

Topic 9 Part 2 [514 marks]

A particle moves in a straight line with velocity v metres per second. At any time t seconds,
 $0 \leq t < \frac{3\pi}{4}$, the velocity is given by the differential equation

$$\frac{dv}{dt} + v^2 + 1 = 0.$$

It is also given that $v = 1$ when $t = 0$.

1a. Find an expression for v in terms of t . [7 marks]

1b. Sketch the graph of v against t , clearly showing the coordinates of any intercepts, and the equations of any asymptotes. [3 marks]

1c. (i) Write down the time T at which the velocity is zero. [3 marks]
(ii) Find the distance travelled in the interval $[0, T]$.

1d. Find an expression for s , the displacement, in terms of t , given that $s = 0$ when $t = 0$. [5 marks]

1e. Hence, or otherwise, show that [4 marks]
 $s = \frac{1}{2} \ln \frac{2}{1+v^2}.$

2. Use L'Hôpital's Rule to find [6 marks]
 $\lim_{x \rightarrow 0} \frac{e^x - 1 - x \cos x}{\sin^2 x}.$

Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2}{1+x}, \text{ where } x > -1 \text{ and } y = 1 \text{ when } x = 0.$$

3a. Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 0.5$. [7 marks]

3b. (i) Show that [8 marks]
 $\frac{d^2 y}{dx^2} = \frac{2y^3 - y^2}{(1+x)^2}.$
(ii) Hence find the Maclaurin series for y , up to and including the term in x^2 .

3c. (i) Solve the differential equation. [6 marks]
(ii) Find the value of a for which
 $y \rightarrow \infty$ as
 $x \rightarrow a.$

4. Find the general solution of the differential equation [7 marks]
 $t \frac{dy}{dt} = \cos t - 2y$, for $t > 0$.

The sequence

$\{u_n\}$ is defined by

$$u_n = \frac{3n+2}{2n-1}, \text{ for}$$

$$n \in \mathbb{Z}^+.$$

5a. Show that the sequence converges to a limit L , the value of which should be stated. [3 marks]

- 5b. Find the least value of the integer N such that [4 marks]
 $|u_n - L| < \varepsilon$, for all $n > N$ where
 (i)
 $\varepsilon = 0.1$;
 (ii)
 $\varepsilon = 0.00001$.

- 5c. For each of the sequences [6 marks]
 $\left\{\frac{u_n}{n}\right\}$, $\left\{\frac{1}{2u_n-2}\right\}$ and
 $\{(-1)^n u_n\}$, determine whether or not it converges.

- 5d. Prove that the series [2 marks]
 $\sum_{n=1}^{\infty} (u_n - L)$ diverges.

- 6a. Find the set of values of k for which the improper integral [6 marks]
 $\int_2^{\infty} \frac{dx}{x(\ln x)^k}$ converges.

- 6b. Show that the series [5 marks]
 $\sum_{r=2}^{\infty} \frac{(-1)^r}{r \ln r}$ is convergent but not absolutely convergent.

- 7a. Use the limit comparison test to prove that [5 marks]
 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

- 7b. Using the Maclaurin series for [3 marks]
 $\ln(1+x)$, show that the Maclaurin series for
 $(1+x)\ln(1+x)$ is
 $x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)}.$

- 8a. Prove by mathematical induction that, for [8 marks]
 $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \dots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

8b. (a) Using integration by parts, show that

[17 marks]

$$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C.$$

(b) Solve the differential equation

$$\frac{dy}{dx} = \sqrt{1 - y^2} e^{2x} \sin x, \text{ given that } y = 0 \text{ when } x = 0,$$

writing your answer in the form

$$y = f(x).$$

(c) (i) Sketch the graph of

$$y = f(x), \text{ found in part (b), for}$$

$$0 \leq x \leq 1.5.$$

Determine the coordinates of the point P, the first positive intercept on the x -axis, and mark it on your sketch.

(ii) The region bounded by the graph of

$$y = f(x) \text{ and the } x\text{-axis, between the origin and P, is rotated } 360^\circ \text{ about the } x\text{-axis to form a solid of revolution.}$$

Calculate the volume of this solid.

