0$

$4x + 3y + 14z = 4$

$2x - 2y + (\alpha - 2)z = \beta - 12$

attempt at row reduction   *M1*

eg $R_2 - 2R_1$ and $R_3 - R_1$

$2x + y + 6z = 0$

$y + 2z = 4$

$-3y + (\alpha - 8)z = \beta - 12$ *A1*

eg $R_3 + 3R_2$

$2x + y + 6z = 0$

$y + 2z = 4$ *A1*

$(\alpha - 2)z = \beta$

(i)   no solutions if $\alpha = 2,\ \beta \neq 0$ *A1*

(ii)   one solution if $\alpha \neq 2$ *A1*

(iii) infinite solutions if $\alpha = 2,\ \beta = 0$ *A1*

Note: Accept alternative methods e.g. determinant of a matrix

Note: Award *A1A1A0* if all three consistent with their reduced form, *A1A0A0* if two or one answer consistent with their reduced form.

[6 marks]

Examiners report

[N/A]

20b.   *[3 marks]*

Markscheme

$y + 2z = 4 \Rightarrow y = 4 - 2z$

$2x = -y - 6z = 2z - 4 - 6z = -4z - 4 \Rightarrow x = -2z - 2$ *A1*

therefore Cartesian equation is $\frac{x+2}{-2} = \frac{y-4}{-2} = \frac{z}{1}$ or equivalent   *A1*

[3 marks]

Total [9 marks]

Examiners report

[N/A]

21a.   *[2 marks]*

Markscheme

modulus $= \sqrt{8}$ *A1*

argument $= \frac{\pi}{4}$ (accept 45°)   *A1*

Note: *A0* if extra values given.

[2 marks]

Examiners report

Those who tackled this question were generally very successful. A few, with varying success, tried to work out the powers of the complex numbers by multiplying the Cartesian form rather than using de Moivre's Theorem.

Markscheme

METHOD 1

$w^4 z^6 - 64e^{\pi i} \times e^{5\pi i}$ *(A1)(A1)*

Note: Allow alternative notation.

$= 64e^{6\pi i}$ *(M1)*

$= 64$ *A1*

METHOD 2

$w^4 = -64$ *(M1)(A1)*

$z^6 = -1$ *(A1)*

$w^4 z^6 = 64$ *A1*

[4 marks]

Examiners report

Those who tackled this question were generally very successful. A few, with varying success, tried to work out the powers of the complex numbers by multiplying the Cartesian form rather than using de Moivre's Theorem.

Markscheme

METHOD 1

$d = \left. \frac{1}{\log_8 x} - \frac{1}{\log_2 x} \right|$ *(M1)*

$= \left. \frac{\log_2 8}{\log_2 x} - \frac{1}{\log_2 x} \right|$ *(M1)*

Note: Award this *M1* for a correct change of base anywhere in the question.

$= \left. \frac{2}{\log_2 x} \right|$ *(A1)*

$\left. \frac{20}{2} \left(2 \times \frac{1}{\log_2 x} + 19 \times \frac{2}{\log_2 x} \right) \right|$ *M1*

$= \left. \frac{400}{\log_2 x} \right|$ *(A1)*

$100 = \left. \frac{400}{\log_2 x} \right|$

$\log_2 x = 4 \Rightarrow x = 2^4 = \left. 16 \right|$ *A1*

METHOD 2

$20^{\text{th}} \text{ term} = \left. \frac{1}{\log_{2^{39}} x} \right|$ *A1*

$100 = \left. \frac{20}{2} \left(\frac{1}{\log_2 x} + \frac{1}{\log_{2^{39}} x} \right) \right|$ *M1*

$100 = \left. \frac{20}{2} \left(\frac{1}{\log_2 x} + \frac{\log_2 2^{39}}{\log_2 x} \right) \right|$ *M1(A1)*

Note: Award this *M1* for a correct change of base anywhere in the question.

$100 = \left. \frac{400}{\log_2 x} \right|$ *(A1)*

$\log_2 x = 4 \Rightarrow x = 2^4 = \left. 16 \right|$ *A1*

METHOD 3

$\left. \frac{1}{\log_2 x} + \frac{1}{\log_8 x} + \frac{1}{\log_{32} x} + \frac{1}{\log_{128} x} + \ldots \right|$

$\left. \frac{1}{\log_2 x} + \frac{\log_2 8}{\log_2 x} + \frac{\log_2 32}{\log_2 x} + \frac{\log_2 128}{\log_2 x} + \ldots \right|$ *(M1)(A1)*

Note: Award this *M1* for a correct change of base anywhere in the question.

