Mechanics 2

2.1 Motion

This section is an introduction to the basic concepts used in describing motion. We will begin with motion in a straight line with constant velocity and then constant acceleration. Knowledge of uniformly accelerated motion allows analysis of more complicated motions, such as the motion of projectiles.

Learning objectives

- Understand the difference between distance and displacement.
- Understand the difference between speed and velocity.
- Understand the concept of acceleration.
- Analyse graphs describing motion.
- Solve motion problems using the equations for constant acceleration.
- Discuss the motion of a projectile.
- Show a qualitative understanding of the effects of a fluid resistance force on motion.
- Understand the concept of terminal speed.

Kinematical quantities

We will begin our discussion of motion with straight line motion in one dimension. This means that the particle that moves is constrained to move along a straight line. The **position** of the particle is then described by its coordinate on the straight line (Figure **2.1a**). If the line is horizontal, we may use the symbol x to represent the coordinate and hence the position. If the line is vertical, the symbol y is more convenient. In general, for an arbitrary line we may use a generic name, s, for position. So in Figure **2.1**, $x=6\,\text{m}$, $y=-4\,\text{m}$ and $s=0$.

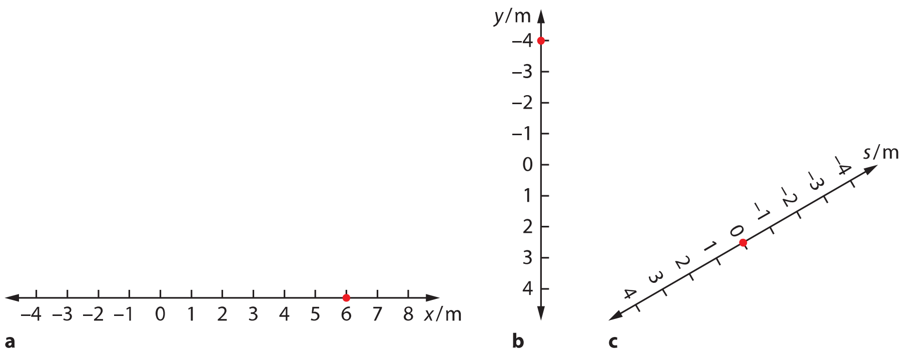


Figure 2.1 The position of a particle is determined by the coordinate on the number line.

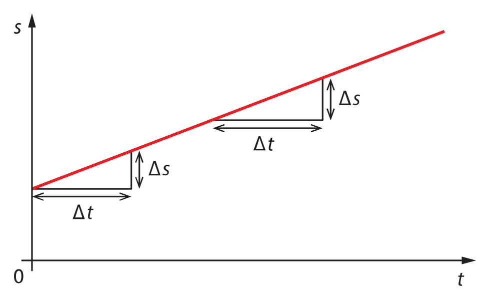


Figure 2.2 In uniform motion the graph of position versus time is a straight line.

As the particle moves on the straight line its position changes. In **uniform motion** the graph of position against time is a straight line (Figure **2.2**). In equal intervals of time, the position changes by the same amount. This means that the slope of the position–time graph is constant. This slope is defined to be the **average velocity** of the particle:

$$v=\frac{\Delta s}{\Delta t}$$

where Δs is the change in position.

The average velocity during an interval of time Δt is the ratio of the change in position Δs during that time interval to Δt.

(In uniform motion velocity is constant so the term 'average' is unnecessary. The velocity is the same at all times.)

Positive velocity means that the coordinate s that gives the position is increasing. Negative velocity means that s is decreasing.

Suppose we choose a time interval from $t=0$ to some arbitrary time t later. Let the position at $t=0$ (the initial position) be s_i and the position at time t be s. Then:

$$v=\frac{s-s_i}{t-0}$$

which can be re-arranged to give:

$$s=s_i+vt$$

This formula gives, in uniform motion, the position s of the moving object t seconds after time zero, given that the velocity is v and the initial position is s_i.

Worked example

2.1 Two cyclists, A and B, start moving at the same time. The initial position of A is 0 m and her velocity is $+20\,\text{km h}^{-1}$. The initial position of B is 150 km and he cycles at a velocity of $-30\,\text{km h}^{-1}$. Determine the time and position at which they will meet.

The position of A is given by the formula: $s_A=0+20t$

The position of B is given by the formula: $s_B=150-30t$

They will meet when they are the same position, i.e. when $s_A=s_B$. This implies:

$$20t=150-30t$$

$$50t=150$$

$$t=3.0\text{ hours}$$

The common position is found from either $s_A=20\times 3.0=60\,\text{km}$ or $s_B=150-30\times 3.0=60\,\text{km}$.

Consider two motions shown in Figure **2.3**. In the first, the particle leaves its initial position s_i at -4 m and continues to its final position at 16 m. The change in position is called **displacement** and in this case equals $16-(-4)=20$ m. The **distance** travelled is the actual length of the path followed and in this case is also 20 m.

Displacement = change in position
Distance = length of path followed

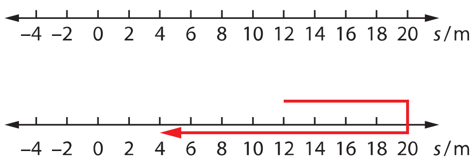


Figure 2.3 A motion in which the particle changes direction.

In the second motion, the particle leaves its initial position at 12 m, arrives at position 20 m and then comes back to its final position at 4.0 m.

The second motion is an example of motion with changing direction. The change in the position of this particle, i.e. the displacement is $\Delta s = s_f - s_i = 4.0 - 12 = -8.0$ m. But the distance travelled by the particle (the length of the path) is 8.0 m in the outward trip and 16 m on the return trip, making a total distance of 24 m. So we must be careful to distinguish distance from displacement. Distance is a scalar but displacement is a vector. Numerically, they are different if there is a change of direction, as in this example.

For constant velocity, the graph of velocity versus time gives a horizontal straight line (Figure **2.4a**). An example of this type of motion is coasting in a straight line on a bicycle on level ground (Figure **2.4b**).

Figure 2.4 **a** In uniform motion the graph of velocity versus time is a horizontal straight line. **b** This motion is a good approximation to uniform motion.

But we now observe that the area under the graph from $t = 0$ to time t is vt. From $s = s_i + vt$ we deduce that this area is the change in position or the displacement.

Uniformly accelerated motion

In the last section we discussed uniform motion. This means motion in a straight line with **constant velocity**. In such motion the graph of position versus time is a straight line.

In most motions velocity is not constant. In **uniformly accelerated motion** the graph of velocity versus time is a non-horizontal straight line (Figure **2.5**).

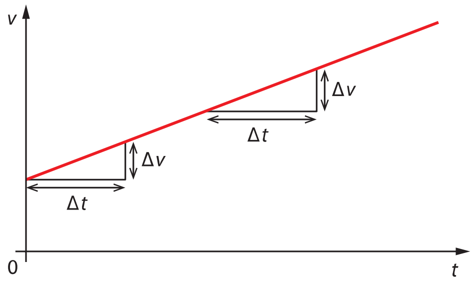


Figure 2.5 In uniformly accelerated motion the graph of velocity versus time is a straight line with non-zero slope.

In equal intervals of time the velocity changes by the same amount. The slope of the velocity–time graph is constant. This slope is defined to be the acceleration of the particle:

$$a = \frac{\Delta v}{\Delta t}$$

Acceleration is the rate of change of velocity.

When the acceleration is positive, the velocity is increasing (Figure **2.6**). Negative acceleration means that v is decreasing. The plane reaches a take-off speed of 260 km h^{-1} (about 72 m s^{-1}) in about 2 seconds, implying an average acceleration of about 36 m s^{-2}. The distance travelled until take-off is about 72 m.

Figure 2.6 This F/ A-18C is accelerating!

Suppose we choose a time interval from $t=0$ to some arbitrary time t later. Let the velocity at $t=0$ (the initial velocity) be u and the velocity at time t be v. Then:

$$a=\frac{v-u}{t-0}$$

which can be re-arranged to:

$$v=u+at$$

For uniformly accelerated motion, this formula gives the velocity v of the moving object t seconds after time zero, given that the initial velocity is u and the acceleration is a.

Worked example

2.2 A particle has initial velocity $12\,\mathrm{m\,s^{-1}}$ and moves with a constant acceleration of $-3.0\,\mathrm{m\,s^{-2}}$. Determine the time at which the particle stops instantaneously.

The particle is getting slower. At some point it will stop instantaneously, i.e. its velocity v will be zero.

We know that $v=u+at$. Just substituting values gives:

$$0=12+(-3.0)\times t$$

$$3.0t=12$$

Hence $t=4.0\,\mathrm{s}$.

Defining velocity in non-uniform motion

But how is velocity defined now that it is not constant? We define the average velocity as before:

$$\bar{v}=\frac{\Delta s}{\Delta t}$$

But since the velocity changes, it has different values at different times. We would like to have a concept of the velocity at an instant of time, the **instantaneous velocity**. We need to make the time interval Δt very small. The instantaneous velocity is then defined as:

$$v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

In other words, instantaneous velocity is the average velocity obtained during an interval of time that is very, very small. In calculus, we learn that $\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$ has the following meaning: look at the graph of position s versus time t shown in Figure **2.7a**. As there is uniform acceleration, the graph is a curve. Choose a point on this curve. Draw the tangent line to the curve at the point. The slope of the tangent line is the meaning of $\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$ and therefore also of velocity.

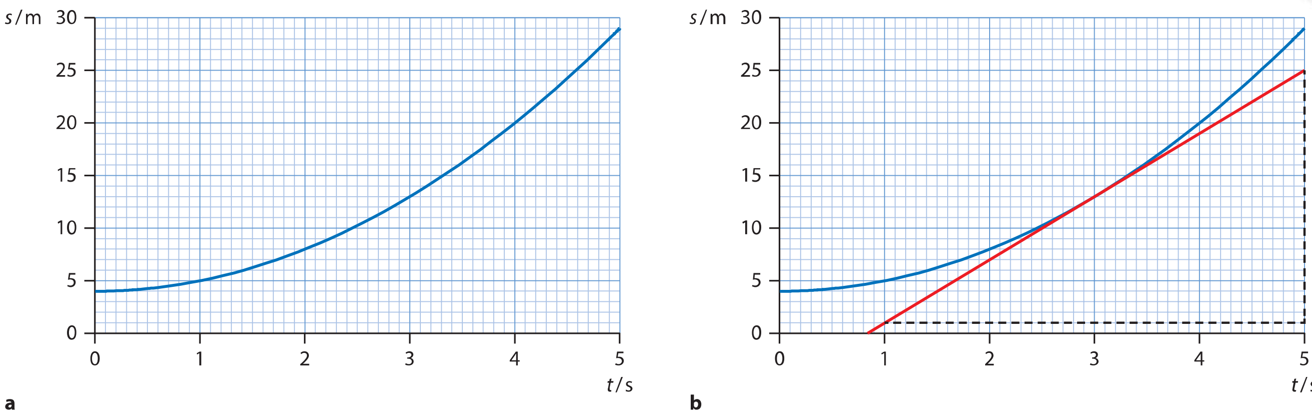


Figure 2.7 **a** In uniformly accelerated motion the graph of position versus time is a curve. **b** The slope of the tangent at a particular point gives the velocity at that point.

In Figure **2.7b** the tangent is drawn at $t = 3.0\,\text{s}$. We can use this to find the instantaneous velocity at $t = 3.0\,\text{s}$. The slope of this tangent line is:

$$\frac{25 - 1.0}{5.0 - 1.0} = 6.0\,\text{m}\,\text{s}^{-1}$$

To find the instantaneous velocity at some other instant of time we must take another tangent and we will find a different instantaneous velocity. At the point at $t = 0$ it is particularly easy to find the velocity: the tangent is horizontal and so the velocity is zero.

The slope of the tangent to the graph of position versus time is velocity

Instantaneous velocity can be positive or negative. The magnitude of the instantaneous velocity is known as the **instantaneous speed**.

We define the **average speed** to be the total distance travelled divided by the total time taken. The **average velocity** is defined as the change in position (i.e. the displacement) divided by the time taken:

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

$$\text{average velocity} = \frac{\text{displacement}}{\text{total time taken}}$$

Consider the graph of velocity versus time in Figure **2.8**. Imagine approximating the straight line with a staircase. The area under the staircase is the change in position since at each step the velocity is constant. If we make the steps of the staircase smaller and smaller, the area under the line and the area under the staircase will be indistinguishable and so we have the general result that:

The area under the curve in a velocity versus time graph is the change in position.

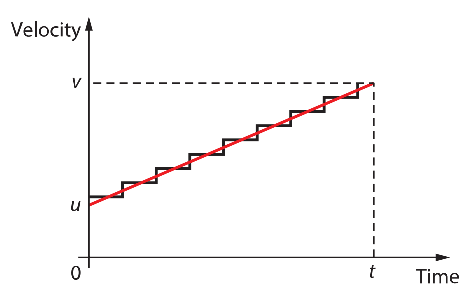


Figure 2.8 The straight-line graph may be approximated by a staircase.

From Figure **2.8** this area is (the shape is a trapezoid):

$$\Delta s = \left(\frac{u + v}{2}\right) t$$

But $v = u + at$, so this becomes:

$$\Delta s = \left(\frac{u + u + at}{2}\right)t = ut + \tfrac{1}{2}at^2$$

So we have two formulas for position in the case of uniformly accelerated motion (recall that $\Delta s = s - s_i$):

$$s = s_i + \left(\frac{u + v}{2}\right)t$$

$$s = s_i + ut + \tfrac{1}{2}at^2$$

We get a final formula if we combine $s = s_i + ut + \frac{1}{2}at^2$ with $v = u + at$. From the second equation write $t = \dfrac{v - u}{a}$ and substitute in the first equation to get:

$$s - s_i = u\frac{v - u}{a} + \frac{1}{2}\left(\frac{v + u}{a}\right)^2$$

After a bit of uninteresting algebra this becomes:

$$v^2 = u^2 + 2a(s - s_i)$$

This is useful in problems in which no information on time is given.

Graphs of position versus time for uniformly accelerated motion are parabolas (Figure 2.9). If the parabola 'holds water' the acceleration is positive. If not, the acceleration is negative.

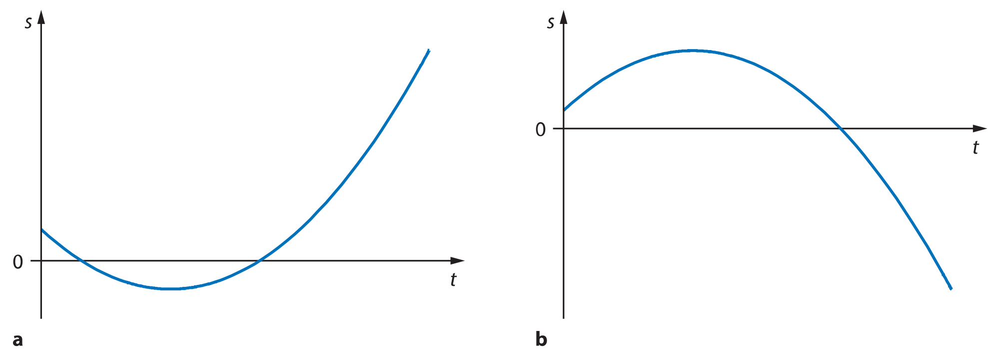


Figure 2.9 Graphs of position *s* against time *t* for uniformly accelerated motion. **a** Positive acceleration. **b** Negative acceleration.

Exam tip

The table summarises the meaning of the slope and area for the different motion graphs.

Graph of …	Slope	Area
position against time	velocity	
velocity against time	acceleration	change in position
acceleration against time		change in velocity

These formulas can be used for constant acceleration only (if the initial position is zero, Δs may be replaced by just s).

$$v = u + at \qquad \Delta s = ut + \tfrac{1}{2}at^2 \qquad \Delta s = \left(\frac{u + v}{2}\right)t \qquad v^2 = u^2 + 2a\Delta s$$

Worked examples

2.3 A particle has initial velocity $2.00\,\mathrm{m\,s^{-1}}$ and acceleration $a = 4.00\,\mathrm{m\,s^{-2}}$. Find its displacement after 10.0 s.

Displacement is the change of position, i.e. $\Delta s = s - s_i$. We use the equation:

$$\Delta s = ut + \tfrac{1}{2}at^2$$

$$\Delta s = 2.00 \times 10.0 + \tfrac{1}{2} \times 4.00 \times 10.0^2$$

$$\Delta s = 220\,\mathrm{m}$$

2.4 A car has an initial velocity of $u = 5.0\,\mathrm{m\,s^{-1}}$. After a displacement of 20 m, its velocity becomes $7.0\,\mathrm{m\,s^{-1}}$. Find the acceleration of the car.