9a. Find the first three terms of the Maclaurin series for

[6 marks]

$$\ln(1 + e^x).$$

9b. Hence, or otherwise, determine the value of

[4 marks]

$$\lim_{x \rightarrow 0} \frac{2 \ln(1 + e^x) - x - \ln 4}{x^2}.$$

Consider the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \text{ where } y = 1 \text{ when } x = 0.$$

10a. Use Euler's method with step length 0.1 to find an approximate value of y when $x = 0.4$.

[7 marks]

10b. Write down, giving a reason, whether your approximate value for y is greater than or less than the actual value of y .

[1 mark]

The curve C with equation

$$y = f(x) \text{ satisfies the differential equation}$$

$$\frac{dy}{dx} = \frac{y}{\ln y} (x + 2), \quad y > 1,$$

$$\text{and } y = e \text{ when } x = 2.$$

11a. Find the equation of the tangent to C at the point (2, e).

[3 marks]

11b. Find

[11 marks]

$$f(x).$$

11c. Determine the largest possible domain of f . [6 marks]

11d. Show that the equation [4 marks]

$$f(x) = f'(x) \text{ has no solution.}$$

12. Find [5 marks]

$$\lim_{x \rightarrow \frac{1}{2}} \left(\frac{\left(\frac{1}{4} - x^2 \right)}{\cot \pi x} \right).$$

13a. Show that [2 marks]

$$n! \geq 2^{n-1}, \text{ for}$$

$$n \geq 1.$$

13b. Hence use the comparison test to determine whether the series [3 marks]

$$\sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges or diverges.}$$

Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \times 2^n}.$$

14a. Find the radius of convergence of the series. [7 marks]

14b. Hence deduce the interval of convergence. [4 marks]

15a. Using the integral test, show that [4 marks]

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+1} \text{ is convergent.}$$

15b. (i) Show, by means of a diagram, that [4 marks]

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+1} < \frac{1}{4 \times 1^2+1} + \int_1^{\infty} \frac{1}{4x^2+1} dx.$$

(ii) Hence find an upper bound for

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$$

16a. Given that [5 marks]

$$y = \ln\left(\frac{1+e^{-x}}{2}\right), \text{ show that}$$

$$\frac{dy}{dx} = \frac{e^{-y}}{2} - 1.$$

16b. Hence, by repeated differentiation of the above differential equation, find the Maclaurin series for y as far as the term in x^3 , showing that two of the terms are zero. [11 marks]

17. The real and imaginary parts of a complex number [15 marks]

$x + iy$ are related by the differential equation

$$(x + y)\frac{dy}{dx} + (x - y) = 0.$$

By solving the differential equation, given that

$y = \sqrt{3}$ when $x = 1$, show that the relationship between the modulus r and the argument

θ of the complex number is

$$r = 2e^{\frac{\pi}{3} - \theta}.$$

An open glass is created by rotating the curve

$y = x^2$, defined in the domain

$x \in [0, 10]$,

2π radians about the y -axis. Units on the coordinate axes are defined to be in centimetres.

18a. When the glass contains water to a height h cm, find the volume V of water in terms of h . [3 marks]

18b. If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that, [6 marks]

$$\frac{dV}{dt} = -3\sqrt{2\pi V}, \text{ where}$$

t is measured in hours.

18c. If the glass is filled completely, how long will it take for all the water to evaporate? [7 marks]

19. Determine whether the series [6 marks]

$$\sum_{n=1}^{\infty} \frac{n^{10}}{10^n} \text{ is convergent or divergent.}$$

20. (a) Using l'Hopital's Rule, show that [9 marks]

$$\lim_{x \rightarrow \infty} xe^{-x} = 0.$$

(b) Determine

$$\int_0^a xe^{-x} dx.$$

(c) Show that the integral

$$\int_0^{\infty} xe^{-x} dx \text{ is convergent and find its value.}$$

Consider the differential equation

$$x \frac{dy}{dx} - 2y = \frac{x^3}{x^2 + 1}.$$

21. (a) Find an integrating factor for this differential equation.