$= \left. \frac{1}{\log_2 x} (1 + 3 + 5 + \ldots) \right|$ *A1*

$= \left. \frac{1}{\log_2 x} \left(\frac{20}{2} (2 + 38) \right) \right|$ *(M1)(A1)*

$100 = \left. \frac{400}{\log_2 x} \right|$

$\log_2 x = 4 \Rightarrow x = 2^4 = \left. 16 \right|$ *A1*

[6 marks]

Examiners report

There were plenty of good answers to this question. Those who realised they needed to make each log have the same base (and a great variety of bases were chosen) managed the question successfully.

23a.   *[3 marks]*

Markscheme

$B\left(6, \frac{2}{3}\right)$ **(M1)**

$p(4) = \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2$ **A1**

$\binom{6}{4} = 15$ **A1**

$= 15 \times \frac{2^4}{3^6} = \frac{80}{243}$ **AG**

[3 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(a) Candidates need to be aware how to work out binomial coefficients without a calculator

23b.

Markscheme

(i)   2 outcomes for each of the 6 games or $2^6 = 64$ **R1**

(ii) $(1+x)^6 = \binom{6}{0} + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6$ **A1**

Note: Accept ^nC_r notation or $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

setting $x = 1$ in both sides of the expression   **R1**

Note: Do not award **R1** if the right hand side is not in the correct form.

$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$ **AG**

(iii)   the total number of outcomes = number of ways Alfred can win no games, plus the number of ways he can win one game

etc.   **R1**

[4 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(b) (ii) A surprising number of candidates chose to work out the values of all the binomial coefficients (or use Pascal's triangle) to make a total of 64 rather than simply putting 1 into the left hand side of the expression.

23c.

Markscheme

(i)   Let $P(x, y)$ be the probability that Alfred wins x games on the first day and y on the second.

$$P(4, 2) = \binom{6}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^2 \times \binom{6}{2} \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^4 \quad \textbf{\textit{M1A1}}$$

$$\binom{6}{2}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 \text{ or } \binom{6}{4}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 \quad \textbf{\textit{A1}}$$

$r = 2$ or 4, $s = t = 6$

(ii) $P(\text{Total} = 6) =$

$$P(0, 6) + P(1, 5) + P(2, 4) + P(3, 3) + P(4, 2) + P(5, 1) + P(6, 0) \quad \textbf{\textit{(M1)}}$$

$$= \binom{6}{0}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 + \binom{6}{1}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 + \ldots + \binom{6}{6}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 \quad \textbf{\textit{A2}}$$

$$= \frac{2^6}{3^{12}} \left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 \right)$$

Note: Accept any valid sum of 7 probabilities.

(iii)   use of $\binom{6}{i} = \binom{6}{6 - i}$ $\quad$ **_(M1)_**

(can be used either here or in (c)(ii))

$$P(\text{wins 6 out of 12}) = \binom{12}{6} \times \left(\frac{2}{3}\right)^6 \times \left(\frac{1}{3}\right)^6 = \frac{2^6}{3^{12}} \binom{12}{6} \quad \textbf{\textit{A1}}$$

$$= \frac{2^6}{3^{12}} \left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 \right) = \frac{2^6}{3^{12}} \binom{12}{6} \quad \textbf{\textit{A1}}$$

therefore $\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 = \binom{12}{6}$ $\quad$ **_AG_**

[9 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

23d.

[6 marks]

Markscheme

(i) $E(A) = \sum_{r=0}^{n} r \binom{n}{r} \left(\frac{2}{3}\right)^r \left(\frac{1}{3}\right)^{n-r} = \sum_{r=0}^{n} r \binom{n}{r} \frac{2^r}{3^n}$

$(a = 2, b = 3)$ **M1A1**

Note: **M0A0** for $a = 2$, $b = 3$ without any method.