Here, $\Delta s = s - s_i = 20\,\mathrm{m}$. So use $v^2 = u^2 + 2a\Delta s$ to find a.

$$7.0^2 = 5.0^2 + 2a \times 20$$

$$24 = 40a$$

Therefore $a = 0.60\,\mathrm{m\,s^{-2}}$.

2.5 A body has initial velocity $4.0\,\mathrm{m\,s^{-1}}$. After 6.0 s the velocity is $12\,\mathrm{m\,s^{-1}}$. Determine the displacement of the body in the 6.0 s.

We know u, v and t. We can use:

$$\Delta s = \left(\frac{v + u}{2}\right)t$$

to get:

$$\Delta s = \left(\frac{12 + 4.0}{2}\right) \times 6.0$$

$$\Delta s = 48\,\mathrm{m}$$

A slower method would be to use $v = u + at$ to find the acceleration:

$$12 = 4.0 + 6.0a$$

$$\Rightarrow \quad a = 1.333\,\mathrm{m\,s^{-2}}$$

Then use the value of a to find Δs:

$$\Delta s = ut + \tfrac{1}{2}at^2$$

$$\Delta s = 4.0 \times 6.0 + \tfrac{1}{2} \times 1.333 \times 36$$

$$\Delta s = 48\,\mathrm{m}$$

2.6 Two balls start out moving to the right with constant velocities of $5.0\,\mathrm{m\,s^{-1}}$ and $4.0\,\mathrm{m\,s^{-1}}$. The slow ball starts first and the other 4.0 s later. Determine the position of the balls when they meet.

Let the two balls meet t s after the first ball starts moving.

The position of the slow ball is: $s = 4t$

The position of the fast ball is: $5(t-4)$

(The factor $t-4$ is there because after t s the fast ball has actually been moving for only $t-4$ seconds.)

These two positions are equal when the two balls meet, and so:

$$4t = 5t - 20$$

$$\Rightarrow \quad t = 20\,\mathrm{s}$$

Substituting into the equation for the position of the slow ball, the position where the balls meet is 80 m to the right of the start.

2.7 A particle starts out from the origin with velocity $10\,\mathrm{m\,s^{-1}}$ and continues moving at this velocity for 5 s. The velocity is then abruptly reversed to $-5\,\mathrm{m\,s^{-1}}$ and the object moves at this velocity for 10 s. For this motion find:

a the change in position, i.e. the displacement

b the total distance travelled

c the average speed

d the average velocity.

The problem is best solved using the velocity–time graph, which is shown in Figure **2.10**.

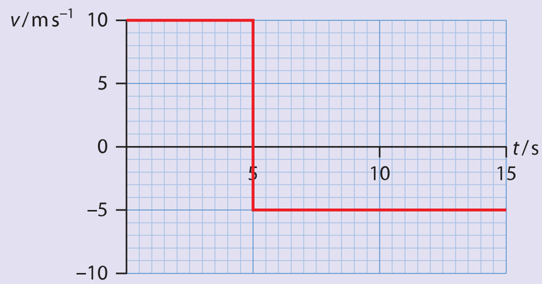


Figure 2.10

a The initial position is zero. Thus, after 5.0 s the position is $10 \times 5.0\,\mathrm{m} = 50\,\mathrm{m}$ (the area under the first part of the graph). In the next 10 s the displacement changes by $-5.0 \times 10 = -50\,\mathrm{m}$ (the area under the second part of the graph). The change in position, i.e. the displacement, is thus $50 - 50 = 0\,\mathrm{m}$.

b Take the initial velocity as moving to the right. The object moved toward the right, stopped and returned to its starting position (we know this because the displacement was 0). The distance travelled is 50 m in moving to the right and 50 m coming back, giving a total distance travelled of 100 m.

c The average speed is $\dfrac{100\,\mathrm{m}}{15\,\mathrm{s}} = 6.7\,\mathrm{m\,s^{-1}}$.

d The average velocity is zero, since the displacement is zero.

2.8 An object with initial velocity $20\,\mathrm{m\,s^{-1}}$ and initial position of $-75\,\mathrm{m}$ experiences a constant acceleration of $-2\,\mathrm{m\,s^{-2}}$. Sketch the position–time graph for this motion for the first 20 s.

Use the equation $s = ut + \frac{1}{2}at^2$. Substituting the values we know, the displacement is given by $s = -75 + 20t - t^2$. This is the function we must graph. The result is shown in Figure **2.11.**

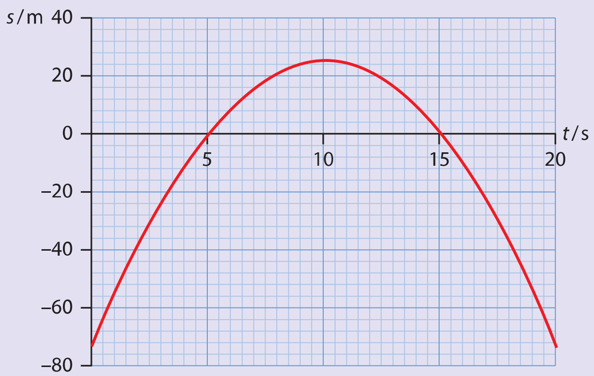


Figure 2.11

At 5 s the object reaches the origin and overshoots it. It returns to the origin 10 s later ($t = 15\,\mathrm{s}$). The furthest it gets from the origin is 25 m. The velocity at 5 s is $10\,\mathrm{m\,s^{-1}}$ and at 15 s it is $-10\,\mathrm{m\,s^{-1}}$. At 10 s the velocity is zero.

A special acceleration

Assuming that we can neglect air resistance and other frictional forces, an object thrown into the air will experience the **acceleration of free fall** while in the air. This is an acceleration caused by the attraction between the Earth and the body. The magnitude of this acceleration is denoted by g. Near the surface of the Earth $g = 9.8\,\mathrm{m\,s^{-2}}$. The direction of this acceleration is always vertically downward. (We will sometimes approximate g by $10\,\mathrm{m\,s^{-2}}$.)

Worked example

2.9 An object is thrown vertically upwards with an initial velocity of $20\,\mathrm{m\,s^{-1}}$ from the edge of a cliff that is 30 m from the sea below, as shown in Figure **2.12**.

Determine:

a the ball's maximum height

b the time taken for the ball to reach its maximum height

c the time to hit the sea

d the speed with which it hits the sea.

(You may approximate g by $10\,\mathrm{m\,s^{-2}}$.)

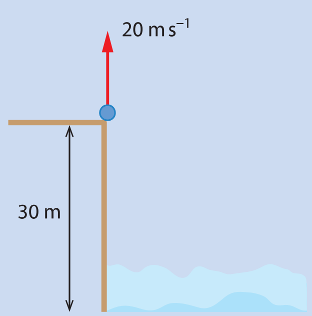


Figure 2.12 A ball is thrown upwards from the edge of a cliff.

We have motion on a vertical line so we will use the symbol y for position (Figure **2.13a**). We make the vertical line point upwards. The zero for displacement is the ball's initial position.

a The quickest way to get the answer to this part is to use $v^2 = u^2 - 2gy$. (The acceleration is $a = -g$.) At the highest point $v = 0$, and so:

$$0 = 20^2 - 2 \times 10y$$

$$\Rightarrow \; y = 20\,\text{m}$$

b At the highest point the object's velocity is zero. Using $v = 0$ in $v = u - gt$ gives:

$$0 = 20 - 10 \times t$$

$$t = \frac{20}{10} = 2.0\,\text{s}$$

c There are many ways to do this. One is to use the displacement arrow shown in blue in Figure **2.13a**. Then when the ball hits the sea, $y = -30\,\text{m}$. Now use the formula $y = ut - \frac{1}{2}gt^2$ to find an equation that only has the variable t:

$$-30 = 20 \times t - 5 \times t^2$$

$$t^2 - 4t - 6 = 0$$

This is a quadratic equation. Using your calculator you can find the two roots as $-1.2\,\text{s}$ and $5.2\,\text{s}$. Choose the positive root to find the answer $t = 5.2\,\text{s}$.

Another way of looking at this is shown in Figure **2.13b**. Here we start at the highest point and make the line along which the ball moves point downwards. Then, at the top $y = 0$, at the sea $y = +50$ and $g = +10\,\text{m}\,\text{s}^{-2}$. Now, the initial velocity is zero because we take our initial point to be at the top.

Using $y = ut + \frac{1}{2}gt^2$ with $u = 0$, we find:

$$50 = 5t^2$$

$$\Rightarrow \quad t = 3.2\,\text{s}$$

This is the time to fall to the sea. It took 2.0 s to reach the highest point, so the total time from launch to hitting the sea is:

$2.0 + 3.2 = 5.2\,\text{s}$.

d Use $v = u - gt$ and $t = 5.2\,\text{s}$ to get $v = 20 - 10 \times 5.2 = -32\,\text{m}\,\text{s}^{-1}$. The speed is then $32\,\text{m}\,\text{s}^{-1}$.

(If you preferred the diagram in Figure **2.13b** for working out part **c** and you want to continue this method for part **d**, then you would write $v = u + gt$ with $t = 3.2\,\text{s}$ and $u = 0$ to get $v = 10 \times 3.2 = +32\,\text{m}\,\text{s}^{-1}$.)

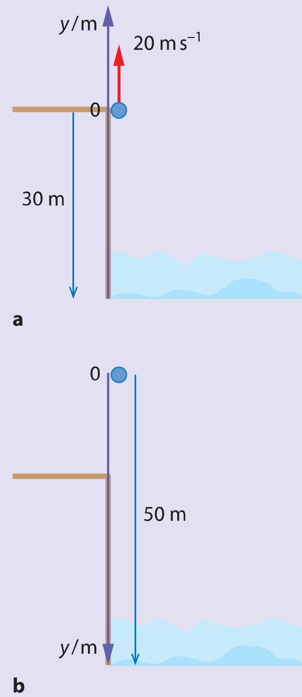


Figure 2.13 Diagrams for solving the ball's motion. **a** Displacement upwards is positive. **b** The highest point is the zero of displacement.

Projectile motion

Figure **2.14** shows the positions of two objects every 0.2 s: the first was simply allowed to drop vertically from rest, the other was launched horizontally with no vertical component of velocity. We see that in the vertical direction, both objects fall the **same distance** in the **same time**.

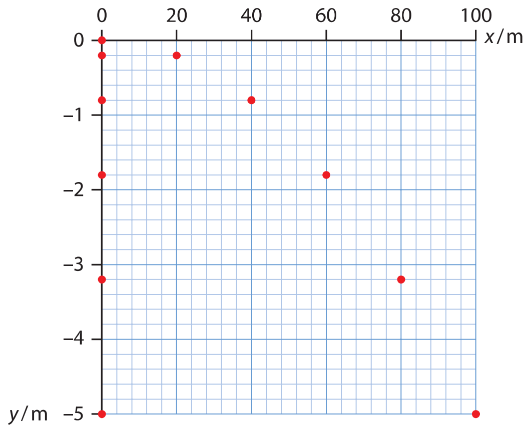


Figure 2.14 A body dropped from rest and one launched horizontally cover the same vertical displacement in the same time.

How do we understand this fact? Consider Figure **2.15**, in which a black ball is projected horizontally with velocity v. A blue ball is allowed to drop vertically from the same height. Figure **2.15a** shows the situation when the balls are released as seen by an observer X at rest on the ground. But suppose there is an observer Y, who moves to the right with velocity $\frac{v}{2}$ with respect to the ground. What does Y see? Observer Y sees the black ball moving to the right with velocity $\frac{v}{2}$ and the blue ball approaching with velocity $-\frac{v}{2}$ (Figure **2.15b**) The motions of the two balls are therefore **identical** (except for direction). So this observer will determine that the two bodies reach the ground at the **same time**. Since time is absolute in Newtonian physics, the two bodies must reach the ground at the same time as far as any other observer is concerned as well.

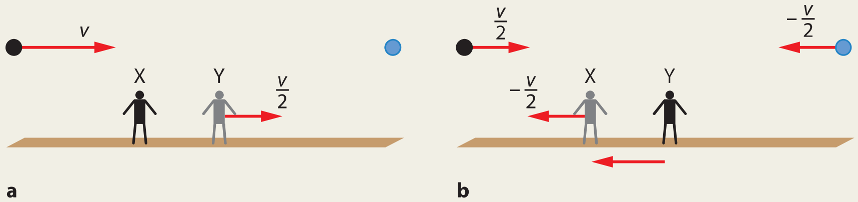


Figure 2.15 **a** A ball projected horizontally and one simply dropped from rest from the point of view of observer X. Observer Y is moving to the right with velocity $\frac{v}{2}$ with respect to the ground. **b** From the point of view of observer Y, the black and the blue balls have identical motions.

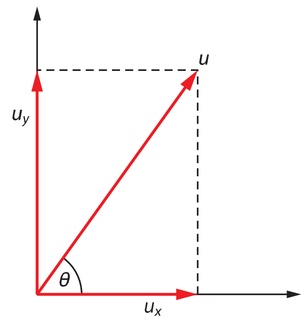


Figure 2.16 A projectile is launched at an angle θ to the horizontal with speed u.

The discussion shows that the motion of a ball that is projected at some angle can be analysed by separately looking at the horizontal and the vertical directions. All we have to do is consider two motions, one in the horizontal direction in which there is no acceleration, and another in the vertical direction in which we have an acceleration, g.

Consider Figure **2.16**, where a projectile is launched at an angle θ to the horizontal with speed u. The components of the *initial* velocity vector are $u_x = u\cos\theta$ and $u_y = u\sin\theta$. At some later time t the components of velocity are v_x and v_y. In the x-direction we do not have any acceleration and so:

$$v_x = u_x$$

$$v_x = u\cos\theta$$

In the y-direction the acceleration is $-g$ and so:

$$v_y = u_y - gt$$

$$v_y = u\sin\theta - gt$$

The green vector in Figure **2.17a** shows the position of the projectile t seconds after launch. The red arrows in Figure **2.17b** show the velocity vectors.

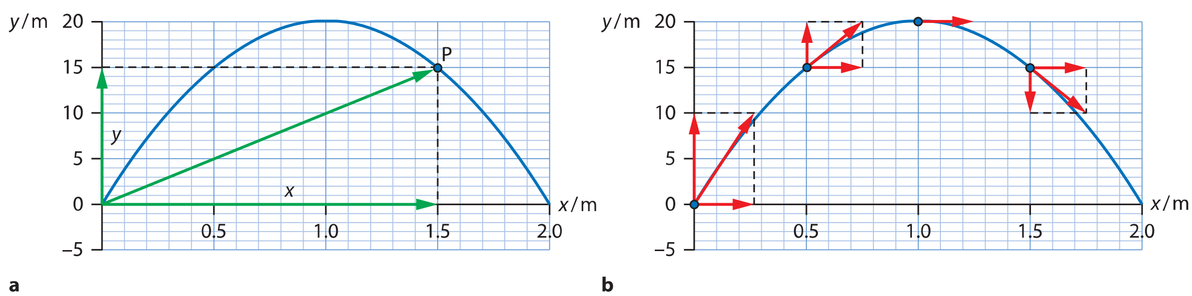


Figure 2.17 **a** The position of the particle is determined if we know the x- and y-components of the position vector. **b** The velocity vectors for projectile motion are tangents to the parabolic path.

Exam tip

All that we are doing is using the formulas from the previous section for velocity and position $v = u + at$ and $s = ut + \frac{1}{2}at^2$ and rewriting them **separately** for each direction x and y.

In the x-direction there is zero acceleration and in the y-direction there is an acceleration $-g$.

We would like to know the x- and y-components of the position vector. We now use the formula for position. In the x-direction:

$$x = u_x t$$

$$x = ut\cos\theta$$

And in the y-direction:

$$y = u_y t - \tfrac{1}{2}gt^2$$

$$y = ut\sin\theta - \tfrac{1}{2}gt^2$$

Let us collect what we have derived so far. We have four equations with which we can solve any problem with projectiles, as we will soon see:

$$\underbrace{v_x = u\cos\theta}_{x\text{-velocity}}, \qquad \underbrace{v_y = u\cos\theta - gt}_{y\text{-velocity}}$$

$$\underbrace{x = ut\cos\theta}_{x\text{-displacement}}, \qquad \underbrace{y = ut\sin\theta - \frac{1}{2}gt^2}_{y\text{-displacement}}$$

The equation with 'squares of speeds' is a bit trickier (carefully review the following steps). It is:

$$v^2 = u^2 - 2gy$$

Since $v^2 = v_x{}^2 + v_y{}^2$ and $u^2 = u_x{}^2 + u_y{}^2$, and in addition $v_x{}^2 = u_x{}^2$, this is also equivalent to:

$$v_y{}^2 = u_y{}^2 - 2gy$$

Exam tip
Always choose your *x*- and *y*-axes so that the origin is the point where the launch takes place.