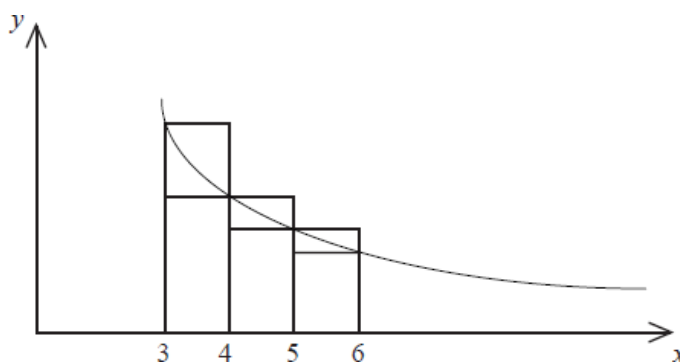
[13 marks]

- (b) Solve the differential equation given that

$$y = 1 \text{ when}$$

$$x = 1, \text{ giving your answer in the form}$$

$$y = f(x).$$



The diagram shows part of the graph of

$$y = \frac{1}{x^3} \text{ together with line segments parallel to the coordinate axes.}$$

22. (a) Using the diagram, show that

[15 marks]

$$\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \dots < \int_3^\infty \frac{1}{x^3} dx < \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \dots$$

- (b) **Hence** find upper and lower bounds for

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

The function f is defined by

$$f(x) = \ln\left(\frac{1}{1-x}\right).$$

23. (a) Write down the value of the constant term in the Maclaurin series for

[17 marks]

$$f(x).$$

- (b) Find the first three derivatives of

$$f(x) \text{ and hence show that the Maclaurin series for}$$

$$f(x) \text{ up to and including the}$$

$$x^3 \text{ term is}$$

$$x + \frac{x^2}{2} + \frac{x^3}{3}.$$

- (c) Use this series to find an approximate value for $\ln 2$.

- (d) Use the Lagrange form of the remainder to find an upper bound for the error in this approximation.

- (e) How good is this upper bound as an estimate for the actual error?

- 24a. Find the value of

[3 marks]

$$\lim_{x \rightarrow 1} \left(\frac{\ln x}{\sin 2\pi x} \right).$$

24b. By using the series expansions for

[7 marks]

e^{x^2} and $\cos x$ evaluate

$$\lim_{x \rightarrow 0} \left(\frac{1 - e^{x^2}}{1 - \cos x} \right) ..$$

25. Find the exact value of

[9 marks]

$$\int_0^\infty \frac{dx}{(x+2)(2x+1)}.$$

A curve that passes through the point (1, 2) is defined by the differential equation

$$\frac{dy}{dx} = 2x(1 + x^2 - y).$$

26. (a) (i) Use Euler's method to get an approximate value of y when $x = 1.3$, taking steps of 0.1. Show intermediate steps to four decimal places in a table. [14 marks]

(ii) How can a more accurate answer be obtained using Euler's method?

(b) Solve the differential equation giving your answer in the form $y = f(x)$.

27. (a) Given that

[14 marks]

$y = \ln \cos x$, show that the first two non-zero terms of the Maclaurin series for y are

$$-\frac{x^2}{2} - \frac{x^4}{12}.$$

(b) Use this series to find an approximation in terms of π for $\ln 2$.

28a. Find the radius of convergence of the series

[6 marks]

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)3^n}.$$

28b. Determine whether the series

[7 marks]

$$\sum_{n=0}^{\infty} \left(\sqrt[3]{n^3 + 1} - n \right)$$
 is convergent or divergent.

29. (a) Show that the solution of the homogeneous differential equation

[12 marks]

$$\frac{dy}{dx} = \frac{y}{x} + 1, \quad x > 0,$$

given that

$$y = 0 \text{ when } x = e, \text{ is } y = x(\ln x - 1).$$

(b) (i) Determine the first three derivatives of the function

$$f(x) = x(\ln x - 1).$$

(ii) Hence find the first three non-zero terms of the Taylor series for $f(x)$ about $x = 1$.

30a. (i) Show that

[8 marks]

$$\int_1^\infty \frac{1}{x(x+p)} dx, \quad p \neq 0 \text{ is convergent if } p > -1 \text{ and find its value in terms of } p.$$

(ii) Hence show that the following series is convergent.

$$\frac{1}{1 \times 0.5} + \frac{1}{2 \times 1.5} + \frac{1}{3 \times 2.5} + \dots$$

[11 marks]

30b. Determine, for each of the following series, whether it is convergent or divergent.

