Topic 6 Part 2 *[456 marks]*

1. Paint is poured into a tray where it forms a circular pool with a uniform thickness of 0.5 cm. If the paint is poured at a constant rate of *[6 marks]*

$4\ \text{cm}^3\text{s}^{-1}$, find the rate of increase of the radius of the circle when the radius is 20 cm.

2. A curve is defined by the equation *[7 marks]*

$8y\ln x - 2x^2 + 4y^2 = 7$. Find the equation of the tangent to the curve at the point where $x = 1$ and

$y > 0$.

3a. Find all values of x for *[2 marks]*

$0.1 \leqslant x \leqslant 1$ such that

$\sin(\pi x^{-1}) = 0$.

3b. Find *[3 marks]*

$\int_{\frac{1}{n+1}}^{\frac{1}{n}} \pi x^{-2}\sin(\pi x^{-1})\mathrm{d}x$, showing that it takes different integer values when n is even and when n is odd.

3c. Evaluate *[2 marks]*

$\int_{0.1}^{1} \left|\pi x^{-2}\sin(\pi x^{-1})\right|\mathrm{d}x$.

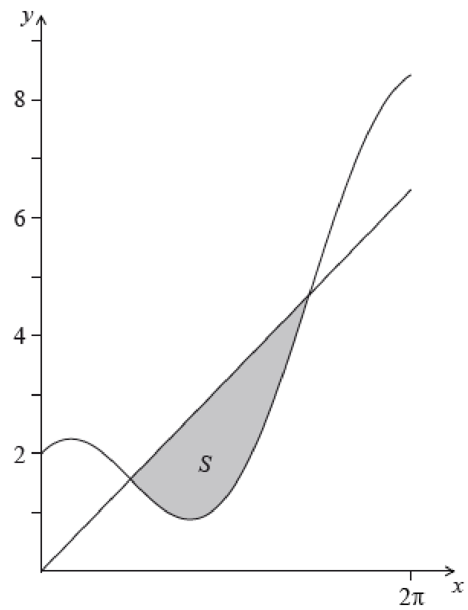
4a. Consider the functions *[5 marks]*
$f(x) = (\ln x)^2,\ x > 1$ and
$g(x) = \ln(f(x)),\ x > 1$.
(i) Find
$f'(x)$.
(ii) Find
$g'(x)$.
(iii) Hence, show that
$g(x)$ is increasing on
$]1, \infty[$.

4b. Consider the differential equation *[12 marks]*

$$(\ln x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{2x-1}{(\ln x)},\ x > 1.$$

(i) Find the general solution of the differential equation in the form
$y = h(x)$.
(ii) Show that the particular solution passing through the point with coordinates
$\left(\mathrm{e},\ \mathrm{e}^2\right)$ is given by
$y = \frac{x^2 - x + \mathrm{e}}{(\ln x)^2}$.
(iii) Sketch the graph of your solution for
$x > 1$, clearly indicating any asymptotes and any maximum or minimum points.

The shaded region S is enclosed between the curve
$y = x + 2\cos x$, for
$0 \leqslant x \leqslant 2\pi$, and the line
$y = x$, as shown in the diagram below.



5a. Find the coordinates of the points where the line meets the curve. *[3 marks]*

5b. The region
S is rotated by
2π about the
x-axis to generate a solid. *[5 marks]*
(i) Write down an integral that represents the volume
V of the solid.
(ii) Find the volume
V.

Let
$f(x) = x(x+2)^6$.

6a. Solve the inequality
$f(x) > x$. *[5 marks]*

6b. Find
$\int f(x)\mathrm{d}x$. *[5 marks]*

Let
$f(x) = \frac{\mathrm{e}^{2x}+1}{\mathrm{e}^{x}-2}$.

7a. Find the equations of the horizontal and vertical asymptotes of the curve
$y = f(x)$. *[4 marks]*

7b. (i) Find $f'(x)$. [8 marks]

(ii) Show that the curve has exactly one point where its tangent is horizontal.