Worked examples

2.10 A body is launched with a speed of $18.0\,\text{m}\,\text{s}^{-1}$ at the following angles:

a 30° to the horizontal
b 0° to the horizontal
c 90° to the horizontal.

Find the *x*- and *y*-components of the initial velocity in each case.

a $v_x = u\cos\theta$ $\qquad v_y = u\sin\theta$

$v_x = 18.0 \times \cos 30°$ $\qquad v_y = 18.0 \times \sin 30°$

$v_x = 15.6\,\text{m}\,\text{s}^{-1}$ $\qquad v_y = 9.00\,\text{m}\,\text{s}^{-1}$

b $v_x = 18.0\,\text{m}\,\text{s}^{-1}$ $\qquad v_y = 0\,\text{m}\,\text{s}^{-1}$

c $v_x = 0$ $\qquad v_y = 18.0\,\text{m}\,\text{s}^{-1}$

2.11 Sketch graphs to show the variation with time of the horizontal and vertical components of velocity for a projectile launched at some angle above the horizontal.

The graphs are shown in Figure **2.18**.

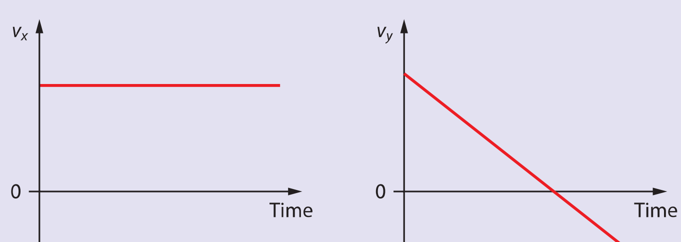


Figure 2.18

2.12 An object is launched horizontally from a height of 20 m above the ground with speed $15\,\mathrm{m\,s^{-1}}$. Determine:

a the time at which it will hit the ground

b the horizontal distance travelled

c the speed with which it hits the ground.

(Take $g = 10\,\mathrm{m\,s^{-2}}$.)

a The launch is horizontal, i.e. $\theta = 0°$, and so the formula for vertical displacement is just $y = -\frac{1}{2}gt^2$.

The object will hit the ground when $y = -20$ m.

Substituting the values, we find:

$$-20 = -5t^2$$

$$\Rightarrow \quad t = 2.0\,\mathrm{s}$$

Exam tip
This is a basic problem – you must know how to do this!

b The horizontal distance is found from $x = ut$. Substituting values:

$$x = 15 \times 2.0 = 30\,\mathrm{m}$$

(Remember that $\theta = 0°$).

c Use $v^2 = u^2 - 2gy$ to get:

$$v^2 = 15^2 - 2 \times 10 \times (-20)$$

$$v = 25\,\mathrm{m\,s^{-1}}$$

2.13 An object is launched horizontally with a velocity of $12\,\mathrm{m\,s^{-1}}$. Determine:

a the vertical component of velocity after 4.0 s

b the x- and y-components of the position vector of the object after 4.0 s.

a The launch is again horizontal, i.e. $\theta = 0°$, so substitute this value in the formulas. The horizontal component of velocity is $12\,\mathrm{m\,s^{-1}}$ at all times.

From $v_y = -gt$, the vertical component after 4.0 s is $v_y = -20\,\mathrm{m\,s^{-1}}$.

b The coordinates after time t are:

$x = ut$ and $y = -\frac{1}{2}gt^2$

$x = 12.0 \times 4.0$ $\quad y = -5 \times 16$

$x = 48\,\mathrm{m}$ $\quad y = -80\,\mathrm{m}$

Figure **2.19** shows an object thrown at an angle of $\theta = 30°$ to the horizontal with initial speed $20\,\mathrm{m\,s^{-1}}$. The position of the object is shown every 0.2 s. Note how the dots get closer together as the object rises (the speed is decreasing) and how they move apart on the way down (the speed is increasing). It reaches a maximum height of 5.1 m and travels a horizontal distance of 35 m. The photo in Figure **2.20** show an example of projectile motion.

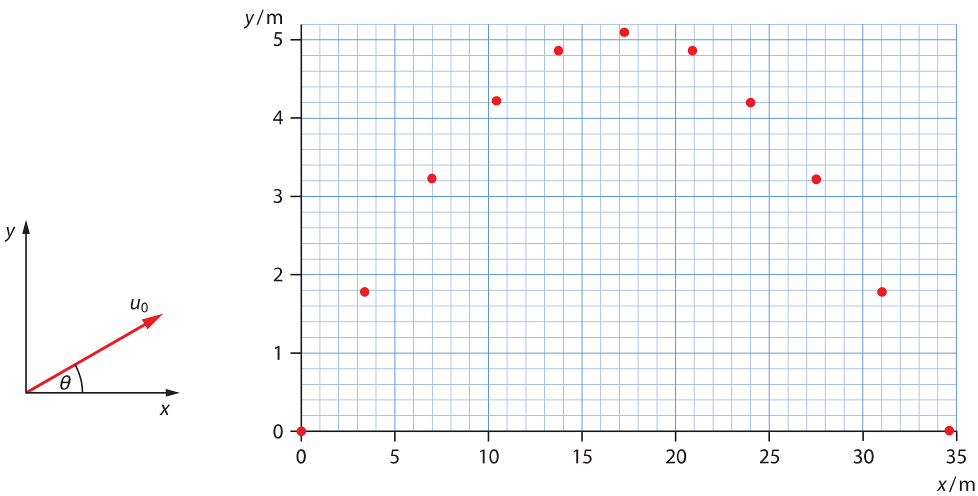


Figure 2.19 A launch at of $\theta = 30°$ to the horizontal with initial speed $20\,\mathrm{m\,s^{-1}}$.

At what point in time does the vertical velocity component become zero? Setting $v_y = 0$ we find:

$$0 = u \sin\theta - gt$$

$$\Rightarrow \quad t = \frac{u \sin\theta}{g}$$

Figure 2.20 A real example of projectile motion!

The time when the vertical velocity becomes zero is, of course, the time when the object attains its maximum height. What is this height? Going back to the equation for the vertical component of displacement, we find that when:

$$t = \frac{u \sin\theta}{g}$$

y is given by:

$$y_{\max} = u \frac{u \sin\theta}{g} \sin\theta - \frac{1}{2} g \left(\frac{u \sin\theta}{g}\right)^2$$

$$y_{\max} = \frac{u^2 \sin^2\theta}{2g}$$

Exam tip
You should not remember these formulas by heart. You should be able to derive them quickly.

What about the maximum displacement in the horizontal direction (sometimes called the range)? At this point the vertical component of displacement y is zero. Setting $y = 0$ in the formula for y gives:

$$0 = ut \sin\theta - \tfrac{1}{2} g t^2$$

$$0 = t\left(u \sin\theta - \tfrac{1}{2} g t\right)$$

and so:

$$t = 0 \qquad \text{and} \qquad t = \frac{2u \sin\theta}{g}$$

The first time $t=0$ is, of course, when the object first starts out. The second time is what we want – the time in which the range is covered. Therefore the range is:

$$x=\frac{2u^2\sin\theta\cos\theta}{g}$$

A bit of trigonometry allows us to rewrite this as:

$$x=\frac{u^2\sin(2\theta)}{g}$$

One of the identities in trigonometry is $2\sin\theta\cos\theta=\sin 2\theta$

The maximum value of $\sin 2\theta$ is 1, and this happens when $2\theta=90°$ (i.e. $\theta=45°$); in other words, we obtain the maximum range with a launch angle of 45°. This equation also says that there are two different angles of launch that give the same range for the same initial speed. These two angles add up to a right angle (can you see why?).

Worked examples

2.14 A projectile is launched at 32.0° to the horizontal with initial speed $25.0\,\mathrm{m\,s^{-1}}$. Determine the maximum height reached. (Take $g=9.81\,\mathrm{m\,s^{-2}}$.)

The vertical velocity is given by $v_y=u\sin\theta-gt$ and becomes zero at the highest point. Thus:

$$t=\frac{u\sin\theta}{g}$$

$$t=\frac{25.0\times\sin 32.0°}{9.81}$$

$$t=1.35\,\mathrm{s}$$

Substituting in the formula for y, $y=ut\sin\theta-\frac{1}{2}gt^2$, we get:

$$y=25\times\sin 32.0°\times 1.35-\frac{1}{2}\times 9.81\times 1.35^2$$

$$y=8.95\,\mathrm{m}$$

2.15 A projectile is launched horizontally from a height of 42 m above the ground. As it hits the ground, the velocity makes an angle of 55° to the horizontal. Find the initial velocity of launch. (Take $g = 9.8\,\mathrm{m\,s^{-2}}$.)

The time it takes to hit the ground is found from $y = \frac{1}{2}gt^2$ (here $\theta = 0°$ since the launch is horizontal).

The ground is at $y = -42\,\mathrm{m}$ and so:

$$-42 = -\frac{1}{2} \times 9.8t^2$$

$$\Rightarrow \quad t = 2.928\,\mathrm{s}$$

Using $v = u - at$, when the projectile hits the ground:

$$v_y = 0 - 9.8 \times 2.928$$

$$v_y = -28.69\,\mathrm{m\,s^{-1}}$$

We know the angle the final velocity makes with the ground (Figure **2.21**). Hence:

$$\tan 55° = \left|\frac{v_y}{v_x}\right|$$

$$\Rightarrow \quad v_x = \frac{28.69}{\tan 55°}$$

$$v_x = 20.03 \approx 20\,\mathrm{m\,s^{-1}}$$

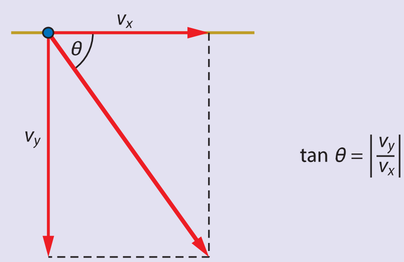


Figure 2.21

Fluid resistance

The discussion of the previous sections has neglected air resistance forces. In general, whenever a body moves through a fluid (gas or liquid) it experiences a **fluid resistance force** that is directed opposite to the velocity. Typically $F = kv$ for low speeds and $F = kv^2$ for high speeds (where k is a constant). The magnitude of this force increases with increasing speed.

Imagine dropping a body of mass m from some height. Assume that the force of air resistance on this body is $F = kv$. Initially, the only force on the body is its weight, which accelerates it downward. As the speed increases, the force of air resistance also increases. Eventually, this force will become equal to the weight and so the acceleration will become zero: the body will then move at constant speed, called **terminal speed**, v_T. This speed can be found from:

$$mg = kv_T$$

which leads to:

$$v_T = \frac{mg}{k}$$

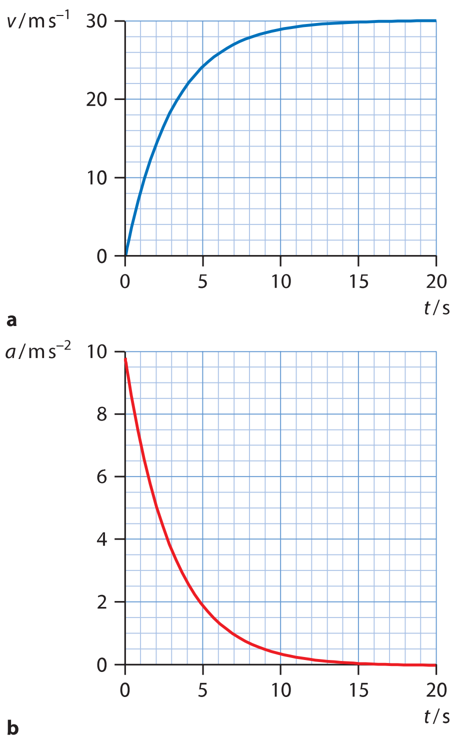


Figure 2.22 The variation with time of **a** speed and **b** acceleration in motion with an air resistance force proportional to speed.

Figure **2.22** shows how the speed and acceleration vary for motion with an air resistance force that is proportional to speed. The speed eventually becomes the terminal speed and the acceleration becomes zero. The initial acceleration is g.

The effect of air resistance forces on projectiles is very pronounced. Figure **2.23** shows the positions of a projectile with (red) and without (blue) air resistance forces. With air resistance forces the range and maximum height are smaller and the shape is no longer symmetrical. The projectile hits the ground with a steeper angle.

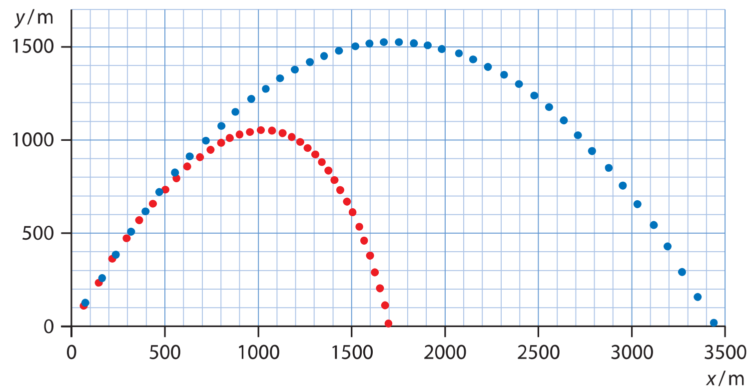


Figure 2.23 The effect of air resistance on projectile motion.

Worked example

2.16 The force of air resistance in the motion described by Figure **2.22** is given by $F = 0.653v$. Determine the mass of the projectile.

The particle is getting slower. At some point it will stop instantaneously, i.e. its velocity v will be zero.

We know that $v = u + at$. Just substituting values gives:

$$0 = 12 + (-3.0) \times t$$

$$3.0t = 12$$

Hence $t = 4.0\,\text{s}$.

The terminal speed is $30\,\text{m}\,\text{s}^{-1}$ and is given by $v_T = \frac{mg}{k}$. Hence:

$$m = \frac{kv_T}{g}$$

$$m = \frac{0.653 \times 30}{9.8}$$

$$m \approx 2.0\,\text{kg}$$

Nature of science

The simple and the complex

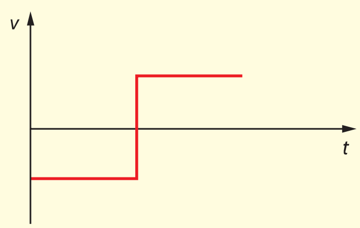
Careful observation of motion in the natural world led to the equations for motion with uniform acceleration along a straight line that we have used in this section. Thinking about what causes an object to move links to the idea of forces. However, although the material in this section is perhaps some of the 'easiest' material in your physics course, it does not enable one to understand the falling of a leaf off a tree. The falling leaf is complicated because it is acted upon by several forces: its weight, but also by air resistance forces that constantly vary as the orientation and speed of the leaf change. In addition, there is wind to consider as well as the fact that turbulence in air greatly affects the motion of the leaf. So the physics of the falling leaf is far away from the physics of motion along a straight line at constant acceleration. But learning the principles of physics in a simpler context allows its application in more involved situations.

? Test yourself

Uniform motion

1 A car must be driven a distance of 120 km in 2.5 h. During the first 1.5 h the average speed was 70 km h^{-1}. Calculate the average speed for the remainder of the journey.

2 Draw the position–time graph for an object moving in a straight line with a velocity–time graph as shown below. The initial position is zero. You do not have to put any numbers on the axes.



3 Two cyclists, **A** and **B**, have displacements 0 km and 70 km, respectively. At $t = 0$ they begin to cycle towards each other with velocities 15 km h^{-1} and 20 km h^{-1}, respectively. At the same time, a fly that was sitting on **A** starts flying towards **B** with a velocity of 30 km h^{-1}. As soon as the fly reaches **B** it immediately turns around and flies towards **A**, and so on until **A** and **B** meet.

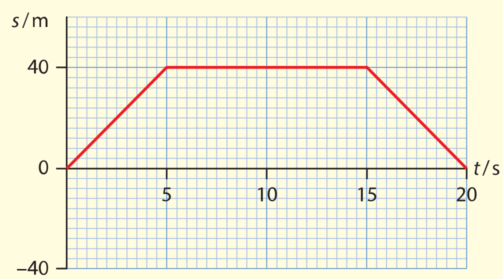
a Find the position of the two cyclists and the fly when all three meet.

b Determine the distance travelled by the fly.

