

Topic 1 Part 1 [440 marks]

1. Find the value of k if [4 marks]

$$\sum_{r=1}^{\infty} k \left(\frac{1}{3}\right)^r = 7.$$

2. If $z_1 = a + a\sqrt{3}i$ and $z_2 = 1 - i$, where a is a real constant, express z_1 and z_2 in the form $r \operatorname{cis} \theta$, and hence find [7 marks]
an expression for $\left(\frac{z_1}{z_2}\right)^6$ in terms of a and i .

3. Given that z is the complex number [6 marks]
 $x + iy$ and that
 $|z| + z = 6 - 2i$, find the value of x
and the value of y .

4. Solve the equation [5 marks]
 $2 - \log_3(x + 7) = \log_{\frac{1}{3}} 2x$.

5a. Expand and simplify [3 marks]
 $\left(x - \frac{2}{x}\right)^4$.

5b. Hence determine the constant term in the expansion [2 marks]
 $(2x^2 + 1)\left(x - \frac{2}{x}\right)^4$.

Given that
 $(4 - 5i)m + 4n = 16 + 15i$, where
 $i^2 = -1$, find m and n if

6a. m and n are real numbers; [3 marks]

6b. m and n are conjugate complex numbers. [4 marks]

7. Expand and simplify [4 marks]
 $\left(\frac{x}{y} - \frac{y}{x}\right)^4$.

Consider the following equations, where a ,
 $b \in \mathbb{R}$:

$$x + 3y + (a - 1)z = 1$$
$$2x + 2y + (a - 2)z = 1$$
$$3x + y + (a - 3)z = b.$$

8a. If each of these equations defines a plane, show that, for any value of a , the planes do not intersect at a unique point. [3 marks]

8b. Find the value of b for which the intersection of the planes is a straight line. [4 marks]

Consider the complex numbers

$$z_1 = 2\sqrt{3}\text{cis}\frac{3\pi}{2} \text{ and}$$

$$z_2 = -1 + \sqrt{3}i .$$

- 9a. (i) Write down $\overline{z_1}$ in Cartesian form. [3 marks]
 (ii) Hence determine $(z_1 + z_2)^*$ in Cartesian form.

- 9b. (i) Write z_2 in modulus-argument form. [6 marks]
 (ii) Hence solve the equation $z^3 = z_2$.

- 9c. Let $z = r \text{cis}\theta$, where $r \in \mathbb{R}^+$ and $0 \leq \theta < 2\pi$. Find all possible values of r and θ , [6 marks]
 (i) if $z^2 = (1 + z_2)^2$;
 (ii) if $z = -\frac{1}{z_2}$.

- 9d. Find the smallest positive value of n for which $\left(\frac{z_1}{z_2}\right)^n \in \mathbb{R}^+$. [4 marks]

Consider a function f , defined by

$$f(x) = \frac{x}{2-x} \text{ for } 0 \leq x \leq 1 .$$

- 10a. Find an expression for $(f \circ f)(x)$. [3 marks]

- 10b. Let $F_n(x) = \frac{x}{2^n - (2^n - 1)x}$, where $0 \leq x \leq 1$. [8 marks]
 Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \dots \circ f)}_{n \text{ times}}(x) = F_n(x)$$

- 10c. Show that $F_{-n}(x)$ is an expression for the inverse of F_n . [6 marks]

- 10d. (i) State $F_n(0)$ and $F_n(1)$. [6 marks]
 (ii) Show that $F_n(x) < x$, given $0 < x < 1$, $n \in \mathbb{Z}^+$.
 (iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1} , the x -axis and the line $x = 1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n .

11. Find the sum of all the multiples of 3 between 100 and 500. [4 marks]

12. A metal rod 1 metre long is cut into 10 pieces, the lengths of which form a geometric sequence. The length of the longest piece is 8 times the length of the shortest piece. Find, to the nearest millimetre, the length of the shortest piece. [6 marks]

13. Let $\omega = \cos\theta + i\sin\theta$. Find, in terms of θ , the modulus and argument of $(1 - \omega^2)^*$. [7 marks]

14. Find the value of k such that the following system of equations does not have a unique solution. [5 marks]

$$kx + y + 2z = 4$$

$$-y + 4z = 5$$

$$3x + 4y + 2z = 1$$

Three boys and three girls are to sit on a bench for a photograph.

15a. Find the number of ways this can be done if the three girls must sit together. [3 marks]

15b. Find the number of ways this can be done if the three girls must all sit apart. [4 marks]

16a. (i) Express each of the complex numbers [9 marks]

$$z_1 = \sqrt{3} + i, z_2 = -\sqrt{3} + i \text{ and}$$

$$z_3 = -2i \text{ in modulus-argument form.}$$

(ii) Hence show that the points in the complex plane representing

z_1 ,

z_2 and

z_3 form the vertices of an equilateral triangle.