(ii) $n(1+x)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r x^{r-1}$ **A1A1**

(sigma notation not necessary)

(if sigma notation used also allow lower limit to be $r = 0$)

let $x = 2$ **M1**

$n3^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r 2^{r-1}$

multiply by 2 and divide by 3^n **(M1)**

$\frac{2n}{3} = \sum_{r=1}^{n} \binom{n}{r} r \frac{2^r}{3^n} \left(= \sum_{r=0}^{n} \binom{n}{r} \frac{2^r}{3^n} \right)$ **AG**

[6 marks]

Examiners report

This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(d) This was poorly done. Candidates were not able to manipulate expressions given using sigma notation.

24.

Markscheme

clear attempt at binomial expansion for exponent 5   **M1**

$2^5 + 5 \times 2^4 \times (-3x) + \frac{5 \times 4}{2} \times 2^3 \times (-3x)^2 + \frac{5 \times 4 \times 3}{6} \times 2^2 \times (-3x)^3$

$+ \frac{5 \times 4 \times 3 \times 2}{24} \times 2 \times (-3x)^4 + (-3x)^5$ **(A1)**

Note: Only award **M1** if binomial coefficients are seen.

$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$ **A2**

Note: Award **A1** for correct moduli of coefficients and powers. **A1** for correct signs.

Total [4 marks]

Examiners report

Generally well done. The majority of candidates obtained a quintic with correct alternating signs. A few candidates made arithmetic errors. A small number of candidates multiplied out the linear expression, often correctly.

25.

Markscheme

for the first series $\frac{a}{1-r} = 76$ *A1*

for the second series $\frac{a}{1-r^3} = 36$ *A1*

attempt to eliminate a e.g. $\frac{76(1-r)}{1-r^3} = 36$ *M1*

simplify and obtain $9r^2 + 9r - 10 = 0$ *(M1)A1*

Note: Only award the *M1* if a quadratic is seen.

obtain $r = \frac{12}{18}$ and $-\frac{30}{18}$ *(A1)*

$r = \frac{12}{18}$ $\left(= \frac{2}{3} = 0.666\ldots\right)$ *A1*

Note: Award *A0* if the extra value of r is given in the final answer.

Total [7 marks]

Examiners report

Almost all candidates obtained the cubic equation satisfied by the common ratio of the first sequence, but few were able to find its roots. One of the roots was $r = 1$.

26a.

Markscheme

$|z_1| = \sqrt{10}$; $\arg(z_2) = -\frac{3\pi}{4}$ $\left(\text{accept } \frac{5\pi}{4}\right)$ *A1A1*

[2 marks]

Examiners report

Disappointingly, few candidates obtained the correct argument for the second complex number, mechanically using arctan(1) but not thinking about the position of the number in the complex plane.

26b.

Markscheme

$|z_1 + \alpha z_2| = \sqrt{(1-\alpha)^2 + (3-\alpha)^2}$ or the squared modulus   *(M1)(A1)*

attempt to minimise $2\alpha^2 - 8\alpha + 10$ or their quadratic or its half or its square root   *M1*

obtain $\alpha = 2$ at minimum   *(A1)*

state $\sqrt{2}$ as final answer   *A1*

[5 marks]

Examiners report

Most candidates obtained the correct quadratic or its square root, but few knew how to set about minimising it.

27.

Markscheme

METHOD 1

substituting

$-5 + 12i + a(2 + 3i) + b = 0$ *(A1)*

equating real or imaginary parts   *(M1)*

$12 + 3a = 0 \Rightarrow a = -4$ *A1*

$-5 + 2a + b = 0 \Rightarrow b = 13$ *A1*

METHOD 2

other root is $2 - 3i$ *(A1)*

considering either the sum or product of roots or multiplying factors   *(M1)*

$4 = -a$ (sum of roots) so $a = -4$ *A1*

$13 = b$ (product of roots)   *A1*

[4 marks]

Examiners report

[N/A]

28.

Markscheme

the number of ways of allocating presents to the first child is $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ $\left(\text{or} \begin{pmatrix} 7 \\ 2 \end{pmatrix} \right)$ *(A1)*

multiplying by $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\left(\text{or} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \text{ or } \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right)$ *(M1)(A1)*

Note: Award *M1* for multiplication of combinations.

$\begin{pmatrix} 7 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 210$ *A1*

[4 marks]

Examiners report

[N/A]

29.