4 An object moving in a straight line has the displacement–time graph shown.

a Find the average speed for the trip.

b Find the average velocity for the trip.



Accelerated motion

5 The initial velocity of a car moving on a straight road is 2.0 m s^{-1}. It becomes 8.0 m s^{-1} after travelling for 2.0 s under constant acceleration. Find the acceleration.

6 A car accelerates from rest to 28 m s^{-1} in 9.0 s. Find the distance it travels.

7 A particle has an initial velocity of 12 m s^{-1} and is brought to rest over a distance of 45 m. Find the acceleration of the particle.

8 A particle at the origin has an initial velocity of -6.0 m s^{-1} and moves with an acceleration of 2.0 m s^{-2}. Determine when its position will become 16 m.

9 A plane starting from rest takes 15.0 s to take off after speeding over a distance of 450 m on the runway with constant acceleration. Find the take-off velocity.

10 A car is travelling at $40.0\,\text{m}\,\text{s}^{-1}$. The driver sees an emergency ahead and 0.50 s later slams on the brakes. The deceleration of the car is $4.0\,\text{m}\,\text{s}^{-2}$.

a Find the distance travelled before the car stops.

b Calculate the stopping distance if the driver could apply the brakes instantaneously without a reaction time.

c Calculate the difference in your answers to **a** and **b**.

d Assume now that the car was travelling at $30.0\,\text{m}\,\text{s}^{-1}$ instead. Without performing any calculations, state whether the answer to **c** would now be less than, equal to or larger than before. Explain your answer.

11 Two balls are dropped from rest from the same height. One of the balls is dropped 1.00 s after the other.

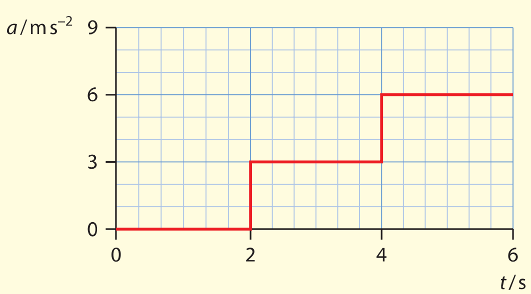
a Find the distance that separates the two balls 2.00 s after the second ball is dropped.

b State what happens to the distance separating the balls as time goes on.

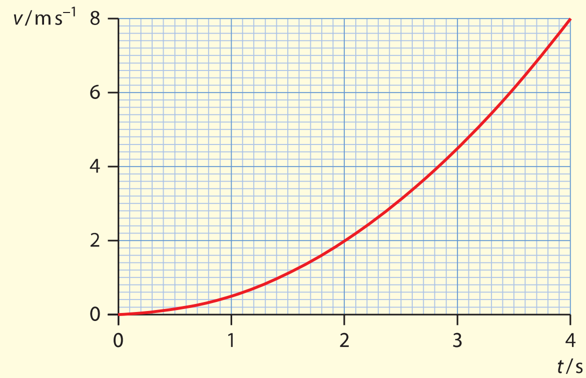
12 A particle moves in a straight line with an acceleration that varies with time as shown in the diagram. Initially the velocity of the object is $2.00\,\text{m}\,\text{s}^{-1}$.

a Find the maximum velocity reached in the first 6.00 s of this motion.

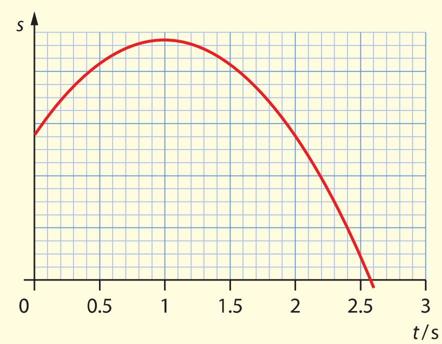
b Draw a graph of the velocity versus time.



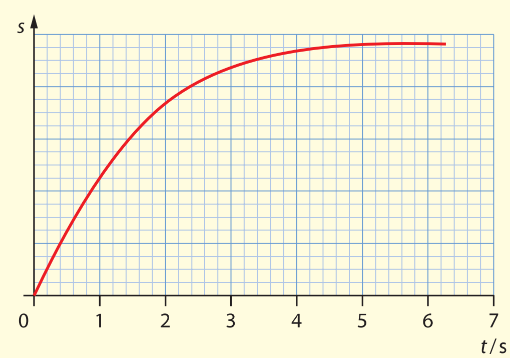
13 The graph shows the variation of velocity with time of an object. Find the acceleration at 2.0 s.



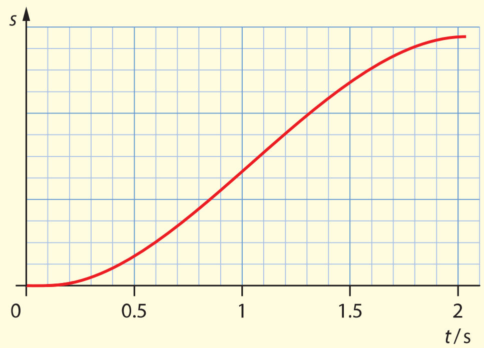
14 The graph shows the variation of the position of a moving object with time. Draw the graph showing the variation of the velocity of the object with time.



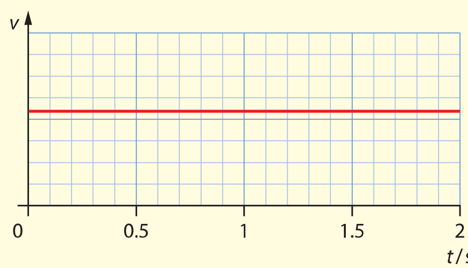
15 The graph shows the variation of the position of a moving object with time. Draw the graph showing the variation of the velocity of the object with time.



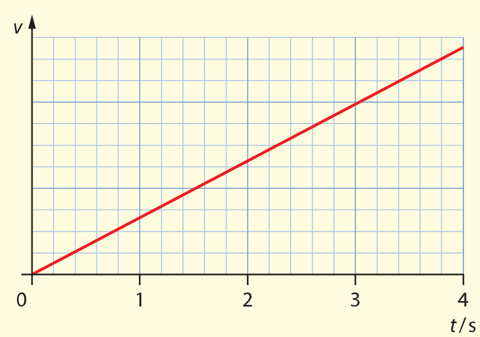
16 The graph shows the variation of the position of a moving object with time. Draw the graph showing the variation of the velocity of the object with time.



17 The graph shows the variation of the velocity of a moving object with time. Draw the graph showing the variation of the position of the object with time.



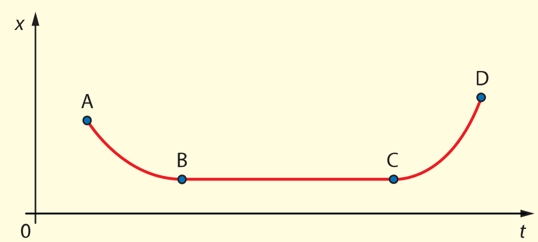
18 The graph shows the variation of the velocity of a moving object with time. Draw the graph showing the variation of the position of the object with time (assuming a zero initial position).



19 The graph shows the variation of the velocity of a moving object with time. Draw the graph showing the variation of the acceleration of the object with time.

20 Your brand new convertible Ferrari is parked 15 m from its garage when it begins to rain. You do not have time to get the keys, so you begin to push the car towards the garage. The maximum acceleration you can give the car is $2.0\,\mathrm{m\,s^{-2}}$ by pushing and $3.0\,\mathrm{m\,s^{-2}}$ by pulling back on the car. Find the least time it takes to put the car in the garage. (Assume that the car, as well as the garage, are point objects.)

21 The graph shows the displacement versus time of an object moving in a straight line. Four points on this graph have been selected.



a Is the velocity between **A** and **B** positive, zero or negative?

b What can you say about the velocity between **B** and **C**?

c Is the acceleration between **A** and **B** positive, zero or negative?

d Is the acceleration between **C** and **D** positive, zero or negative?

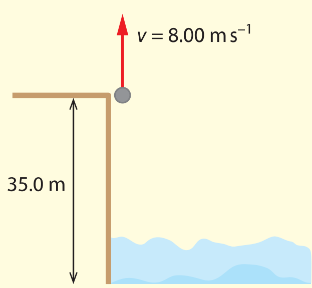
22 Sketch velocity–time sketches (no numbers are necessary on the axes) for the following motions.

a A ball is dropped from a certain height and bounces off a hard floor. The speed just before each impact with the floor is the same as the speed just after impact. Assume that the time of contact with the floor is negligibly small.

b A cart slides with negligible friction along a horizontal air track. When the cart hits the ends of the air track it reverses direction with the same speed it had right before impact. Assume the time of contact of the cart and the ends of the air track is negligibly small.

c A person jumps from a hovering helicopter. After a few seconds she opens a parachute. Eventually she will reach a terminal speed and will then land.

23 A stone is thrown vertically up from the edge of a cliff 35.0 m from the sea. The initial velocity of the stone is $8.00\,\text{m s}^{-1}$.



Determine:

a the maximum height of the stone

b the time when it hits the sea

c the velocity just before hitting the sea

d the distance the stone covers

e the average speed and the average velocity for this motion.

24 A ball is thrown upward from the edge of a cliff with velocity $20.0\,\text{m s}^{-1}$. It reaches the bottom of the cliff 6.0 s later.

a Determine the height of the cliff.

b Calculate the speed of the ball as it hits the ground.

Projectile motion

25 A ball rolls off a table with a horizontal speed of $2.0\,\text{m s}^{-1}$. The table is 1.3 m high. Calculate how far from the table the ball will land.

26 Two particles are on the same vertical line. They are thrown horizontally with the same speed, $4.0\,\text{m s}^{-1}$, from heights of 4.0 m and 8.0 m.

a Calculate the distance that will separate the two objects when both land on the ground.

b The particle at the 4.0 m height is now launched with horizontal speed u such that it lands at the same place as the particle launched from 8.0 m. Calculate u.

27 For an object thrown at an angle of 40° to the horizontal at a speed of $20\,\text{m s}^{-1}$, draw graphs of:

a horizontal velocity against time

b vertical velocity against time

c acceleration against time.

28 Determine the maximum height reached by an object thrown with speed $24\,\text{m s}^{-1}$ at 40° to the horizontal.

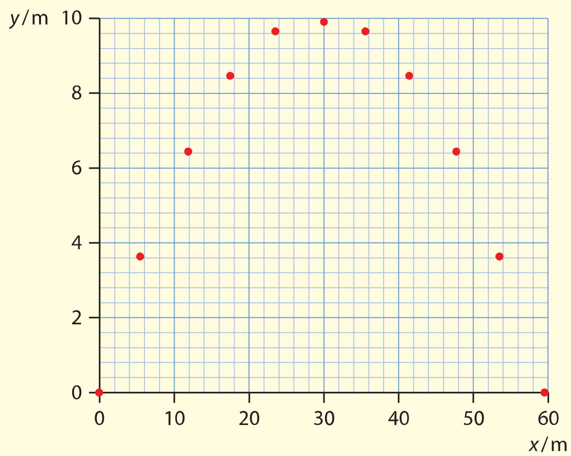
29 An object is thrown with speed $20.0\,\text{m s}^{-1}$ at an angle of 50° to the horizontal. Draw graphs to show the variation with time of:

a the horizontal position

b the vertical position.

30 A cruel hunter takes aim horizontally at a chimp that is hanging from the branch of a tree, as shown in the diagram. The chimp lets go of the branch as soon as the hunter pulls the trigger. Treating the chimp and the bullet as point particles, determine if the bullet will hit the chimp.

31 A ball is launched from the surface of a planet. Air resistance and other frictional forces are neglected. The graph shows the position of the ball every 0.20 s.



a Use this graph to determine:

i the components of the initial velocity of the ball

ii the angle to the horizontal the ball was launched at

iii the acceleration of free fall on this planet.

b Make a copy of the graph and draw two arrows to represent the velocity and the acceleration vectors of the ball at $t = 1.0$ s.

c The ball is now launched under identical conditions from the surface of a **different** planet where the acceleration due to gravity is twice as large. Draw the path of the ball on your graph.

32 A stone is thrown with a speed of $20.0\,\text{m}\,\text{s}^{-1}$ at an angle of 48° to the horizontal from the edge of a cliff 60.0 m above the surface of the sea.

a Calculate the velocity with which the stone hits the sea.

b Discuss qualitatively the effect of air resistance on your answer to **a**.

33 a State what is meant by **terminal speed**.

b A ball is dropped from rest. The force of air resistance in the ball is proportional to the ball's speed. Explain why the ball will reach terminal speed.

2.2 Forces

This section is an introduction to Newton's laws of motion. Classical physics is based to a great extent on these laws. It was once thought that knowledge of the present state of a system and all forces acting on it would enable the complete prediction of the state of that system in the future. This classical version of determinism has been modified partly due to quantum theory and partly due to chaos theory.

Learning objectives

- Treat bodies as point particles.
- Construct and interpret free-body force diagrams.
- Apply the equilibrium condition, $\Sigma F = 0$.
- Understand and apply Newton's three laws of motion.
- Solve problems involving solid friction.

Forces and their direction

A **force** is a vector quantity. It is important that we are able to correctly identify the **direction** of forces. In this section we will deal with the following forces.

Weight

This force is the result of the gravitational attraction between the mass m of a body and the mass of the planet on which the body is placed. The **weight** of a body is given by the formula:

$$W = mg$$

where m is the mass of the body and g is gravitational field strength of the planet (Subtopic **6.2**). The unit of g is newton per kilogram, N kg^{-1}. The gravitational field strength is also known as 'the acceleration due to gravity' or the 'acceleration of free fall'. Therefore the unit of g is also m s^{-2}.

If m is in kg and g in N kg^{-1} or m s^{-2} then W is in newtons, N. On the **surface** of the Earth, $g = 9.81\,\text{N kg}^{-1}$ – a number that we will often approximate by the more convenient $10\,\text{N kg}^{-1}$. This force is always directed vertically downward, as shown in Figure **2.24**.

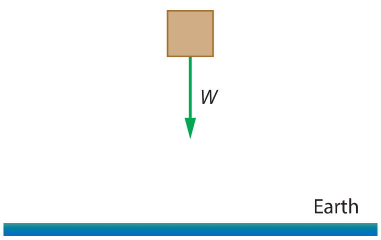


Figure 2.24 The weight of an object is always directed vertically downward.

The mass of an object is the same everywhere in the universe, but its weight depends on the **location** of the body. For example, a mass of 70 kg has a weight of 687 N on the surface of the Earth ($g = 9.81\,\text{N kg}^{-1}$) and a weight of 635 N at a height of 250 km from the Earth's surface (where $g = 9.07\,\text{N kg}^{-1}$). However, on the surface of Venus, where the gravitational field strength is only $8.9\,\text{N kg}^{-1}$, the weight is 623 N.

Tension

The force that arises in any body when it is stretched is called **tension**. A string that is taut is said to be under tension. The tension force is the result of electromagnetic interactions between the molecules of the material making up the string. A tension force in a string is created when two forces are applied in opposite directions at the ends of the string (Figure **2.25**).

Figure 2.25 A tension force in a string.

To say that there is tension in a string means that an arbitrary point on the string is acted upon by two forces (the tension T) as shown in Figure **2.26**. If the string hangs from a ceiling and a mass m is tied at the other end, tension develops in the string. At the point of support at the ceiling, the tension force pulls down on the ceiling and at the point where the mass is tied the tension acts upwards on the mass.

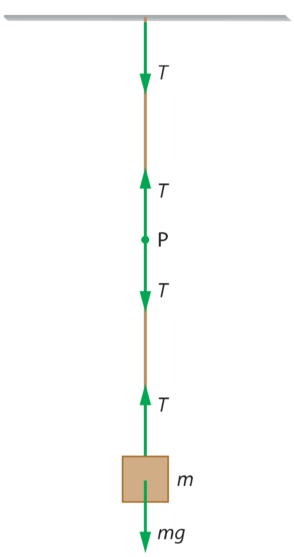


Figure 2.26 The tension is directed along the string.

In most cases we will idealise the string by assuming it is massless. This does not mean that the string really is massless, but rather that its mass is so small compared with any other masses in the problem that we can neglect it. In that case, the tension T is the same at all points on the string. The direction of the tension force is along the string. Further examples of tension forces in a string are given in Figure **2.27**. A string or rope that is not taut has zero tension in it.