(iii) Show that

$$z_1^{3n} + z_2^{3n} = 2z_3^{3n} \text{ where}$$

$n \in \mathbb{N}$.

[9 marks]

16b. (i) State the solutions of the equation

$$z^7 = 1 \text{ for}$$

$z \in \mathbb{C}$, giving them in modulus-argument form.

(ii) If w is the solution to

$z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of

π .

(iii) Show that

$z^2 - 2z \cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial

$z^7 - 1$. State the two other quadratic factors with real coefficients.

Consider the system of equations

$$0.1x - 1.7y + 0.9z = -4.4$$

$$-2.4x + 0.3y + 3.2z = 1.2$$

$$2.5x + 0.6y - 3.7z = 0.8.$$

17a. Express the system of equations in matrix form.

[2 marks]

17b. Find the solution to the system of equations.

[3 marks]

The arithmetic sequence

$\{u_n : n \in \mathbb{Z}^+\}$ has first term

$u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence

$\{v_n : n \in \mathbb{Z}^+\}$ has first term

$v_1 = 3$ and common ratio $r = 1.2$.

18a. Find an expression for

[2 marks]

$u_n - v_n$ in terms of n .

18b. Determine the set of values of n for which

[3 marks]

$$u_n > v_n.$$

18c. Determine the greatest value of

[1 mark]

$u_n - v_n$. Give your answer correct to four significant figures.

19. Use the method of mathematical induction to prove that

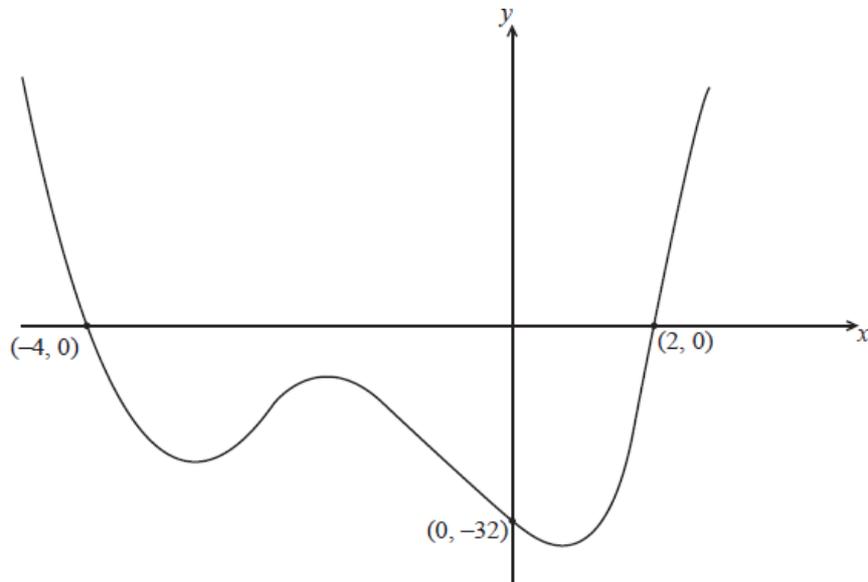
[7 marks]

$5^{2n} - 24n - 1$ is divisible by 576 for

$n \in \mathbb{Z}^+$.

- 20a. (i) Express the sum of the first n positive odd integers using sigma notation. [4 marks]
- (ii) Show that the sum stated above is n^2 .
- (iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers.
- 20b. A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points. [7 marks]
- (i) Show on a diagram all diagonals if there are 5 points.
- (ii) Show that the number of diagonals is $\frac{n(n-3)}{2}$ if there are n points, where $n > 2$.
- (iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible.
- 20c. The random variable [8 marks]
- $X \sim B(n, p)$ has mean 4 and variance 3.
- (i) Determine n and p .
- (ii) Find the probability that in a single experiment the outcome is 1 or 3.
21. Let [5 marks]
- $f(x) = \ln x$. The graph of f is transformed into the graph of the function g by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, followed by a reflection in the x -axis. Find an expression for $g(x)$, giving your answer as a single logarithm.
22. Find the constant term in the expansion of [7 marks]
- $(x - \frac{2}{x})^4 (x^2 + \frac{2}{x})^3$.
- 23a. Given that [2 marks]
- $(x + iy)^2 = -5 + 12i$, $x, y \in \mathbb{R}$. Show that
- (i) $x^2 - y^2 = -5$;
- (ii) $xy = 6$.
- 23b. Hence find the two square roots of [5 marks]
- $-5 + 12i$.
- 23c. For any complex number z , show that [3 marks]
- $(z^*)^2 = (z^2)^*$.
- 23d. Hence write down the two square roots of [2 marks]
- $-5 - 12i$.