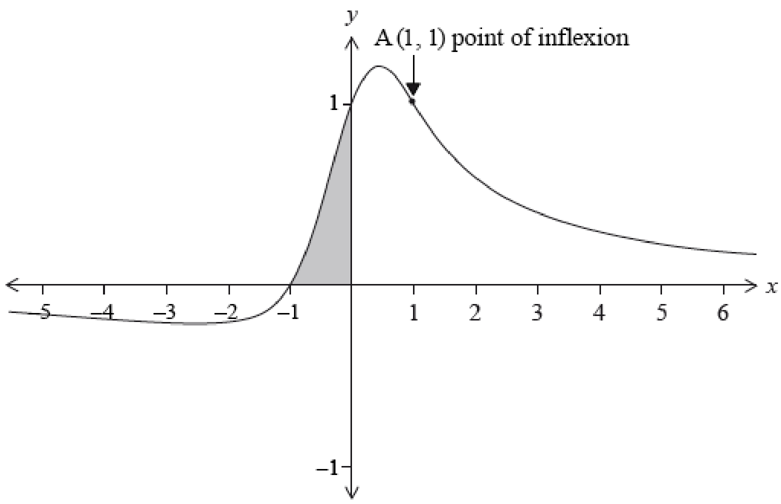
(iii) Find the coordinates of this point.

7c. Find the equation of L_1, the normal to the curve at the point where it crosses the y-axis. [4 marks]

The line L_2 is parallel to L_1 and tangent to the curve $y = f(x)$.

7d. Find the equation of the line L_2. [5 marks]

The graph of the function $f(x) = \frac{x+1}{x^2+1}$ is shown below.



8a. Find $f'(x)$. [2 marks]

8b. Hence find the x-coordinates of the points where the gradient of the graph of f is zero. [1 mark]

8c. Find $f''(x)$ expressing your answer in the form $\frac{p(x)}{(x^2+1)^3}$, where $p(x)$ is a polynomial of degree 3. [3 marks]

The point (1, 1) is a point of inflexion. There are two other points of inflexion.

8d. Find the x-coordinates of the other two points of inflexion. [4 marks]

8e. Find the area of the shaded region. Express your answer in the form $\frac{\pi}{a} - \ln\sqrt{b}$, where a and b are integers. *[6 marks]*

Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$

$g(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$

9a. Sketch the graph of $y = h(x)$. *[2 marks]*

9b. Find an expression for the composite function $h \circ g(x)$ and state its domain. *[2 marks]*

9c. Given that $f(x) = h(x) + h \circ g(x)$, *[7 marks]*

(i) find $f'(x)$ in simplified form;

(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

9d. Nigel states that f is an odd function and Tom argues that f is an even function. *[3 marks]*

(i) State who is correct and justify your answer.

(ii) Hence find the value of $f(x)$ for $x < 0$.

The graphs of $y = x^2 \mathrm{e}^{-x}$ and $y = 1 - 2\sin x$ for $2 \leqslant x \leqslant 7$ intersect at points A and B. The x-coordinates of A and B are x_{A} and x_{B}.

10a. Find the value of x_{A} and the value of x_{B}. *[2 marks]*

10b. Find the area enclosed between the two graphs for $x_{\mathrm{A}} \leqslant x \leqslant x_{\mathrm{B}}$. *[3 marks]*

11. Sand is being poured to form a cone of height h cm and base radius r cm. The height remains equal to the base radius at all times. The height of the cone is increasing at a rate of $0.5\ \mathrm{cm\,min}^{-1}$. *[5 marks]*

Find the rate at which sand is being poured, in $\mathrm{cm}^3\ \mathrm{min}^{-1}$, when the height is 4 cm.