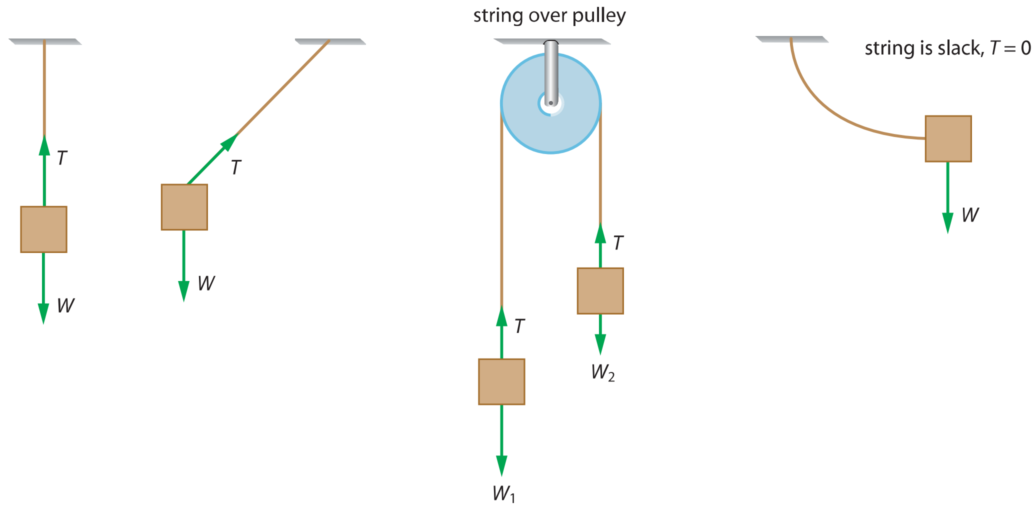


Figure 2.27 More examples of tension forces.

Forces in springs

A spring that is pulled so that its length increases will develop a tension force inside the spring that will tend to bring the length back to its original value. Similarly, if it is compressed a tension force will again try to restore the length of the spring, Figure **2.28**. Experiments show that for a range of extensions of the spring, the tension force is proportional to the extension, $T = kx$, where k is known as the spring constant. This relation between tension and extension is known as **Hooke's law**.

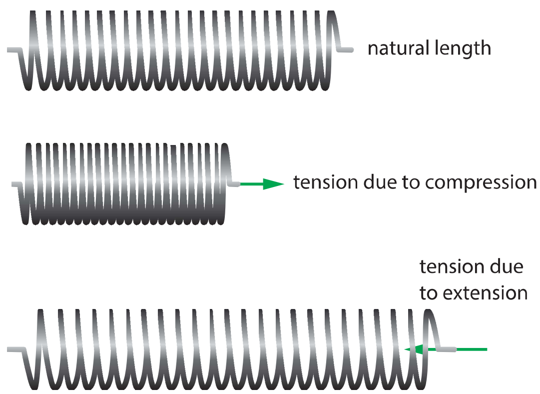


Figure 2.28 Tension forces in a spring.

Normal reaction contact forces

If a body touches another body, there is a **force of reaction** or **contact force** between the two bodies. This force is perpendicular to the surface of the body exerting the force. Like tension, the origin of this force is also electromagnetic. In Figure **2.29** we show the reaction force on several bodies.

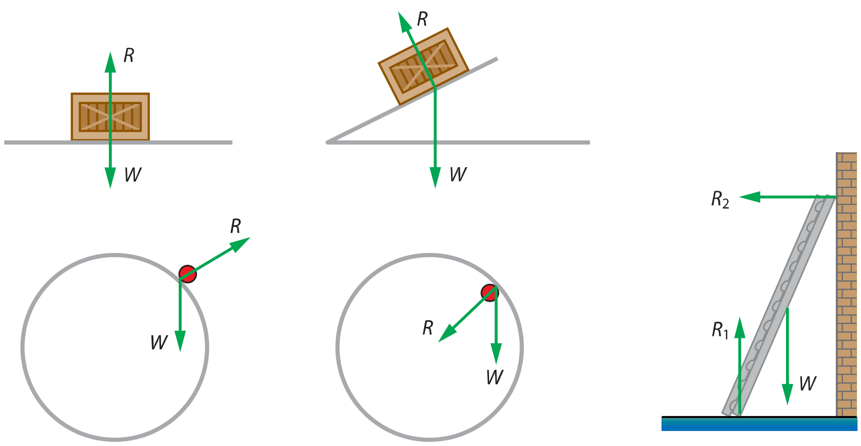


Figure 2.29 Examples of reaction forces, *R*.

We can understand the existence of contact reaction forces in a simple model in which atoms are connected by springs. The block pushes down on the atoms of the table, compressing the springs under the block (Figure 2.30). This creates the normal reaction force on the block.

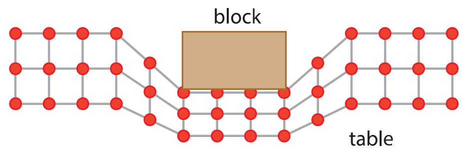


Figure 2.30 A simple model of contact forces.

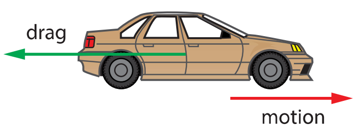


Figure 2.31 The drag force on a moving car.

Drag forces

Drag forces are forces that oppose the motion of a body through a fluid (a gas or a liquid). Typical examples are the air resistance force experienced by a car (Figure 2.31) or plane, or the resistance force experienced by a steel marble dropped into a jar of honey. Drag forces are directed opposite to the velocity of the body and in general depend on the speed and shape of the body. The higher the speed, the higher the drag force.

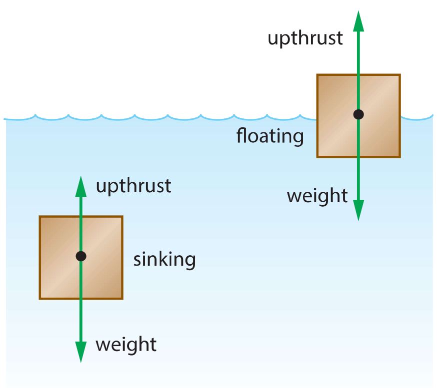


Figure 2.32 Upthrust.

Upthrust

Any object placed in a fluid experiences an upward force called **upthrust** (Figure 2.32). If the upthrust force equals the weight of the body, the body will float in the fluid. If the upthrust is less than the weight, the body will sink. Upthrust is caused by the pressure that the fluid exerts on the body.

Frictional forces

Frictional forces generally oppose the motion of a body (Figure 2.33). These forces are also electromagnetic in origin.

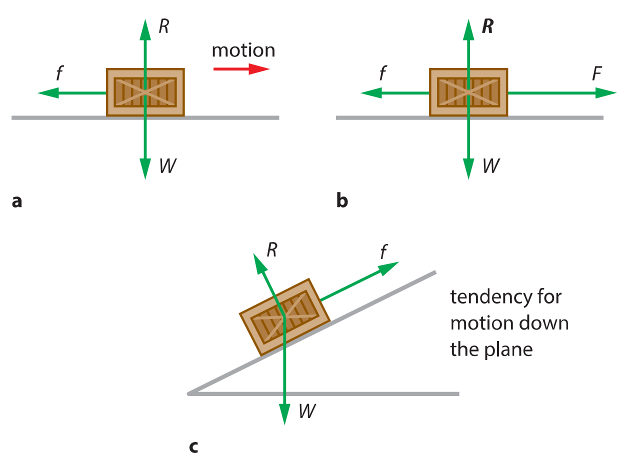


Figure 2.33 Examples of frictional forces, *f*. In **a** there is motion to the right, which is opposed by a single frictional force that will eventually stop the body. In **b** the force accelerating the body is opposed by a frictional force. In **c** the body does not move; but it does have a tendency to move down the plane and so a frictional force directed up the plane opposes this tendency, keeping the body in equilibrium.

Friction arises whenever one body slides over another. In this case we have **dynamic or kinetic friction**. Friction also arises whenever there is a tendency for motion, not necessarily motion itself. For example a block that rests on an inclined plane has a tendency to slide down the plane, so there is a force of friction up the plane. Similarly, if you pull on a block on a level rough road with a small force the block will not move. This is because a force of friction develops that is equal and opposite to the pulling force. In this case we have **static friction**.

In the simple model of matter consisting of atoms connected by springs, pushing the block to the right results in springs stretching and compressing. The net result is a force opposing the motion: friction (Figure **2.34**).

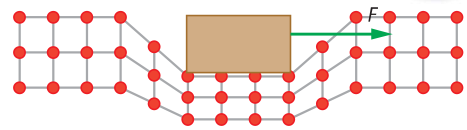


Figure 2.34 Friction in the simple atoms-and-springs model of matter.

A more realistic model involves irregularities (called **asperities**) in the surfaces which interlock, opposing sliding, as shown in Figure **2.35**.

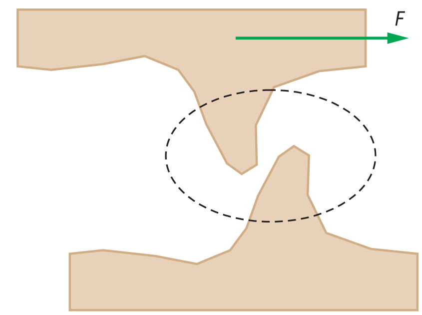


Figure 2.35 Exaggerated view of how asperities oppose the sliding of one surface over the other.

Frictional forces are still not very well understood and there is no theory of friction that follows directly from the fundamental laws of physics. However, a number of simple, empirical 'laws' of friction have been discovered. These are not always applicable and are only approximately true, but they are useful in describing frictional forces in general terms.

These so-called **friction laws** may be summarised as follows:

- The area of contact between the two surfaces does not affect the frictional force.
- The force of dynamic friction is equal to:
  $f_d = \mu_d R$
  where R is the normal reaction force between the surfaces and μ_d is the **coefficient of dynamic friction**.
- The force of dynamic friction does not depend on the speed of sliding.
- The **maximum** force of static friction that can develop between two surfaces is given by:
  $f_s = \mu_s R$
  where R is the normal reaction force between the surfaces and μ_s is the **coefficient of static friction**, with $\mu_s > \mu_d$.

Exam tip
One of the most common mistakes is to think that $\mu_s R$ is the formula that gives the static friction force. This is not correct. This formula gives the maximum possible static friction force that can develop between two surfaces.

Figure **2.36** shows how the frictional force f varies with a pulling force F. The force F pulls on a body on a horizontal rough surface. Initially the static frictional force matches the pulling force and we have no motion, $f_s = F$. When the pulling force exceeds the maximum possible static friction force, $\mu_s R$, the frictional force drops abruptly to the dynamic value of $\mu_d R$ and stays at that constant value as the object accelerates. This is a well-known phenomenon of everyday life: it takes a lot of force to get a heavy piece of furniture to start moving (you must exceed the maximum value of the static friction force), but once you get it moving, pushing it along becomes easier (you are now opposed by the smaller dynamic friction force).

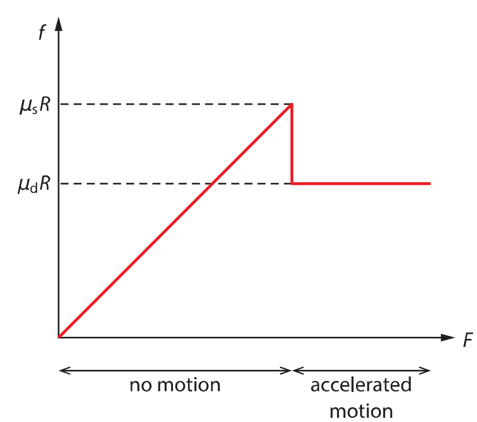


Figure 2.36 The variation of the frictional force f between surfaces with the pulling force F.

Worked example

2.17 A brick of weight 50 N rests on a horizontal surface. The coefficient of static friction between the brick and the surface is 0.60 and the coefficient of dynamic friction is 0.20. A horizontal force F is applied to the brick, its magnitude increasing uniformly from zero. Once the brick starts moving the pulling force no longer increases. Estimate the net force on the moving brick.

The maximum frictional force that can develop between the brick and the surface is:

$$f_s = \mu_s R$$

which evaluates to:

$$0.60 \times 50 = 30\,\text{N}$$

So motion takes place when the pulling force is just barely larger than 30 N.

Once motion starts the frictional force will be equal to $\mu_d R$, i.e.

$$0.20 \times 50 = 10\,\text{N}$$

The net force on the brick in that case will be just larger than $30 - 10 = 20\,\text{N}$.

Free-body diagrams

A **free-body diagram** is a diagram showing the magnitude and direction of all the forces acting on a chosen body. The body is shown on its own, free of its surroundings and of any other bodies it may be in contact with. We treat the body as a **point particle**, so that all forces act through the same point. In Figure **2.37** we show three situations in which forces are acting; below each is the corresponding free-body diagram for the coloured bodies.

In any mechanics problem, it is important to be able to draw correctly the free-body diagrams for all the bodies of interest. It is also important that the length of the arrow representing a given force is proportional to the magnitude of the force.

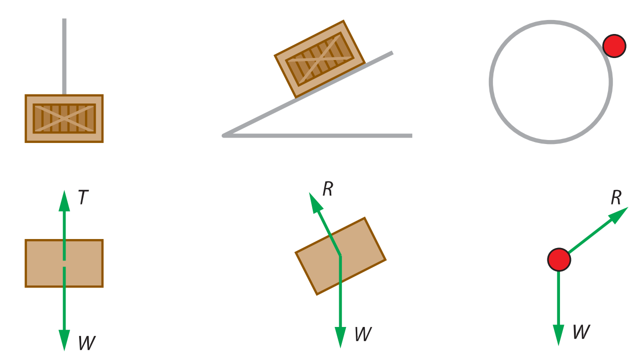


Figure 2.37 Free-body diagrams for the coloured bodies.

Newton's first law of motion

Suppose you have two identical train carriages. Both are equipped with all the apparatus you need to do physics experiments. One train carriage is at rest at the train station. The other moves in a straight line with constant speed – the ride is perfectly smooth, there are no bumps, there is no noise and there are no windows to look outside. Every physics experiment conducted in the train at rest will give identical results to similar experiments made in the moving train. We have no way of determining whether a carriage is 'really at rest' or 'really moving'. We find it perfectly natural to believe, correctly, that no net force is present in the case of the carriage at rest. Therefore no net force is required in the case of the carriage moving in a straight line with constant speed.

Newton's first law (with a big help from Galileo) states that:

> **When the net force on a body is zero, the body will move with constant velocity (which may be zero).**

In effect, Newton's first law defines what a force is. A force is what changes a body's velocity. A force is *not* what is required to keep something moving, as Aristotle thought.

Using the law in reverse allows us to conclude that if a body is not moving with constant velocity (which may mean not moving in a straight line, or not moving with constant speed, or both) then a force must be acting on the body. So, since the Earth revolves around the Sun we know that a force must be acting on the Earth.

Newton's first law is also called the law of **inertia**. Inertia is what keeps the body in the same state of motion when no forces act on the body. When a car accelerates forward, the passengers are thrown back into their seats because their original state of motion was motion with low speed. If a car brakes abruptly, the passengers are thrown forward (Figure **2.38**). This implies that a mass tends to stay in the state of motion it was in before the force acted on it. The reaction of a body to a change in its state of motion (acceleration) is inertia.

Figure 2.38 The car was originally travelling at high speed. When it hits the wall the car stops but the passenger stays in the original high speed state of motion. This results in the crash dummy hitting the steering wheel and the windshield (which is why it is a good idea to have safety belts and air bags).

Newton's third law of motion

Newton's third law states that if body A exerts a force on body B, then body B will exert an equal and opposite force on body A. These forces are known as **force pairs**. Make sure you understand that these equal and opposite forces act on different bodies. Thus, you cannot use this law to claim that it is impossible to ever have a net force on a body because for every force on it there is also an equal and opposite force. Here are a few examples of this law:

- You stand on roller skates facing a wall. You push on the wall and you move away from it. This is because you exerted a force on the wall and in turn the wall exerted an equal and opposite force on you, making you move away (Figure **2.39**).

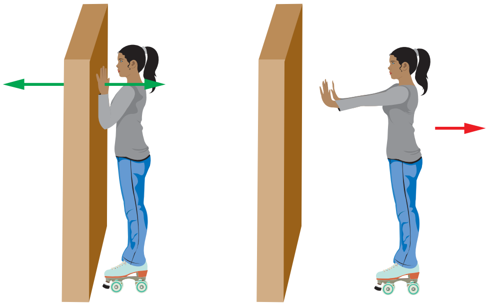


Figure 2.39 The girl pushes on the wall so the wall pushes on her in the opposite direction.