Markscheme

(a)
$$\begin{bmatrix} x + 2y - z = 2 \\ 2x + y + z = 1 \\ -x + 4y + az = 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x + 2y - z = 2 \\ -3y + 3z = -3 \\ 6y + (a-1)z = 6 \end{bmatrix} \quad \textbf{\textit{M1A1}}$$

$$\rightarrow \begin{bmatrix} x + 2y - z = 2 \\ -3y + 3z = -3 \\ (a+5)z = 0 \end{bmatrix} \quad \textbf{\textit{A1}}$$

(or equivalent)

if not a unique solution then $a = -5$ *A1*

Note: The first *M1* is for attempting to eliminate a variable, the first *A1* for obtaining two expression in just two variables (plus *a*), and the second *A1* for obtaining an expression in just *a* and one other variable

[4 marks]

(b)   if $a = -5$ there are an infinite number of solutions as last equation always true   *R1*
and if $a \neq -5$ there is a unique solution   *R1*
hence always a solution   *AG*
[2 marks]

Total [6 marks]

Examiners report

[N/A]

30.   *[8 marks]*

Markscheme

if $n = 0$

$7^3 + 2 = 345$ which is divisible by 5, hence true for $n = 0$ *A1*

Note: Award *A0* for using $n = 1$ but do not penalize further in question.

assume true for $n = k$ *M1*

Note: Only award the *M1* if truth is assumed.

so $7^{8k+3} + 2 = 5p, \ p \in \bullet$ *A1*

if $n = k + 1$

$7^{8(k+1)+3} + 2$ *M1*

$= 7^8 7^{8k+3} + 2$ *M1*

$= 7^8 (5p - 2) + 2$ *A1*

$= 7^8 .5p - 2.7^8 + 2$

$= 7^8 .5p - 11\,529\,600$

$= 5(7^8 p - 2\,305\,920)$ *A1*

hence if true for $n = k$, then also true for $n = k + 1$. Since true for $n = 0$, then true for all $n \in \bullet$ *R1*

Note: Only award the *R1* if the first two *M1*s have been awarded.

[8 marks]

Examiners report

[N/A]

31. a. [4 marks]

Markscheme

$\left(A \begin{pmatrix} 6 \\ 5 \end{pmatrix} 2^5 B + 3 \begin{pmatrix} 6 \\ 4 \end{pmatrix} 2^4 B^2 \right) x^5$ *M1A1A1*

$= \left(192AB + 720B^2 \right) x^5$ *A1*

[4 marks]

Examiners report

[N/A]

31. b. [4 marks]

Markscheme

METHOD 1

$x = \frac{1}{6}$, $A = \frac{3}{6}$ $\left(= \frac{1}{2}\right)$, $B = \frac{4}{6}$ $\left(= \frac{2}{3}\right)$ *A1A1A1*

probability is $\frac{4}{81}$ $(= 0.0494)$ *A1*

METHOD 2

P (5 eaten) =P (M eats 1) P (N eats 4) + P (M eats 0) P (N eats 5) *(M1)*

$= \frac{1}{2} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + \frac{1}{2} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)$ *(A1)(A1)*

$= \frac{4}{81}$ $(= 0.0494)$ *A1*

[4 marks]

Examiners report

[N/A]

32.

Markscheme

(a) **METHOD 1**

$\frac{1}{2+3i} + \frac{1}{3+2i} = \frac{2-3i}{4+9} + \frac{3-2i}{9+4}$ *M1A1*

$\frac{10}{w} = \frac{5-5i}{13}$ *A1*

$w = \frac{130}{5-5i}$

$= \frac{130 \times 5 \times (1+i)}{50}$

$w = 13 + 13i$ *A1*

[4 marks]

METHOD 2

$\frac{1}{2+3i} + \frac{1}{3+2i} = \frac{3+2i+2+3i}{(2+3i)(3+2i)}$ *M1A1*

$\frac{10}{w} = \frac{5+5i}{13i}$ *A1*

$\frac{w}{10} = \frac{13i}{5+5i}$

$w = \frac{130i}{(5+5i)} \times \frac{(5-5i)}{(5-5i)}$

$= \frac{650+650i}{50}$

$= 13 + 13i$ *A1*

[4 marks]

(b) $w^* = 13 - 13i$ *A1*

$z = \sqrt{338} e^{-\frac{\pi}{4}i}$ $\left(= 13\sqrt{2} e^{-\frac{\pi}{4}i}\right)$ *A1A1*

Note: Accept $\theta = \frac{7\pi}{4}$.