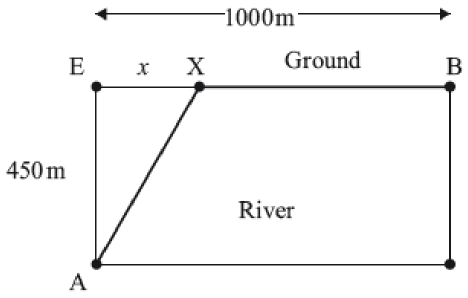
Consider the curve with equation $\left(x^2+y^2\right)^2 = 4xy^2$.

12a. Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[5 marks]*

12b. Find the equation of the normal to the curve at the point (1, 1). *[3 marks]*

13. Engineers need to lay pipes to connect two cities A and B that are separated by a river of width 450 metres as shown in the following diagram. They plan to lay the pipes under the river from A to X and then under the ground from X to B. The cost of laying the pipes under the river is five times the cost of laying the pipes under the ground. *[15 marks]*

Let $\mathrm{EX} = x$.



Let k be the cost, in dollars per metre, of laying the pipes under the ground.

(a) Show that the total cost C, in dollars, of laying the pipes from A to B is given by $C = 5k\sqrt{202\,500 + x^2} + (1000 - x)k$.

(b) (i) Find $\frac{\mathrm{d}C}{\mathrm{d}x}$.

(ii) Hence find the value of x for which the total cost is a minimum, justifying that this value is a minimum.

(c) Find the minimum total cost in terms of k.

The angle at which the pipes are joined is $\mathrm{A\hat{X}B} = \theta$.

(d) Find θ for the value of x calculated in (b).

For safety reasons θ must be at least 120°.

Given this new requirement,

(e) (i) find the new value of x which minimises the total cost;

(ii) find the percentage increase in the minimum total cost.

Particle A moves such that its velocity v ms^{-1}, at time t seconds, is given by $v(t) = \frac{t}{12+t^4}$, $t \geqslant 0$.

14a. Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. *[2 marks]*

14b. Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4} \mathrm{d}t$. *[4 marks]*

14c. Find the exact distance travelled by particle A between $t = 0$ and $t = 6$ seconds. *[3 marks]*

Give your answer in the form $k \arctan(b)$, k, $b \in \mathbb{R}$.

Particle B moves such that its velocity v ms^{-1} is related to its displacement s m, by the equation $v(s) = \arcsin(\sqrt{s})$.

14d. Find the acceleration of particle B when $s = 0.1$ m. *[3 marks]*

15. A curve has equation $x^3y^2 + x^3 - y^3 + 9y = 0$. Find the coordinates of the three points on the curve where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[7 marks]*

The function f is given by $f(x) = x\mathrm{e}^{-x}$ $(x \geqslant 0)$.

16a. (i) Find an expression for $f'(x)$. *[3 marks]*

(ii) Hence determine the coordinates of the point A, where $f'(x) = 0$.

16b. Find an expression for $f''(x)$ and hence show the point A is a maximum. *[3 marks]*

16c. Find the coordinates of B, the point of inflexion. *[2 marks]*

16d. The graph of the function g is obtained from the graph of f by stretching it in the x-direction by a scale factor 2. *[5 marks]*

(i) Write down an expression for $g(x)$.

(ii) State the coordinates of the maximum C of g.

(iii) Determine the x-coordinates of D and E, the two points where $f(x) = g(x)$.

16e. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, showing clearly the points A, B, C, D and E. *[4 marks]*

16f. Find an exact value for the area of the region bounded by the curve $y = g(x)$, the x-axis and the line $x = 1$. *[3 marks]*

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

17a. Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta,\ n \in \mathbb{Z}^+$. *[2 marks]*

17b. Expand $\left(z + z^{-1}\right)^4$. *[1 mark]*

17c. Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where $p,\ q$ and r are constants to be determined. *[4 marks]*

17d. Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. *[3 marks]*

17e. Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \mathrm{d}\theta$. *[3 marks]*

The region S is bounded by the curve $y = \sin x\cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

17f. S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4 marks]*

17g. (i) Write down an expression for the constant term in the expansion of $\left(z+z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^{+}$. [3 marks]

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k} \theta \mathrm{d}\theta$ in terms of k.