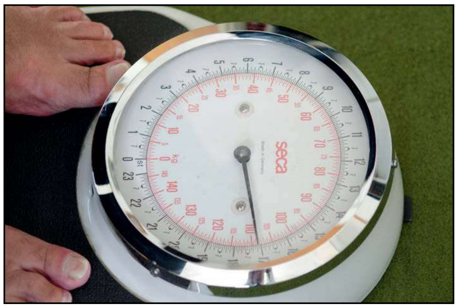


Figure 2.40 The familiar bathroom scales do not measure mass. They measure the force that you exert on the scales. This force is equal to the weight only when the scales are at rest.

Figure 2.41 The upward force on the rotor is due to the force the rotor exerts on the air downward.

- You step on the bathroom scales. The scales exert an upward force on you and so you exert a downward force on the scales. This is the force shown on the scales (Figure **2.40**).
- A helicopter hovers in air (Figure **2.41**). Its rotors exert a force downward on the air. Thus, the air exerts the upward force on the helicopter that keeps it from falling.
- A book of mass 2 kg is allowed to fall feely. The Earth exerts a force on the book, namely the weight of the book of about 20 N. Thus, the book exerts an equal and opposite force on the Earth – a force upward equal to 20 N.

You must be careful with situations in which two forces are equal and opposite; they do not always have to do with the third law. For example, a block of mass 3 kg resting on a horizontal table has two forces acting on it – its weight of about 30 N and the normal reaction force from the table that is also 30 N. These two forces are equal and opposite, but they are acting on the same body and so have nothing to do with Newton's third law. (We have seen in the last bullet point above the force that pairs with the weight of the block. The force that pairs with the reaction force is a downward force on the table.)

Newton's third law also applies to cases where there is no contact between the bodies. Examples are the **electric** force between two electrically charged particles or the **gravitational** force between any two massive particles. These forces must be equal and opposite (Figure **2.42**).

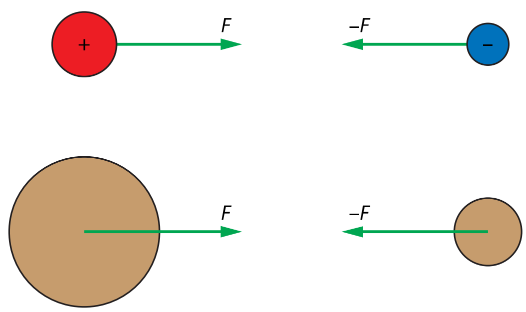


Figure 2.42 The two charges and the two masses are different, but the forces are equal and opposite.

Equilibrium

Equilibrium of a point particle means that the **net force** on the particle is zero. The net force on a particle is the one single force whose effect is the same as the combined effect of individual forces acting on the particle. We denote it by ΣF. Finding the net force is easy when the forces are in the same or opposite directions (Figure **2.43**).

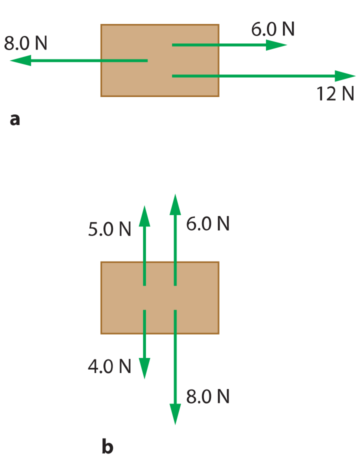


Figure 2.43 The net force is found by plain addition and/or subtraction when the forces are in the same or opposite direction.

In Figure **2.43a**, the net force is (if we take the direction to the right to be positive) $\Sigma F = 12 + 6.0 - 8.0 = 10$ N. This is positive, indicating a direction to the right.

In Figure **2.43b**, the net force is (we take the direction upward to be positive) $\Sigma F = 5.0 + 6.0 - 4.0 - 8.0 = -1.0$ N. The negative sign indicates a direction vertically down.

Worked example

2.18 Determine the magnitude of the force F in Figure **2.44**, given that the block is in equilibrium.

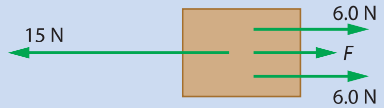


Figure 2.44

For equilibrium, $\Sigma F = 0$, and so:

$$6.0 + F + 6.0 - 15 = 0$$

This gives $F = 3.0\,\text{N}$.

Solving equilibrium problems

When there are angles between the various forces, solving equilibrium problems will involve finding components of forces using vector methods. We choose a set of axes whose origin is the body in question and find the components of all the forces on the body. Figure **2.45** shows three forces acting at the same point. We have equilibrium, which means the net force acting at the point is zero. We need to find the unknown magnitude and direction of force F_1. This situation could represent three people pulling on three ropes that are tied at a point.

Finding components along the horizontal (x) and vertical (y) directions for the known forces F_2 and F_3, we have:

$F_{2x} = 0$

$F_{2y} = -22.0\,\text{N}$ (add minus sign to show the direction)

$F_{3x} = -29.0 \cos 37° = -23.16\,\text{N}$ (add minus sign to show the direction)

$F_{3y} = 29.0 \,\text{sins}\, 37° = 17.45\,\text{N}$

Equilibrium demands that $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

$\Sigma F_x = 0$ implies:

$$F_{1x} + 0 - 23.16 = 0 \Rightarrow F_{1x} = 23.16\,\text{N}$$

$\Sigma F_y = 0$ implies:

$$F_{1y} - 22.0 + 17.45 = 0 \Rightarrow F_{1y} = 4.55\,\text{N}$$

Therefore, $F_1 = \sqrt{23.16^2 + 4.55^2} = 23.6\,\text{N}$

The angle is found from $\tan\theta = \frac{F_{1y}}{F_{1x}} \Rightarrow \theta = \tan^{-1}\left(\frac{4.55}{23.16}\right) = 11.1°$

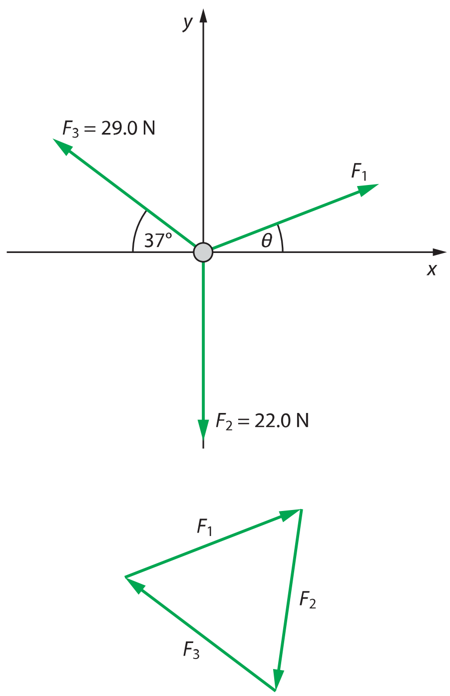


Figure 2.45 Force diagram of three forces in equilibrium pulling a common point. Notice that the three vectors representing the three forces form a triangle.

Exam tip

If we know the x- and y-components of a force we can find the magnitude of the force from $F = \sqrt{F_x^2 + F_y^2}$.

Worked example

2.19 A body of weight 98.0 N hangs from two strings that are attached to the ceiling as shown in Figure **2.46**. Determine the tension in each string.

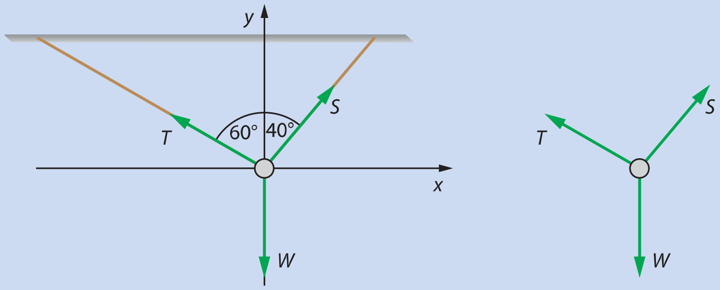


Figure 2.46

The three forces acting on the body are as shown, with T and S being the tensions in the two strings and W its weight. Taking components about horizontal and vertical axes through the body we find:

$T_x = -T\cos 30°$ (add minus sign to show the direction) $\quad S_x = S\cos 50° \quad W_x = 0$

$T_y = T\sin 30° \quad S_y = S\sin 50° \quad W_y = -98.0\,\text{N}$

Equilibrium thus demands $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

$\Sigma F_x = 0$ implies:

$$-T\cos 30° + S\cos 50° = 0$$

$\Sigma F_y = 0$ implies:

$$T\sin 30° + S\sin 50° - 98.0 = 0$$

From the first equation we find that:

$$S = T\frac{\cos 30°}{\cos 50°} = 1.3473 \times T$$

Substituting this in the second equation gives:

$$T(\sin 30° + 1.3473\sin 50°) = 98$$

which solves to give:

$$T = 63.96 \approx 64.0\,\text{N}$$

Hence $S = 1.3473 \times 63.96 = 86.17 \approx 86.2\,\text{N}$.

2.20 A mass of 125 g is attached to a spring of spring constant $k = 58\,\mathrm{N\,m^{-1}}$ that is hanging vertically.

a Find the extension of the spring.

b If the mass and the spring are placed on the Moon, will there be any change in the extension of the spring?

a The forces on the hanging mass are its weight and the tension in the spring. By Hooke's law, the tension in the spring is kx, where x is the extension and k the spring constant. Since we have equilibrium, the two forces are equal in magnitude. Therefore:

$$kx = mg$$

$$x = \frac{mg}{k}$$

$$x = \frac{0.125 \times 10}{58} \quad (\text{taking } g = 10\,\mathrm{N\,kg^{-1}})$$

$$x = 0.022\,\mathrm{m}$$

The extension is 2.2 cm.

b The extension will be less, since the acceleration of gravity is less.

Newton's second law of motion

Newton's second law states that:

> **The net force on a body of constant mass is proportional to that body's acceleration and is in the same direction as the acceleration.**

Mathematically:

$$F = ma$$

where the constant of proportionality, m, is the *mass* of the body.

Figure **2.47** shows the net force on a freely falling body, which happens to be its weight, $W = mg$. By Newton's second law, the net force equals the mass times the acceleration, and so:

$$mg = ma$$

$$a = g$$

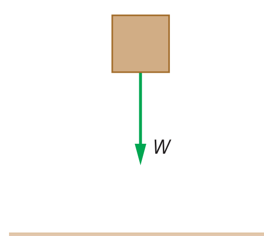


Figure 2.47 A mass falling to the ground acted upon by gravity.

That is, the acceleration of the freely falling body is exactly g. Experiments going back to Galileo show that indeed all bodies fall with the same acceleration in a vacuum (the acceleration of free fall) irrespective of their density, their mass, their shape and the material from which they are made. Look for David Scott's demonstration dropping a hammer and feather on the Moon in Apollo 15's mission in 1971. You can do the same demonstration without going to the Moon by placing a hammer and a

Exam tip

To solve an '$F = ma$' problem:

- Make a diagram.
- Identify the forces on the body of interest.
- Find the net force on each body, taking the direction of acceleration to be the positive direction.
- Apply $F_{net} = ma$ to each body.

feather on a book and dropping the book. If the heavy and the light object fell with different accelerations the one with the smaller acceleration would lift off the book – but it doesn't.

The equation $F = ma$ defines the unit of force, the newton (N). One newton is the force required to accelerate a mass of 1 kg by $1\,\text{m}\,\text{s}^{-2}$ in the direction of the force.

It is important to realise that the force in the second law is the net force ΣF on the body.

Worked examples

2.21 A man of mass $m = 70\,\text{kg}$ stands on the floor of an elevator. Find the force of reaction he experiences from the elevator floor when the elevator:

a is standing still

b moves up at constant speed $3.0\,\text{m}\,\text{s}^{-1}$

c moves up with acceleration $4.0\,\text{m}\,\text{s}^{-2}$

d moves down with acceleration $4.0\,\text{m}\,\text{s}^{-2}$

e moves down, slowing down with deceleration $4.0\,\text{m}\,\text{s}^{-2}$.

Take $g = 10\,\text{m}\,\text{s}^{-2}$.

Two forces act on the man: his weight mg vertically down and the reaction force R from the floor vertically up.

a There is no acceleration and so by Newton's second law the net force on the man must be zero. Hence:

$$R = mg$$
$$R = 7.0 \times 10^2\,\text{N}$$

b There is no acceleration and so again:

$$R = mg$$
$$R = 7.0 \times 10^2\,\text{N}$$

c There is acceleration upwards. The net force in the direction of the acceleration is given by:

$$\Sigma F = R - mg$$

So: $ma = R - mg$

$$\Rightarrow \quad R = mg + ma$$
$$R = 700\,\text{N} + 280\,\text{N}$$
$$R = 9.8 \times 10^2\,\text{N}$$

d We again have acceleration, but this time in the downward direction. We need to find the net force in the direction of the acceleration:

$$\Sigma F = mg - R$$

So: $ma = mg - R$

$$\Rightarrow \quad R = mg - ma$$
$$R = 700\,\text{N} - 280\,\text{N}$$
$$R = 4.2 \times 10^2\,\text{N}$$

e The deceleration is equivalent to an upward acceleration, so this case is identical to part **c**.

2.22 A man of mass 70 kg is standing in an elevator. The elevator is moving **upward** at a speed of $3.0\,\mathrm{m\,s^{-1}}$. The elevator comes to rest in a time of 2.0 s. Determine the reaction force on the man from the elevator floor during the period of deceleration.

Use $a = v - \frac{u}{t}$ to find the acceleration experienced by the man:

$$a = -\frac{3.0}{2.0} = -1.5\,\mathrm{m\,s^{-2}}$$

The minus sign shows that this acceleration is directed in the **downward** direction. So we must find the net force in the down direction, which is $\Sigma F = mg - R$. (We then use the **magnitude** of the accelerations, as the form of the equation takes care of the direction.)

$$ma = mg - R$$

$$\Rightarrow \quad R = mg - ma$$

$$R = 700 - 105$$

$$R = 595 \approx 6.0 \times 10^2\,\mathrm{N}$$

If, instead, the man was moving **downward** and then decelerated to rest, the acceleration is directed upward and $\Sigma F = R - mg$.

So: $ma = R - mg$

$$\Rightarrow \quad R = mg + ma$$

$$R = 700 + 105$$

$$R = 805 \approx 8.0 \times 10^2\,\mathrm{N}$$

Both cases are easily experienced in daily life. When the elevator goes up and then stops we feel 'lighter' during the deceleration period. When going down and about to stop, we feel 'heavier' during the deceleration period. The feeling of 'lightness' or 'heaviness' has to do with the reaction force we feel from the floor.

2.23 **a** Two blocks of mass 4.0 kg and 6.0 kg are joined by a string and rest on a frictionless horizontal table (Figure **2.48**). A force of 100 N is applied horizontally on one of the blocks. Find the acceleration of each block and the tension in the string.

b The 4.0 kg block is now placed on top of the other block. The coefficient of static friction between the two blocks is 0.45. The bottom block is pulled with a horizontal force F. Calculate the magnitude of the maximum force F that will result in both blocks moving together without slipping.

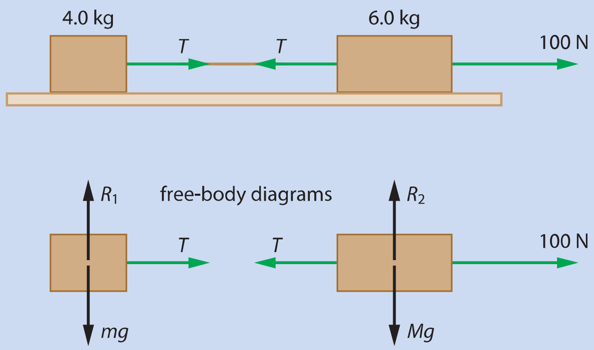


Figure 2.48

a This can be done in **two** ways.

Method 1

Let the acceleration of the system be a. The net horizontal force on the 6.0 kg mass is $100 - T$ and the net horizontal force on the 4.0 kg mass is just T. Thus, applying Newton's second law separately on each mass:

$$100 - T = 6.0a$$

$$T = 4.0a$$

Solving for a (by adding the two equations) gives:

$$100 = 10a$$

$$\Rightarrow \quad a = 10\,\text{m}\,\text{s}^{-2}$$

The tension in the string is therefore:

$$T = 4.0 \times 10$$

$$T = 40\,\text{N}$$

Note: The free-body diagram makes it clear that the 100 N force acts only on the body to the right. It is a common mistake to say that the body to the left is also acted upon by the 100 N force.