The graph of a polynomial function f of degree 4 is shown below.



23e. Explain why, of the four roots of the equation $f(x) = 0$, two are real and two are complex. [2 marks]

23f. The curve passes through the point $(-1, -18)$. Find $f(x)$ in the form $f(x) = (x - a)(x - b)(x^2 + cx + d)$, where $a, b, c, d \in \mathbb{Z}$. [5 marks]

23g. Find the two complex roots of the equation $f(x) = 0$ in Cartesian form. [2 marks]

23h. Draw the four roots on the complex plane (the Argand diagram). [2 marks]

23i. Express each of the four roots of the equation in the form $re^{i\theta}$. [6 marks]

24a. Using the definition of a derivative as $f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$, show that the derivative of $\frac{1}{2x+1}$ is $-\frac{2}{(2x+1)^2}$. [4 marks]

24b. Prove by induction that the n^{th} derivative of $(2x+1)^{-1}$ is $(-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$. [9 marks]

The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8.

25a. Find the first term and the common difference. [4 marks]

25b. Find the smallest value of n such that the sum of the first n terms is greater than 600. [3 marks]

Fifteen boys and ten girls sit in a single line.

26a. In how many ways can they be seated in a single line so that the boys and girls are in two separate groups? [3 marks]

26b. Two boys and three girls are selected to go to the theatre. In how many ways can this selection be made? [3 marks]

Each time a ball bounces, it reaches 95 % of the height reached on the previous bounce. Initially, it is dropped from a height of 4 metres.

27a. What height does the ball reach after its fourth bounce? [2 marks]

27b. How many times does the ball bounce before it no longer reaches a height of 1 metre? [3 marks]

27c. What is the total distance travelled by the ball? [3 marks]

28a. Find the values of k for which the following system of equations has no solutions and the value of k for the system to have an infinite number of solutions. [5 marks]

$$x - 3y + z = 3$$

$$x + 5y - 2z = 1$$

$$16y - 6z = k$$

28b. Given that the system of equations can be solved, find the solutions in the form of a vector equation of a line, $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, where the components of \mathbf{b} are integers. [7 marks]

28c. The plane \div is parallel to both the line in part (b) and the line [5 marks]

$$\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-2}{0}.$$

Given that

\div contains the point $(1, 2, 0)$, show that the Cartesian equation of \div is $16x + 24y - 11z = 64$.

28d. The z -axis meets the plane \div at the point P. Find the coordinates of P. [2 marks]

28e. Find the angle between the line $\frac{x-2}{3} = \frac{y+5}{4} = \frac{z}{2}$ and the plane \div . [5 marks]

29. Find the cube roots of i in the form [6 marks]

$$a + bi, \text{ where}$$

$$a, b \in \mathbb{R}.$$

30. Given that [7 marks]

$y = \frac{1}{1-x}$, use mathematical induction to prove that

$$\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}, \quad n \in \mathbb{Z}^+.$$

31. The complex numbers [7 marks]
- z_1 and z_2 have arguments between 0 and π radians. Given that $z_1 z_2 = -\sqrt{3} + i$ and $\frac{z_1}{z_2} = 2i$, find the modulus and argument of z_1 and of z_2 .

- 32a. Find the set of values of x for which the series [4 marks]
- $$\sum_{n=1}^{\infty} \left(\frac{2x}{x+1} \right)^n$$
- has a finite sum.

- 32b. Hence find the sum in terms of x . [2 marks]

33. Given that [7 marks]
- $$z = \frac{2-i}{1+i} - \frac{6+8i}{u+i},$$
- find the values of u , $u \in \mathbb{R}$, such that $\operatorname{Re} z = \operatorname{Im} z$.

- 34a. In an arithmetic sequence the first term is 8 and the common difference is [9 marks]
- $$\frac{1}{4}.$$
- If the sum of the first $2n$ terms is equal to the sum of the next n terms, find n .

- 34b. If [7 marks]
- a_1, a_2, a_3, \dots are terms of a geometric sequence with common ratio $r \neq 1$, show that
- $$(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + \dots + (a_n - a_{n+1})^2 = \frac{a_1^2(1-r)(1-r^{2n})}{1+r}.$$

- 35a. Show that [1 mark]
- $$|e^{i\theta}| = 1.$$

- 35b. Consider the geometric series [2 marks]
- $$1 + \frac{1}{3}e^{i\theta} + \frac{1}{9}e^{2i\theta} + \dots$$
- Write down the common ratio, z , of the series, and show that
- $$|z| = \frac{1}{3}.$$

- 35c. Find an expression for the sum to infinity of this series. [2 marks]