18. By using the substitution $x = 2\tan u$, show that $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}} = \frac{-\sqrt{x^2+4}}{4x} + C$. [7 marks]

A function f is defined by $f(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$, $x \in \mathbb{R}$.

19a. (i) Explain why the inverse function f^{-1} does not exist. [14 marks]

(ii) Show that the equation of the normal to the curve at the point P where $x = \ln 3$ is given by $9x + 12y - 9\ln 3 - 20 = 0$.

(iii) Find the x-coordinates of the points Q and R on the curve such that the tangents at Q and R pass through $(0, 0)$.

19b. The domain of f is now restricted to $x \geqslant 0$. [8 marks]

(i) Find an expression for $f^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y = f(x)$ and the lines $x = 0$ and $y = 5$ is rotated through an angle of 2π radians about the y-axis.

The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

20a. Find the value of p and the value of q . [4 marks]

20b. The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph. [2 marks]

The curve C has equation $y = \frac{1}{8}(9 + 8x^2 - x^4)$.

21a. Find the coordinates of the points on C at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [4 marks]

21b. The tangent to C at the point P(1, 2) cuts the x-axis at the point T. Determine the coordinates of T. *[4 marks]*

21c. The normal to C at the point P cuts the y-axis at the point N. Find the area of triangle PTN. *[7 marks]*

22a. (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \leqslant x \leqslant \frac{\pi}{2}$. *[9 marks]*

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \leqslant x \leqslant \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

22b. Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$. *[8 marks]*

22c. The increasing function f satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$. *[8 marks]*

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

A skydiver jumps from a stationary balloon at a height of 2000 m above the ground.

Her velocity,

v ms^{-1} , t seconds after jumping, is given by

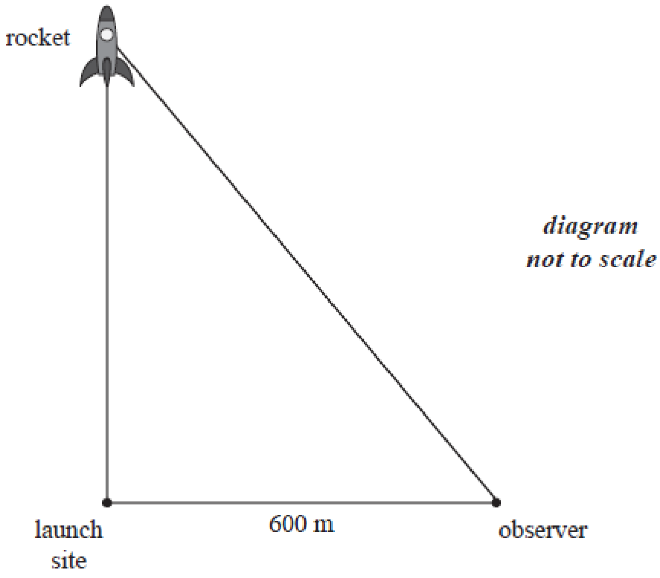
$v = 50(1 - \mathrm{e}^{-0.2t})$.

23a. Find her acceleration 10 seconds after jumping. *[3 marks]*

23b. How far above the ground is she 10 seconds after jumping? *[3 marks]*

24. A rocket is rising vertically at a speed of *[6 marks]*

300 ms^{-1} when it is 800 m directly above the launch site. Calculate the rate of change of the distance between the rocket and an observer, who is 600 m from the launch site and on the same horizontal level as the launch site.



25. The point P, with coordinates *[8 marks]*

$(p,\ q)$, lies on the graph of

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$,

$a > 0$.

The tangent to the curve at P cuts the axes at $(0,\ m)$ and $(n,\ 0)$. Show that $m + n = a$.