Method 2

We may consider the two bodies as one of mass 10 kg. The net force on the body is 100 N. Note that the tensions are irrelevant now since they cancel out. (They did not in Method 1, as they acted on different bodies. Now they act on the same body. They are now **internal** forces and these are irrelevant.)

Applying Newton's second law on the single body we have:

$$100 = 10a$$

$$\Rightarrow \quad a = 10\,\mathrm{m\,s^{-2}}$$

But to find the tension we must break up the combined body into the original two bodies. Newton's second law on the 4.0 kg body gives:

$$T = 4a = 40\,\mathrm{N}$$

(the tension on this block is the net force on the block). If we used the other block, we would see that the net force on it is $100 - T$ and so:

$$100 - T = 6 \times 10 = 60\,\mathrm{N}$$

This gives $T = 40\,\mathrm{N}$, as before.

b If the blocks move together they must have the same acceleration. Treating the two blocks as one (of mass 10 kg), the acceleration will be $a = \frac{F}{10}$ (Figure **2.49a**).

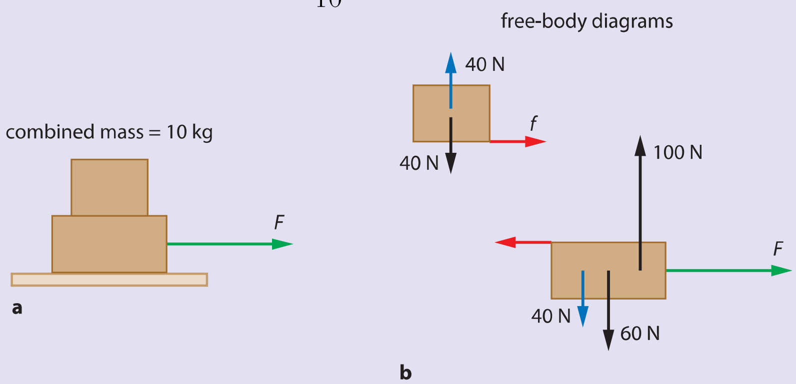


Figure 2.49 **a** Treating the blocks as one. **b** The free-body diagram for each block.

The forces on each block are shown in Figure **2.49b**. The force pushing the smaller block forward is the frictional force f that develops between the blocks. The **maximum** value f can take is:

$$f = \mu_s R = 0.45 \times 40 = 18\,\mathrm{N}$$

So the acceleration of the small block is:

$$a = \frac{18}{4.0} = 4.5\,\mathrm{m\,s^{-2}}$$

But $a = \frac{F}{10}$, so:

$$\frac{F}{10} = 4.5\,\mathrm{m\,s^{-2}}$$

$$\Rightarrow \quad F = 45\,\mathrm{N}$$

2.24 Two masses of $m = 4.0\,\text{kg}$ and $M = 6.0\,\text{kg}$ are joined together by a string that passes over a pulley (this arrangement is known as Atwood's machine). The masses are held stationary and suddenly released. Determine the acceleration of each mass.

Intuition tells us that the larger mass will start moving downward and the small mass will go up. So if we say that the larger mass's acceleration is a, then the other mass's acceleration will also be a in magnitude but, of course, in the opposite direction. The two accelerations are the same because the string cannot be extended.

Method 1

The forces on each mass are weight mg and tension T on m and weight Mg and tension T on M (Figure **2.50**).

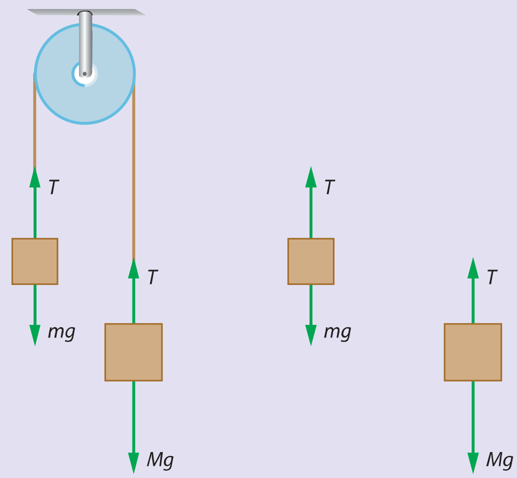


Figure 2.50

Newton's second law applied to each mass gives:

$$T - mg = ma \quad (1)$$

$$Mg - T = Ma \quad (2)$$

Note these equations carefully. Each says that the net force on the mass in question is equal to that mass times that mass's acceleration. In the first equation, we find the net force in the upward direction, because that is the direction of acceleration. In the second, we find the net force in downward direction, since that is the direction of acceleration in that case. We want to find the acceleration, so we simply add these two equations to find:

$$Mg - mg = (m + M)a$$

Hence:

$$a = \frac{M - m}{M + m} g$$

(Note that if $M \gg m$ the acceleration tends to g. Can you think why this is?) This shows clearly that if the two masses are equal, then there is no acceleration. This is a convenient method for measuring g. Atwood's machine effectively 'slows down' g so the falling mass has a much smaller acceleration from which g can then be determined. Putting in the numbers for our example we find $a = 2.0\,\text{m}\,\text{s}^{-2}$.

Having found the acceleration we may, if we wish, also find the tension in the string, T. Putting the value for a in formula (1) we find:

$$T = m\left(\frac{M - m}{M + m}\right)g + mg$$

$$T = 2\left(\frac{Mm}{M + m}\right)g$$

(If $M \gg m$ the tension tends to $2mg$. Can you see why?)

Method 2

We treat the two masses as one body and apply Newton's second law on this body (but this is trickier than in the previous example) – see Figure **2.51**.

In this case the net force is $Mg - mg$ and, since this force acts on a body of mass $M + m$, the acceleration is found as before from $F =$ mass × acceleration. Note that the tension T does not appear, as it is now an internal force.

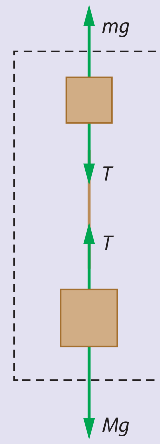


Figure 2.51

2.25 In Figure **2.52**, a block of mass M is connected to a smaller mass m through a string that goes over a pulley. Ignoring friction, find the acceleration of each mass and the tension in the string.

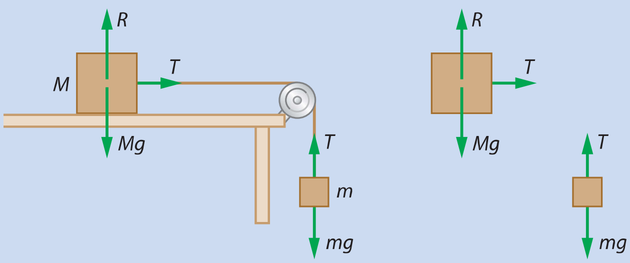


Figure 2.52

Method 1

The forces are shown in Figure **2.52**. The acceleration must be the same magnitude for both masses, but the larger mass accelerates horizontally and the smaller mass accelerates vertically downwards. The free-body diagrams on the right show the forces on the individual masses. Taking each mass separately:

$mg - T = ma$ (small mass accelerating downwards)

$T = Ma$ (large mass accelerating horizontally to the right)

Adding the two equations, we get:

$$mg = ma + Ma$$

$$\Rightarrow \quad a = \frac{mg}{M + m}$$

(If $M \gg m$ the acceleration tends to zero. Why?)

From the expression for T for the larger mass, we have:

$$T = Ma = \frac{Mmg}{M + m}$$

Method 2

Treating the two bodies as one results in the situation shown in Figure 2.53.

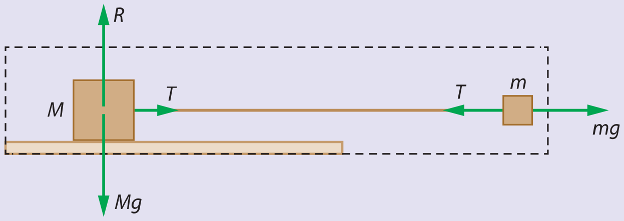


Figure 2.53

The net horizontal force on the combined mass $M+m$ is mg. Hence:

$$mg=(M+m)a$$

$$\Rightarrow \quad a=\frac{mg}{M+m}$$

The tension can then be found as before.

2.26 A block of mass 2.5 kg is held on a rough inclined plane, as shown in Figure 2.54. When released, the block stays in place. The angle of the incline is slowly increased and when the angle becomes slightly larger than 38° the block begins to slip down the plane.

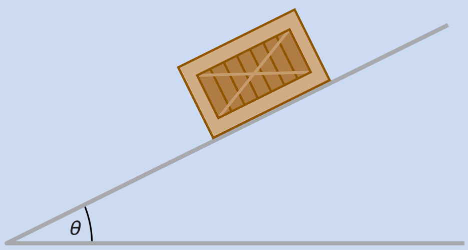


Figure 2.54

a Calculate the coefficient of static friction between the block and the inclined plane.

b The angle of the incline is increased to 49°. The coefficient of dynamic friction between the block and the incline is 0.26. Calculate the force that must be applied to the block along the plane so it moves up the plane with an acceleration of $1.2\,\mathrm{m\,s^{-2}}$.

a The forces on the block just before slipping are shown in Figure 2.55. The frictional force is f and the normal reaction is R. The components of the weight are $mg\sin\theta$ down the plane and $mg\cos\theta$ at right angles to the plane.

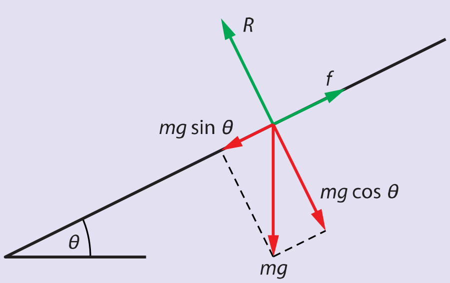


Figure 2.55

Because the block is about to slip, the frictional force is the maximum possible static frictional force and so $f=\mu_s R$. Equilibrium demands that:

$$mg\sin\theta=f$$

$$mg\cos\theta=R$$

Divide the first equation by the second to get:

$$\tan\theta = \frac{f}{R}$$

Now use the fact that $f = \mu_s R$ to find:

$$\tan\theta = \frac{\mu_s R}{R}$$

$$\tan\theta = \mu_s$$

Hence $\mu_s = \tan\theta = \tan 38° = 0.78$

b Let F be the required force up the plane. The net force up the plane is $F - mg\sin 49° - f_d$, since the force of friction now opposes F.

We have that:

$$f_d = \mu_s R = \mu_s mg\cos 49°$$

Therefore:

$$F - mg\sin 49° - \mu_s mg\cos 49° = ma$$

$$F = ma + mg\sin 49° + \mu_s mg\cos 49°$$

Substituting values:

$$F = 2.5 \times 1.2 + 2.5 \times 9.8 \times \sin 49° + 0.26 \times 2.5 \times 9.8\cos 49°$$

$$F = 25.67 \approx 26\,\text{N}$$

Exam tip
Notice that for a block on a frictionless inclined plane the net force down the plane is $mg\sin\theta$, leading to an acceleration of $g\sin\theta$, independent of the mass.

Nature of science

Physics and mathematics

In formulating his laws of motion, published in 1687 in *Philosophiæ Naturalis Principia Mathematica*, Newton used mathematics to show how the work of earlier scientists could be applied to forces and motion in the real world. Newton's second law (for particle of constant mass) is written as $F = ma$. In this form, this equation does not seem particularly powerful. However, using calculus, Newton showed that acceleration is given by:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

The second law then becomes:

$$\frac{d^2x}{dt^2} - \frac{F}{m} = 0$$

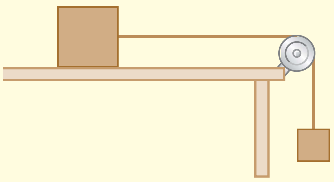
This is a differential equation that can be solved to give the actual path that the particle will move on under the action of the force. Newton showed that if the force depends on position as $F \propto \frac{1}{x^2}$, then the motion has to be along a conic section (ellipse, circle, etc.).

Newton used a flash of inspiration, triggered by observing an apple falling from a tree, to relate the motion of planets to that of the apple, leading to his law of gravitation (which you will meet in Topic **6**).

Test yourself

Equilibrium

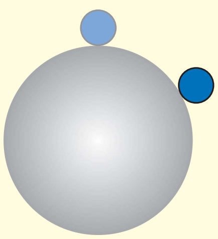
34 A block rests on a rough table and is connected by a string that goes over a pulley to a second hanging block, as shown in the diagram. Draw the forces on each body.



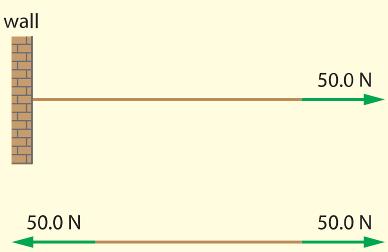
35 A bead rolls on the surface of a sphere, having started from the top, as shown in the diagram. On a copy of the diagram, draw the forces on the bead:

a at the top

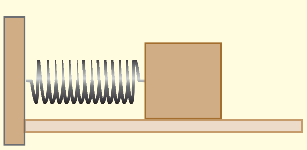
b at the point where it is about to leave the surface of the sphere.



36 Look at the diagram. State in which case the tension in the string is largest.



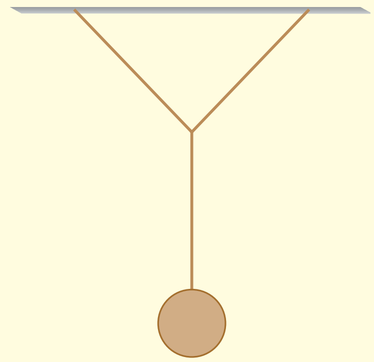
37 A spring is compressed by a certain distance and a mass is attached to its right end, as shown in the diagram. The mass rests on a rough table. On a copy of the diagram, draw the forces acting on the mass.



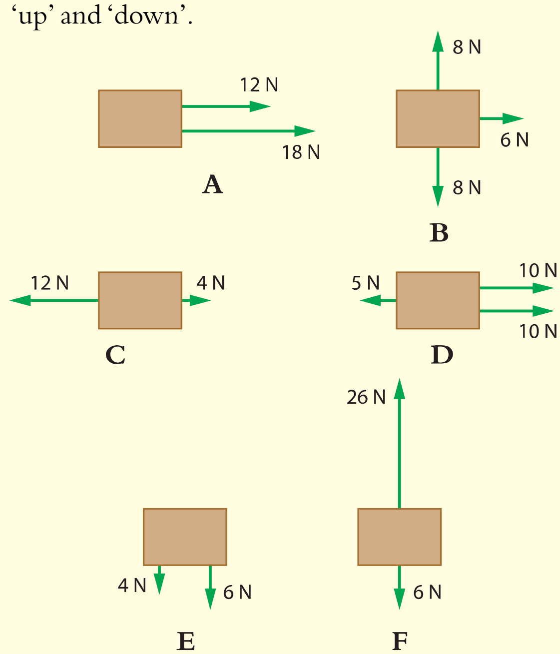
38 A mass hangs attached to three strings, as shown in the diagram. On a copy of the diagram, draw the forces on:

a the hanging mass

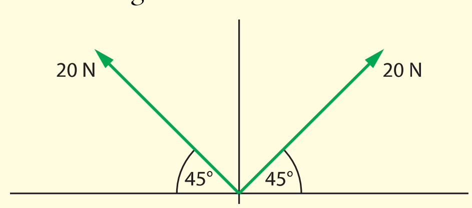
b the point where the strings join.



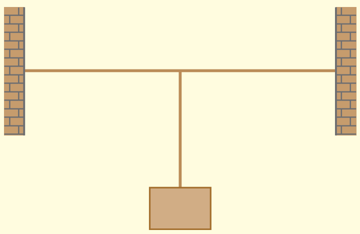
39 Find the net force on each of the bodies shown in the diagrams. The only forces acting are the ones shown. Indicate direction by 'right', 'left', 'up' and 'down'.



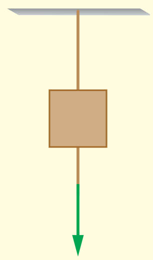
40 Find the magnitude and direction of the net force in the diagram.



41 Explain why is it impossible for a mass to hang attached to two horizontal strings as shown in the diagram.



42 A mass is hanging from a string that is attached to the ceiling. A second piece of string (identical to the first) hangs from the lower end of the mass (see diagram).