Do not accept answers for θ given in degrees.

[3 marks]

Total [7 marks]

Examiners report

[N/A]

33.

Markscheme

(a) $\sin x,\ \sin 2x$ and $4\sin x\cos^2 x|$

$r = \frac{2\sin x \cos x}{\sin x} = 2\cos x|$ **A1**

Note: Accept $\frac{\sin 2x}{\sin x}|$

[1 mark]

(b)   **EITHER**

$|r| < 1 \Rightarrow |2\cos x| < 1|$ **M1**

OR

$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1|$ **M1**

THEN

$0 < \cos x < \frac{1}{2}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}|$

$-\frac{\pi}{2} < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \frac{\pi}{2}|$ **A1A1**

[3 marks]

(c) $S_\infty = \frac{\sin x}{1 - 2\cos x}|$ **M1**

$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1 - 2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}|$

$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}|$ **A1A1**

Note: Award **A1** for correct numerator and **A1** for correct denominator.

$= \frac{\sqrt{15}}{2}|$ **AG**

[3 marks]

Total [7 marks]

Examiners report

[N/A]

34.

Markscheme

(a)   (i) $n = 27$ *(A1)*

METHOD 1

$S_{27} = \frac{14+196}{2} \times 27$ *(M1)*

$= 2835$ *A1*

METHOD 2

$S_{27} = \frac{27}{2}\left(2 \times 14 + 26 \times 7\right)$ *(M1)*

$= 2835$ *A1*

METHOD 3

$S_{27} \sum\limits_{n=1}^{27} 7 + 7n$ *(M1)*

$= 2835$ *A1*

(ii) $\sum\limits_{n=1}^{27}(7 + 7n)$ or equivalent   *A1*

Note: Accept $\sum\limits_{n=2}^{28} 7n$

[4 marks]

(b) $\frac{n}{2}\left(2000 - 6(n-1)\right) < 0$ *(M1)*

$n > 334.333$

$n = 335$ *A1*

Note: Accept working with equalities.

[2 marks]

Total [6 marks]

Examiners report

[N/A]

35.   *[6 marks]*

Markscheme

expanding $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ *A1*

expanding $\left(\frac{1}{2} + 2x\right)^6$ gives

$64x^6 + 192x^4 + 240x^2 + \frac{60}{x^2} + \frac{12}{x^4} + \frac{1}{x^6} + 160$ *(M1)A1A1*

Note: Award *(M1)* for an attempt at expanding using binomial.

Award *A1* for $\frac{60}{x^2}$.

Award *A1* for $\frac{12}{x^4}$.

$\frac{60}{x^2} \times -1 + \frac{12}{x^4} \times -3x^2$ *(M1)*

Note: Award *(M1)* only if both terms are considered.

therefore coefficient x^{-2} is -96 *A1*

Note: Accept $-96x^{-2}$

Note: Award full marks if working with the required terms only without giving the entire expansion.

[6 marks]

Examiners report

[N/A]

Markscheme

let $\mathrm{P}(n)$ be the proposition $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta), n \in \mathbb{C}^+$

let $n = 1 \Rightarrow$

LHS $= r(\cos\theta + \mathrm{i}\sin\theta)$

RHS $= r(\cos\theta + \mathrm{i}\sin\theta), \;\therefore \mathrm{P}(1)$ is true   *R1*

assume true for $n = k \Rightarrow r^k(\cos\theta + \mathrm{i}\sin\theta)^k = r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)$ *M1*

Note: Only award the *M1* if truth is assumed.

now show $n = k$ true implies $n = k + 1$ also true

$r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ *M1*

$= r^{k+1}\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)(\cos\theta + \mathrm{i}\sin\theta)$

$= r^{k+1}\left(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)\right)$ *A1*

$= r^{k+1}\left(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)\right)$ *A1*

$= r^{k+1}\left(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\right) \Rightarrow n = k + 1$ is true   *A1*

$\mathrm{P}(k)$ true implies $\mathrm{P}(k+1)$ true and $\mathrm{P}(1)$ is true, therefore by mathematical induction statement is true for $n \geqslant 1$ *R1*

Note: Only award the final *R1* if the first 4 marks have been awarded.