35d. Hence, show that [8 marks]

$$\sin \theta + \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta + \dots = \frac{9\sin \theta}{10-6\cos \theta}.$$

36a. Show that [2 marks]

$$n! \geq 2^{n-1}, \text{ for}$$

$$n \geq 1.$$

36b. Hence use the comparison test to determine whether the series [3 marks]

$$\sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges or diverges.}$$

37. Given that [4 marks]

$$\frac{z}{z+2} = 2 - i,$$

$z \in \mathbb{C}$, find z in the form

$$a + ib.$$

38. Determine the first three terms in the expansion of [5 marks]

$$(1 - 2x)^5(1 + x)^7 \text{ in ascending powers of } x.$$

39. Find, in its simplest form, the argument of [7 marks]

$$(\sin \theta + i(1 - \cos \theta))^2 \text{ where}$$

θ is an acute angle.

40. $z_1 = (1 + i\sqrt{3})^m$ and $z_2 = (1 - i)^n$. [14 marks]

(a) Find the modulus and argument of

z_1 and

z_2 in terms of m and n , respectively.

(b) **Hence**, find the smallest positive integers m and n such that

$$z_1 = z_2.$$

41. Consider [7 marks]

$$w = \frac{z}{z^2+1} \text{ where } z = x + iy, y \neq 0 \text{ and } z^2 + 1 \neq 0.$$

Given that

$\text{Im } w = 0$, show that

$$|z| = 1.$$

Consider the equation

$$9x^3 - 45x^2 + 74x - 40 = 0 .$$

42a. Write down the numerical value of the sum and of the product of the roots of this equation.

[1 mark]

42b. The roots of this equation are three consecutive terms of an arithmetic sequence.

[6 marks]

Taking the roots to be
 α , $\alpha \pm \beta$, solve the equation.

Consider the following system of equations:

$$x + y + z = 1$$

$$2x + 3y + z = 3$$

$$x + 3y - z = \lambda$$

where

$$\lambda \in \mathbb{R} .$$

43a. Show that this system does not have a unique solution for any value of

[4 marks]

λ .

43b. (i) Determine the value of

[4 marks]

λ for which the system is consistent.

(ii) For this value of

λ , find the general solution of the system.

Consider the complex numbers

$$z_1 = 2\text{cis}150^\circ \text{ and}$$

$$z_2 = -1 + i .$$

44a. Calculate

[7 marks]

$\frac{z_1}{z_2}$ giving your answer both in modulus-argument form and Cartesian form.

44b. Using your results, find the exact value of $\tan 75^\circ$, giving your answer in the form

[5 marks]

$$a + \sqrt{b} , a ,$$

$$b \in \mathbb{Z}^+ .$$

The function f is defined by

$$f(x) = e^x \sin x .$$

45a. Show that

[3 marks]

$$f''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right) .$$

45b. Obtain a similar expression for

[4 marks]

$$f^{(4)}(x) .$$

45c. Suggest an expression for

[8 marks]

$$f^{(2n)}(x),$$

$n \in \mathbb{Z}^+$, and prove your conjecture using mathematical induction.

46. The first term and the common ratio of a geometric series are denoted, respectively, by a and r where $a, r \in \mathbb{Q}$. Given that the third term is 9 and the sum to infinity is 64, find the value of a and the value of r .

[5 marks]

The complex number

$$z = -\sqrt{3} + i .$$

47a. Find the modulus and argument of z , giving the argument in degrees.

[2 marks]

47b. Find the cube root of z which lies in the first quadrant of the Argand diagram, giving your answer in Cartesian form.

[2 marks]

47c. Find the smallest positive integer n for which

[2 marks]

z^n is a positive real number.

A bank offers loans of $\$P$ at the beginning of a particular month at a monthly interest rate of I . The interest is calculated at the end of each month and added to the amount outstanding. A repayment of $\$R$ is required at the end of each month. Let

$\$S_n$ denote the amount outstanding immediately after the

n^{th} monthly repayment.

48a. (i) Find an expression for

[7 marks]

S_1 and show that

$$S_2 = P\left(1 + \frac{I}{100}\right)^2 - R\left(1 + \left(1 + \frac{I}{100}\right)\right).$$

(ii) Determine a similar expression for

S_n . Hence show that

$$S_n = P\left(1 + \frac{I}{100}\right)^n - \frac{100R}{I}\left(\left(1 + \frac{I}{100}\right)^n - 1\right)$$

48b. Sue borrows \$5000 at a monthly interest rate of 1 % and plans to repay the loan in 5 years (*i.e.* 60 months).

[6 marks]

- (i) Calculate the required monthly repayment, giving your answer correct to two decimal places.
- (ii) After 20 months, she inherits some money and she decides to repay the loan completely at that time. How much will she have to repay, giving your answer correct to the nearest \$?