(i)

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n(n+3)}\right)$$

(ii)

$$\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{12}} + \sqrt{\frac{1}{20}} + \dots$$

[12 marks]

31. The function

$f(x) = \frac{1+ax}{1+bx}$ can be expanded as a power series in x , within its radius of convergence R , in the form

$$f(x) \equiv 1 + \sum_{n=1}^{\infty} c_n x^n.$$

(a) (i) Show that

$$c_n = (-b)^{n-1}(a-b).$$

(ii) State the value of R .

(b) Determine the values of a and b for which the expansion of $f(x)$ agrees with that of

e^x up to and including the term in

$$x^2.$$

(c) Hence find a rational approximation to

$$e^{\frac{1}{3}}.$$

32. (a) Show that the solution of the differential equation

[17 marks]

$$\frac{dy}{dx} = \cos x \cos^2 y,$$

given that

$$y = \frac{\pi}{4} \text{ when } x = \pi, \text{ is } y = \arctan(1 + \sin x).$$

(b) Determine the value of the constant a for which the following limit exists

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\arctan(1 + \sin x) - a}{\left(x - \frac{\pi}{2}\right)^2}$$

and evaluate that limit.

33. A gourmet chef is renowned for her spherical shaped soufflé. Once it is put in the oven, its volume increases at a rate proportional to its radius.

[13 marks]

(a) Show that the radius r cm of the soufflé, at time t minutes after it has been put in the oven, satisfies the differential equation

$$\frac{dr}{dt} = \frac{k}{r}, \text{ where } k \text{ is a constant.}$$

(b) Given that the radius of the soufflé is 8 cm when it goes in the oven, and 12 cm when it's cooked 30 minutes later, find, to the nearest cm, its radius after 15 minutes in the oven.

34. The population of mosquitoes in a specific area around a lake is controlled by pesticide. The rate of decrease of the number of mosquitoes is proportional to the number of mosquitoes at any time t . Given that the population decreases from 500 000 to 400 000 in a five year period, find the time it takes in years for the population of mosquitoes to decrease by half. [8 marks]

35. Consider the differential equation [20 marks]

$$\frac{dy}{dx} = \frac{y^2 + x^2}{2x^2} \text{ for which } y = -1 \text{ when } x = 1.$$

- (a) Use Euler's method with a step length of 0.25 to find an estimate for the value of y when $x = 2$.

- (b) (i) Solve the differential equation giving your answer in the form

$$y = f(x).$$

- (ii) Find the value of y when $x = 2$.

The function f is defined on the domain

$$\left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\text{ by } f(x) = \ln(1 + \sin x).$$

- 36a. Show that [4 marks]

$$f''(x) = -\frac{1}{(1 + \sin x)}.$$

- 36b. (i) Find the Maclaurin series for [7 marks]

$f(x)$ up to and including the term in

$$x^4.$$

- (ii) Explain briefly why your result shows that f is neither an even function nor an odd function.

- 36c. Determine the value of [3 marks]

$$\lim_{x \rightarrow 0} \frac{\ln(1 + \sin x) - x}{x^2}.$$

Consider the differential equation

$$x \frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0, \quad x^2 > y^2.$$

- 37a. Show that this is a homogeneous differential equation. [1 mark]

- 37b. Find the general solution, giving your answer in the form [7 marks]

$$y = f(x).$$

Consider the differential equation

$$\frac{dy}{dx} = 2e^x + y \tan x, \text{ given that } y = 1 \text{ when } x = 0.$$

The domain of the function y is

$$\left[0, \frac{\pi}{2}\right[.$$

38a. By finding the values of successive derivatives when $x = 0$, find the Maclaurin series for y as far as the term in x^3 . [6 marks]

38b. (i) Differentiate the function $e^x(\sin x + \cos x)$ and hence show that [9 marks]

$$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + c.$$

(ii) Find an integrating factor for the differential equation and hence find the solution in the form $y = f(x)$.

Let

$$f(x) = 2x + |x|,$$

$$x \in \mathbb{R}.$$

39a. Prove that f is continuous but not differentiable at the point $(0, 0)$. [7 marks]

39b. Determine the value of $\int_{-a}^a f(x) dx$ where [3 marks]

$$a > 0.$$

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(n-1)x^n}{n^2 \times 2^n}.$$

40a. Find the radius of convergence. [4 marks]

40b. Find the interval of convergence. [9 marks]