[7 marks]

36b.

[4 marks]

Markscheme

(i) $u = 2\text{cis}\left(\frac{\pi}{3}\right)$ **A1**

$v = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$ **A1**

Notes: Accept 3 sf answers only. Accept equivalent forms.

Accept $2e^{\frac{\pi}{3}i}$ and $\sqrt{2}e^{-\frac{\pi}{4}i}$.

(ii) $u^3 = 2^3\text{cis}(\pi) = -8$

$v^4 = 4\text{cis}(-\pi) = -4$ **(M1)**

$u^3v^4 = 32$ **A1**

Notes: Award **(M1)** for an attempt to find u^3 and v^4.

Accept equivalent forms.

[4 marks]

36c.

[1 mark]

Markscheme

A1
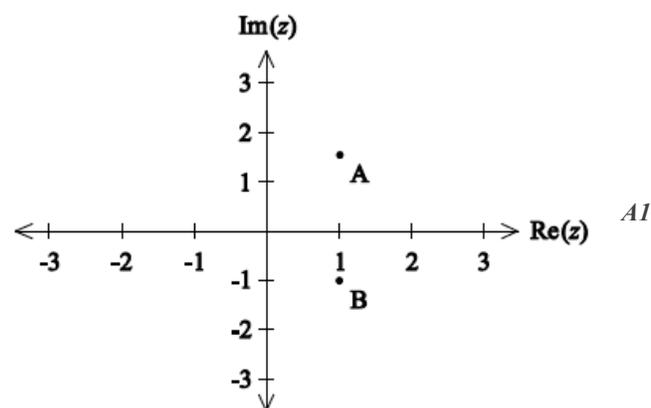
Note: Award **A1** if A or $1 + \sqrt{3}i$ and B or $1 - i$ are in their correct quadrants, are aligned vertically and it is clear that $|u| > |v|$

.

[1 mark]

Examiners report

36d.

[3 marks]

Markscheme

Area $= \frac{1}{2} \times 2 \times \sqrt{2} \times \sin\left(\frac{5\pi}{12}\right)$ *M1A1*

$= 1.37 \left(= \frac{\sqrt{2}}{4}\left(\sqrt{6} + \sqrt{2}\right)\right)$ *A1*

Notes: Award *M1A0A0* for using $\frac{7\pi}{12}$.

[3 marks]

Examiners report

36e.

[5 marks]

Markscheme

$(z - 1 + \mathrm{i})(z - 1 - \mathrm{i}) = z^2 - 2z + 2$ *M1A1*

Note: Award *M1* for recognition that a complex conjugate is also a root.

$\left(z - 1 - \sqrt{3}\mathrm{i}\right)\left(z - 1 + \sqrt{3}\mathrm{i}\right) = z^2 - 2z + 4$ *A1*

$\left(z^2 - 2z + 2\right)\left(z^2 - 2z + 4\right) = z^4 - 4z^3 + 10z^2 - 12z + 8$ *M1A1*

Note: Award *M1* for an attempt to expand two quadratics.

[5 marks]

Examiners report

37.

[7 marks]

Markscheme

$n = 1 : 1^3 + 11 = 12$
$= 3 \times 4$ **or** a multiple of 3   *A1*
assume the proposition is true for $n = k$ $(ie \; k^3 + 11k = 3 \text{ m})$ *M1*

Note: Do not award *M1* for statements with "Let $n = k$".

consider $n = k + 1 :$ $(k + 1)^3 + 11(k + 1)$ *M1*
$= k^3 + 3k^2 + 3k + 1 + 11k + 11$ *A1*
$= k^3 + 11k + (3k^2 + 3k + 12)$ *M1*
$= 3(m + k^2 + k + 4)$ *A1*

Note: Accept $k^3 + 11k + 3(k^2 + k + 4)$ or statement that $k^3 + 11k + (3k^2 + 3k + 12)$ is a multiple of 3.

true for $n = 1$ and $n = k$ true $\Rightarrow n = k + 1$ true
hence true for all $n \in \mathbb{Z}^+$ *R1*

Note: Only award the final *R1* if at least 4 of the previous marks have been achieved.