26a. Prove by mathematical induction that, for *[8 marks]*

$n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

26b. (a) Using integration by parts, show that *[17 marks]*

$\int e^{2x} \sin x \mathrm{d}x = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$.

(b) Solve the differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 - y^2}\, e^{2x} \sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form

$y = f(x)$.

(c) (i) Sketch the graph of

$y = f(x)$, found in part (b), for

$0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the x-axis, and mark it on your sketch.

(ii) The region bounded by the graph of

$y = f(x)$ and the x-axis, between the origin and P, is rotated 360° about the x-axis to form a solid of revolution.

Calculate the volume of this solid.

The integral

I_n is defined by

$I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x}|\sin x|\mathrm{d}x,\ \text{for } n \in \mathbb{N}$.

27a. Show that *[6 marks]*

$I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$.

27b. By letting *[4 marks]*

$y = x - n\pi$, show that

$I_n = \mathrm{e}^{-n\pi} I_0$.

27c. Hence determine the exact value of *[5 marks]*

$\int_0^{\infty} \mathrm{e}^{-x}|\sin x|\mathrm{d}x$.

28a. A particle P moves in a straight line with displacement relative to origin given by *[10 marks]*

$$s = 2\sin(\pi t) + \sin(2\pi t),\ t \geqslant 0,$$

where t is the time in seconds and the displacement is measured in centimetres.

(i) Write down the period of the function s.

(ii) Find expressions for the velocity, v, and the acceleration, a, of P.

(iii) Determine all the solutions of the equation $v = 0$ for

$0 \leqslant t \leqslant 4$.

28b. Consider the function *[8 marks]*

$$f(x) = A\sin(ax) + B\sin(bx),\ A,\ a,\ B,\ b,\ x \in \mathbb{R}.$$

Use mathematical induction to prove that the

$(2n)^{\text{th}}$ derivative of f is given by

$(f^{(2n)}(x) = (-1)^n \left(Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx)\right)$, for all

$n \in \mathbb{Z}^+$.

Consider the graph of

$y = x + \sin(x - 3),\ \ -\pi \leqslant x \leqslant \pi.$

29a. Sketch the graph, clearly labelling the x and y intercepts with their values. *[3 marks]*

29b. Find the area of the region bounded by the graph and the x and y axes. *[2 marks]*

30. A stalactite has the shape of a circular cone. Its height is 200 mm and is increasing at a rate of 3 mm per century. Its base radius is 40 mm and is decreasing at a rate of 0.5 mm per century. Determine if its volume is increasing or decreasing, and the rate at which the volume is changing. *[6 marks]*

31a. Given that *[5 marks]*

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$, show that

$\frac{dy}{dx} = \frac{e^{-y}}{2} - 1$.

31b. Hence, by repeated differentiation of the above differential equation, find the Maclaurin series for y as far as the term in x^3, showing that two of the terms are zero. *[11 marks]*

32. Find the area enclosed by the curve $y = \arctan x$, the x-axis and the line $x = \sqrt{3}$. *[6 marks]*

33. Show that the points (0, 0) and ($\sqrt{2\pi}$, $-\sqrt{2\pi}$) on the curve $e^{(x+y)} = \cos(xy)$ have a common tangent. *[7 marks]*

Consider the function $f(x) = \frac{\ln x}{x}$, $0 < x < e^2$.

34a. (i) Solve the equation $f'(x) = 0$. *[5 marks]*

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f.

34b. Show that there is a point of inflexion on the graph and determine its coordinates. *[5 marks]*

34c. Sketch the graph of $y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum. *[3 marks]*

34d. Now consider the functions $g(x) = \frac{\ln|x|}{x}$ and $h(x) = \frac{\ln|x|}{|x|}$, where $0 < x < e^2$. *[6 marks]*

(i) Sketch the graph of $y = g(x)$.

(ii) Write down the range of g.