[7 marks]

Examiners report

It was pleasing to see a great many clear and comprehensive answers for this relatively straightforward induction question. The inductive step only seemed to pose problems for the very weakest candidates. As in previous sessions, marks were mainly lost by candidates writing variations on 'Let $n = k$', rather than 'Assume true for $n = k$'. The final reasoning step still needs attention, with variations on '$n = k + 1$ true $\Rightarrow n = k$ true' evident, suggesting that mathematical induction as a technique is not clearly understood.

38.   *[7 marks]*

Markscheme

(a)   **METHOD 1**

$a + ar = 10$ *A1*

$a + ar + ar^2 + ar^3 = 30$ *A1*

$a + ar = 10 \Rightarrow ar^2 + ar^3 = 10r^2$ **or** $ar^2 + ar^3 = 20$ *M1*

$10 + 10r^2 = 30$ **or** $r^2(a + ar) = 20$ *A1*

$\Rightarrow r^2 = 2$ *AG*

METHOD 2

$\frac{a(1-r^2)}{1-r} = 10$ and $\frac{a(1-r^4)}{1-r} = 30$ *M1A1*

$\Rightarrow \frac{1-r^4}{1-r^2} = 3$ *M1*

leading to either $1 + r^2 = 3$ (or $r^4 - 3r^2 + 2 = 0$)   *A1*

$\Rightarrow r^2 = 2$ *AG*

[4 marks]

(b)   (i) $a + a\sqrt{2} = 10$

$\Rightarrow a = \frac{10}{1+\sqrt{2}}$ **or** $a = 10\left(\sqrt{2} - 1\right)$ *A1*

(ii) $S_{10} = \frac{10}{1+\sqrt{2}} \left(\frac{\sqrt{2}^{10}-1}{\sqrt{2}-1} \right) \; (= 10 \times 31)$ *M1*

$= 310$ *A1*

[3 marks]

Total [7 marks]

Examiners report

39a.   *[2 marks]*

Markscheme

$z^n + z^{-n} = \cos n\theta + i \sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ *M1*

$= \cos n\theta + \cos n\theta + i \sin n\theta - i \sin n\theta$ *A1*

$= 2 \cos n\theta$ *AG*

[2 marks]

Examiners report

39b.   *[1 mark]*

Markscheme

(b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3 \left(\frac{1}{z}\right) + 6z^2 \left(\frac{1}{z^2}\right) + 4z \left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***A1***

Note: Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

[1 mark]

Examiners report

Part b) was usually answered correctly.

[4 marks]

Markscheme

METHOD 1

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ ***M1***

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$ ***M1***

$= \frac{1}{4}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right)$ ***A1***

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$ ***A1***

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

[4 marks]

Examiners report

Part c) was again often answered correctly, though some candidates often less successfully utilised a trig-only approach rather than taking note of part b).

[3 marks]

Markscheme

$$\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\tfrac{1}{z}\right) + 15z^4\left(\tfrac{1}{z^2}\right) + 20z^3\left(\tfrac{1}{z^3}\right) + 15z^2\left(\tfrac{1}{z^4}\right) + 6z\left(\tfrac{1}{z^5}\right) + \tfrac{1}{z^6} \quad \textbf{\textit{M1}}$$

$$\left(z+z^{-1}\right)^6 = \left(z^6 + \tfrac{1}{z^6}\right) + 6\left(z^4 + \tfrac{1}{z^4}\right) + 15\left(z^2 + \tfrac{1}{z^2}\right) + 20$$

$$\left(2\cos\theta\right)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20 \quad \textbf{\textit{A1A1}}$$

Note: Award **A1** for RHS, **A1** for LHS, independent of the **M1**.

$$\cos^6\theta = \tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16} \quad \textbf{\textit{AG}}$$

Note: Accept a purely trigonometric solution as for (c).

[3 marks]

Examiners report

Part d) was a good source of marks for those who kept with the spirit of using complex numbers for this type of question. Some limited attempts at trig-only solutions were seen, and correct solutions using this approach were extremely rare.

39e.

[3 marks]

Markscheme

$$\int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta = \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16}\right) d\theta$$

$$= \left[\tfrac{1}{192}\sin 6\theta + \tfrac{3}{64}\sin 4\theta + \tfrac{15}{64}\sin 2\theta + \tfrac{5}{16}\theta\right]_0^{\frac{\pi}{2}} \quad \textbf{\textit{M1A1}}$$

$$= \tfrac{5\pi}{32} \quad \textbf{\textit{A1}}$$

[3 marks]

Examiners report

Part e) was well answered, though numerical slips were often common. A small number integrated $\sin n\theta$ as $n\cos n\theta$.
A large number of candidates did not realise the help that part e) inevitably provided for part f). Some correctly expressed the volume as $\pi\int\cos^4 x\,dx - \pi\int\cos^6 x\,dx$ and thus gained the first 2 marks but were able to progress no further. Only a small number of able candidates were able to obtain the correct answer of $\tfrac{\pi^2}{32}$.

39f.

[4 marks]

Markscheme

$$V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\cos^4 x\, dx \quad \textbf{\textit{M1}}$$

$$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x\, dx - \pi\int_0^{\frac{\pi}{2}} \cos^6 x\, dx \quad \textbf{\textit{M1}}$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x\, dx = \tfrac{3\pi}{16} \quad \textbf{\textit{A1}}$$

$$V = \tfrac{3\pi^2}{16} - \tfrac{5\pi^2}{32} = \tfrac{\pi^2}{32} \quad \textbf{\textit{A1}}$$

Note: Follow through from an incorrect r in (c) provided the final answer is positive.

39g.

Markscheme

(i) constant term $= \left.\binom{2k}{k}\right| = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2} \left(\text{accept } C_k^{2k}\right)\Big|$ **A1**

(ii) $2^{2k} \int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{(k!)^2} \left.\frac{\pi}{2}\right|$ **A1**

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \left.\frac{(2k)!\pi}{2^{2k+1}(k!)^2}\right| \left(\text{or } \left.\frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)\right|$ **A1**

[3 marks]

Examiners report

Part g) proved to be a challenge for the vast majority, though it was pleasing to see some of the highest scoring candidates gain all 3 marks.

40.

Markscheme

(a) **METHOD 1**

$34 = a + 3d|$ and $76 = a + 9d|$ **(M1)**

$d = 7|$ **A1**

$a = 13|$ **A1**

METHOD 2

$76 = 34 + 6d|$ **(M1)**

$d = 7|$ **A1**

$34 = a + 21|$

$a = 13|$ **A1**

[3 marks]

(b) $\frac{n}{2}\left(26 + 7(n-1)\right) > 5000|$ **(M1)(A1)**

$n > 36.463\ldots|$ **(A1)**

Note: Award **M1A1A1** for using either an equation, a graphical approach or a numerical approach.

$n = 37|$ **A1** **N3**

[4 marks]

Total [7 marks]

41.   *[6 marks]*

Markscheme

(a) $\dfrac{a+i}{a-i} \times \dfrac{a+i}{a+i}$ *M1*

$= \dfrac{a^2-1+2ai}{a^2+1}$ $\left(= \dfrac{a^2-1}{a^2+1} + \dfrac{2a}{a^2+1}\,i\right)$ *A1*

   (i) z is real when $a = 0$ *A1*

   (ii) z is purely imaginary when $a = \pm 1$ *A1*

Note: Award *M1A0A1A0* for $\dfrac{a^2-1+2ai}{a^2-1}$ $\left(= 1 + \dfrac{2a}{a^2-1}\,i\right)$ leading to $a = 0$ in (i).

[4 marks]

(b) **METHOD 1**

attempting to find either $|z|$ or $|z|^2$ by expanding and simplifying

eg $|z|^2 = \dfrac{\left(a^2-1\right)^2 + 4a^2}{\left(a^2+1\right)^2} = \dfrac{a^4+2a^2+1}{\left(a^2+1\right)^2}$ *M1*

$= \dfrac{\left(a^2+1\right)^2}{\left(a^2+1\right)^2}$

$|z|^2 = 1 \Rightarrow |z| = 1$ *A1*

METHOD 2

$|z| = \dfrac{|a+i|}{|a-i|}$ *M1*

$|z| = \dfrac{\sqrt{a^2+1}}{\sqrt{a^2+1}} \Rightarrow |z| = 1$ *A1*

[2 marks]

Total [6 marks]

Examiners report

Part (a) was reasonably well done. When multiplying and dividing by the conjugate of $a-i$, some candidates incorrectly determined their denominator as $a^2 - 1$.

In part (b), a significant number of candidates were able to correctly expand and simplify $|z|$ although many candidates appeared to not understand the definition of $|z|$.