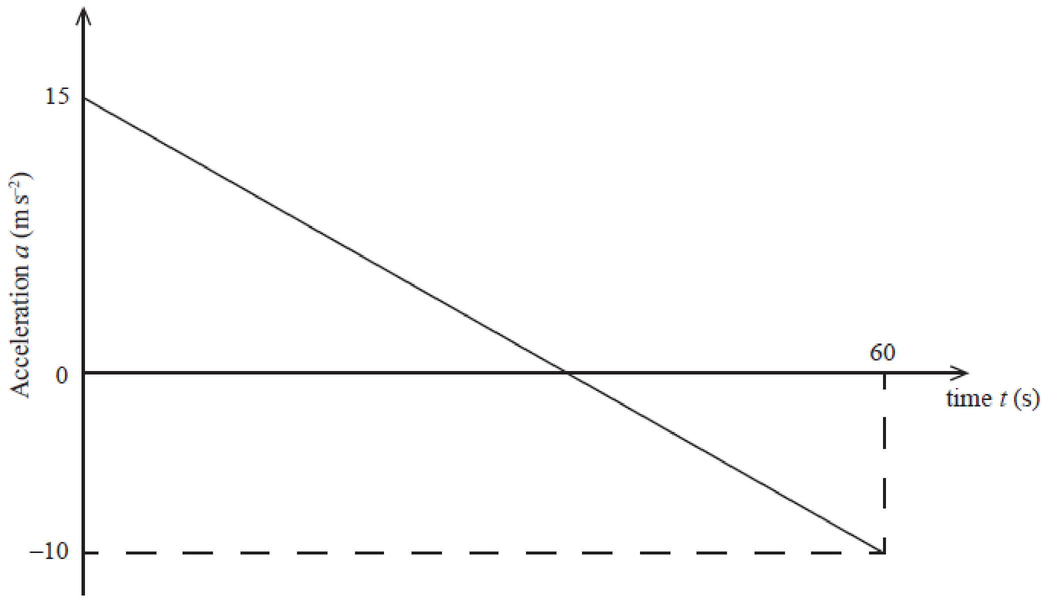
(iii) Find the values of x such that $h(x) > g(x)$.

Consider the function
$f(x) = x^3 - 3x^2 - 9x + 10$,
$x \in \mathbb{R}$.

35a. Find the equation of the straight line passing through the maximum and minimum points of the graph $y = f(x)$. *[4 marks]*

35b. Show that the point of inflexion of the graph $y = f(x)$ lies on this straight line. *[2 marks]*

A jet plane travels horizontally along a straight path for one minute, starting at time
$t = 0$, where
t is measured in seconds. The acceleration,
a , measured in ms^{-2}, of the jet plane is given by the straight line graph below.



36a. Find an expression for the acceleration of the jet plane during this time, in terms of t . *[1 mark]*

36b. Given that when $t = 0$ the jet plane is travelling at $125\ \text{ms}^{-1}$, find its maximum velocity in ms^{-1} during the minute that follows. *[4 marks]*

36c. Given that the jet plane breaks the sound barrier at $295\ \text{ms}^{-1}$, find out for how long the jet plane is travelling greater than this speed. *[3 marks]*

An open glass is created by rotating the curve
$y = x^2$, defined in the domain
$x \in [0, 10]$,
2π radians about the y-axis. Units on the coordinate axes are defined to be in centimetres.

37a. When the glass contains water to a height h cm, find the volume V of water in terms of h . *[3 marks]*

37b. When the glass contains water to a height h cm, find the volume V of water in terms of h. *[3 marks]*

37c. If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that, $\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$, where t is measured in hours. *[6 marks]*

37d. If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that, $\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$, where t is measured in hours. *[6 marks]*

37e. If the glass is filled completely, how long will it take for all the water to evaporate? *[7 marks]*

37f. If the glass is filled completely, how long will it take for all the water to evaporate? *[7 marks]*