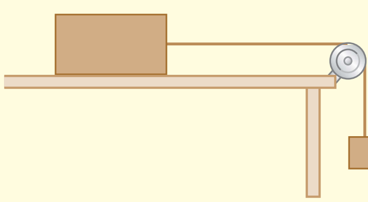


State and explain which string will break if:

a the bottom string is slowly pulled with ever increasing force

b the bottom string is very abruptly pulled down.

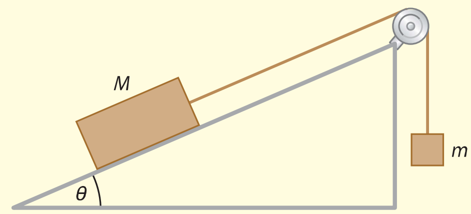
43 A mass of 2.00 kg rests on a rough horizontal table. The coefficient of static friction between the block and the table is 0.60. The block is attached to a hanging mass by a string that goes over a smooth pulley, as shown in the diagram. Determine the largest mass that can hang in this way without forcing the block to slide.



44 A girl tries to lift a suitcase of weight 220 N by pulling upwards on it with a force of 140 N. The suitcase does not move. Calculate the reaction force from the floor on the suitcase.

45 A block of mass 15.0 kg rests on a horizontal table. A force of 50.0 N is applied vertically downward on the block. Calculate the force that the block exerts on the table.

46 A block of mass M is connected with a string to a smaller block of mass m. The big block is resting on a smooth inclined plane as shown in the diagram. Determine the angle of the plane in terms of M and m in order to have equilibrium.



Accelerated motion

47 Describe under what circumstances a constant force would result in **a** an increasing and **b** a decreasing acceleration on a body.

48 A car of mass 1400 kg is on a muddy road. If the force from the engine pushing the car forward exceeds 600 N, the wheels slip (i.e. they rotate without rolling). Estimate the car's maximum acceleration on this road.

49 A man of mass m stands in an elevator.

a Find the reaction force from the elevator floor on the man when:

i the elevator is standing still

ii the elevator moves up at constant speed v

iii the elevator accelerates down with acceleration a

iv the elevator accelerates down with acceleration $a = g$.

b What happens when $a > g$?

50 Get in an elevator and stretch out your arm holding your heavy physics book. Press the button to go up. Describe and explain what is happening to your stretched arm. Repeat as the elevator comes to a stop at the top floor. What happens when you press the button to go down and what happens when the elevator again stops? Explain your observations carefully using the second law of motion.

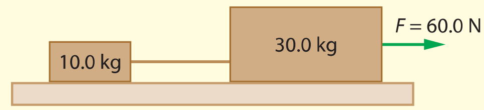
51 The diagram shows a person in an elevator pulling on a rope that goes over a pulley and is attached to the top of the elevator. The mass of the elevator is 30.0 kg and that of the person is 70 kg.

- **a** On a copy of the diagram, draw the forces on the person.
- **b** Draw the forces on the elevator.
- **c** The elevator accelerates upwards at $0.50\,\mathrm{m\,s^{-2}}$. Find the reaction force on the person from the elevator floor.
- **d** The force the person exerts on the elevator floor is 300 N. Find the acceleration of the elevator ($g = 10\,\mathrm{m\,s^{-2}}$).

52 A massless string has the same tension throughout its length. Suggest why.

53 **a** Calculate the tension in the string joining the two masses in the diagram.

b If the position of the masses is interchanged, will the tension change?



54 A mass of 3.0 kg is acted upon by three forces of 4.0 N, 6.0 N and 9.0 N and is in equilibrium. Convince yourself that these forces can indeed be in equilibrium. The 9.0 N force is suddenly removed. Determine the acceleration of the mass.

Learning objectives

- Understand the concepts of kinetic, gravitational potential and elastic potential energy.
- Understand work done as energy transferred.
- Understand power as the rate of energy transfer.
- Understand and apply the principle of energy conservation.
- Calculate the efficiency in energy transfers.

2.3 Work, energy and power

This section deals with energy, one of the most basic concepts in physics. We introduce the principle of energy conservation and learn how to apply it to various situations. We define kinetic and potential energy, work done and power developed.

Energy

Energy is a concept that we all have an intuitive understanding of. Chemical energy derived from food keeps us alive. Chemical energy from gasoline powers our cars. Electrical energy keeps our computers going. Nuclear fusion energy produces light and heat in the Sun that sustains life on Earth. And so on. Very many experiments, from the subatomic to the cosmic scale, appear to be consistent with the principle of **conservation of energy** that states that energy is not created or destroyed but is only transformed from one form into another. This means that any change in the energy of a system must be accompanied by a change in the energy of the surroundings of the system such that:

$$\Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0$$

In other words, if the system's energy increases, the energy of the surroundings must decrease by the same amount and vice-versa.

The energy of the system may change as a result of **interactions** with its surroundings (Figure **2.56**). These interactions mainly involve **work done** W by the surroundings and/or the **transfer of thermal energy** (heat) Q, to or from the surroundings. But there are many other interactions between a system and its surroundings. For example, waves of many kinds may transfer energy to the system (the Sun heats the Earth); gasoline, a chemical fuel, may be added to the system, increasing its energy; wind incident on the blades of a windmill will generate electrical energy as a generator is made to turn, etc. So:

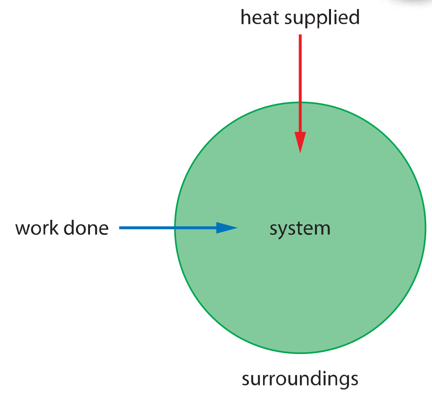


Figure 2.56 The total energy of a system may change as a result of interactions with its surroundings.

$$\Delta E_{\text{system}} = W + Q + \text{other transfers}$$

But in this section we will deal with $Q = 0$ and no other transfers so we must understand and use the relation:

$$\Delta E = W$$

(we dropped the subscript in E_{system}). To do so, we need to define what we mean by work done and what exactly we mean by E, the total energy of the system.

Work done by a force

We first consider the definition of **work done** by a constant force for motion in a straight line. By constant force we mean a force that is constant in magnitude as well as in direction. Figure **2.57** shows a block that is displaced along a straight line. The distance travelled by the body is s. The force makes an angle θ with the displacement.

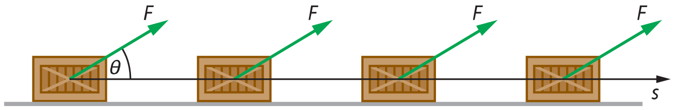


Figure 2.57 A force moving its point of application performs work.

The force acts on the body all the time as it moves. The work done by the force is defined as:

$$W = Fs\cos\theta$$

But $F\cos\theta$ is the component of the force in the direction of the displacement and so:

> **The work done by a force is the product of the force in the direction of the displacement times the distance travelled.**

(Equivalently, since $s\cos\theta$ is the distance travelled in the direction of the force, work may also be defined as the product of the force time, the distance travelled in the direction of the force.)

The cosine here can be positive, negative or zero; thus work can be positive, negative or zero. We will see what that means shortly.

The unit of work is the joule. One joule is the work done by a force of 1 N when it moves a body a distance of 1 m in the direction of the force. 1 J = 1 N m.

Worked examples

2.27 A mass is being pulled along a level road by a rope attached to it in such a way that the rope makes an angle of 34° with the horizontal. The force in the rope is 24 N. Calculate the work done by this force in moving the mass a distance of 8.0 m along the level road.

We just have to apply the formula for work done:

$$W = Fs\cos\theta$$

Substituting the values from the question:

$$W = 24 \times 8.0 \times \cos 34°$$

$$W = 160\,\text{J}$$

2.28 A car with its engine off moves on a horizontal level road. A constant force of 620 N opposes the motion of the car. The car comes to rest after 84 m. Calculate the work done on the car by the opposing force.

We again apply the formula for work done, but now we have to realise that $\theta = 180°$. So:

$$W = 620 \times 84 \times \cos 180°$$

$$W = -52\,\text{kJ}$$

2.29 You stand on roller skates facing a wall. You push against the wall and you move away. Discuss whether the force exerted by the wall on you performed any work.

No work was done because there is no displacement. You moved but the point where the force is applied never moved.

Varying force and curved path

You will meet situations where the force is not constant in magnitude or direction and the path is not a straight line. To find the work done we must break up the curved path into very many small straight segments in a way that approximates the curved path (Figure **2.58**). Think of these segments as the dashes that make up the curve when it is drawn as a dashed line. The large arrowed segments at the bottom of Figure **2.58** show this more clearly. The total work done is the sum of the work done on each segment of the path.

We assume that along each segment the force is constant. The work done on the kth segment is just $F_k s_k \cos\theta_k$. So the work done on all the segments is found by adding up the work done on individual segments, i.e.

$$W = \sum_{k=1} F_k s_k \cos\theta_k$$

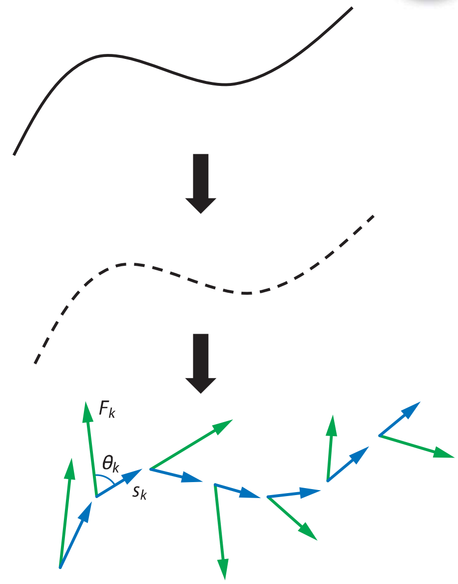


Figure 2.58 The curved path followed by a particle is shown as a dashed line, and then as larger segments, s_k. The green arrows show the varying size and direction of the force acting on the particle as it moves.

Do not be too worried about this formula. You will not be asked to use it, but it can help you to understand one very special and important case: the work done in circular motion. We will learn in Topic **6** that in circular motion there must be a force directed towards the centre of the circle. This is called the **centripetal force**.

Figure **2.59** shows the forces pointing towards the centre of the circular path. When we break the circular path into straight segments the angle between the force and the segment is always a right angle. This means that work done along each segment is zero because $\cos 90° = 0$. So for circular motion the total work done by the centripetal force is zero.

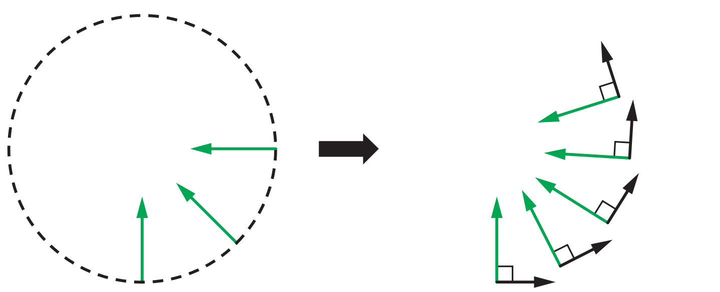


Figure 2.59 The work done by the centripetal force is zero.

In practice, when the force varies in magnitude but is constant in direction, we will be given a graph of how the force varies with distance travelled. The work done can be found from the area under the graph. For the motion shown in Figure **2.60**, the work done in moving a distance of 4.0 m is given by the area of the shaded trapezoid:

$$W = \frac{2.0 + 10}{2} \times 4.0 = 24\,\text{J}$$

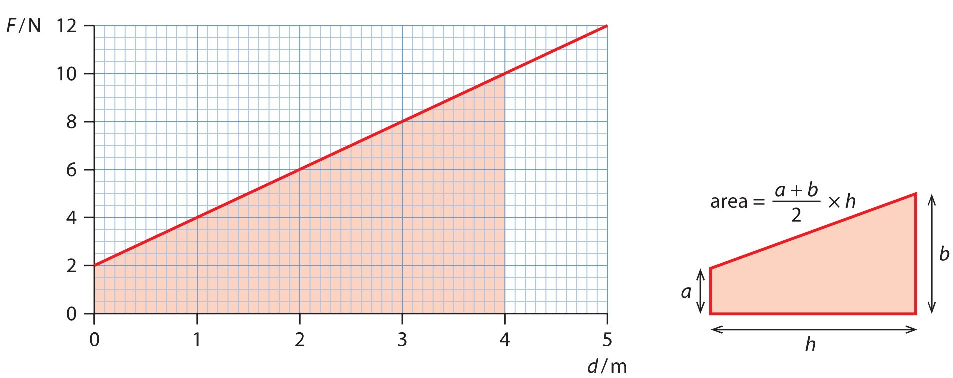


Figure 2.60 The work done is the area under the graph. The area of a trapezoid is half the sum of the parallel sides multiplied by the perpendicular distance between them.

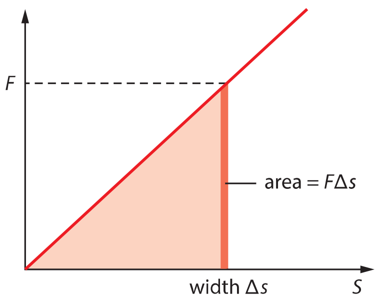


Figure 2.61 The area under the graph is the sum of all the rectangles $F\Delta s$.

The work done by a force is the area under the graph that shows the variation of the magnitude of the force with distance travelled.

How do we know that the area is the work done? For a varying force, consider a very small distance Δs (Figure **2.61**). Because Δs is so small we may assume that the force does not vary during this distance. The work done is then $F\Delta s$ and is the area of the rectangle shown. For the total work we have to add the area of many rectangles under the curve. The sum is the area under the curve.

Work done by a force on a particle

Imagine a net force F that acts on a particle of mass m. The force produces an acceleration a given by:

$$a = \frac{F}{m}$$

Let the initial speed of the particle be u. Because we have acceleration, the speed will change. Let the speed be v after travelling a distance s. We know from kinematics that:

$$v^2 = u^2 + 2as$$

Substituting for the acceleration, this becomes:

$$v^2 = u^2 + 2\frac{F}{m}s$$

We can rewrite this as:

$$Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$

We interpret this as follows: Fs is the work done on the particle by the net force. The quantity $\frac{1}{2} \times \text{mass} \times \text{speed}^2$ is the energy the particle has due to its motion, called kinetic energy. For speed v, **kinetic energy** E_K is defined as:

$$E_K = \tfrac{1}{2}mv^2$$

In our example, the initial kinetic energy of the particle is $\frac{1}{2}mu^2$ and the kinetic energy after travelling distance s is $\frac{1}{2}mv^2$. The result says that the work done has gone into the change in the kinetic energy of the particle.

We can write this as:

$$W_{net} = \Delta E_K$$

where W_{net} is the net work done and ΔE_K is the change in kinetic energy. This is known as the **work–kinetic energy relation**.

We can think of the work done as energy transferred. In this example, the work done has transferred energy to the particle by increasing its kinetic energy.

Worked example

2.30 A block of mass 2.5 kg slides on a rough horizontal surface. The initial speed of the block is $8.6\,\mathrm{m\,s^{-1}}$. It is brought to rest after travelling a distance of 16 m. Determine the magnitude of the frictional force.

We will use the work–kinetic energy relation, $W_{net} = \Delta E_K$.

The only force doing work is the frictional force, f, which acts in the opposite direction to the motion.

$$W_{net} = f \times 16 \times (-1)$$

The angle between the force and the direction of motion is 180°, so we need to multiply by cos 180°, which is −1.

The change in kinetic energy is:

$$\Delta E_K = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = -92.45\,\mathrm{J}$$

So: $-16f = -92.45$

$$f = 5.8\,\mathrm{N}$$

The magnitude of the frictional force is 5.8 N.

Work done in stretching a spring

Consider a horizontal spring whose left end is attached to a vertical wall. If we apply a force F to the other end we will stretch the spring by some amount, x. Experiments show that the force F and the extension x are directly proportional to each other, i.e. $F = kx$ (this is known as **Hooke's law**). How much work does the stretching force F do in stretching the spring from its natural length (i.e. from zero extension) to a length where the extension is x_1, as shown in Figure **2.62**.

Since the force F and the extension x are directly proportional, the graph of force versus extension is a straight line through the origin and work done is the area under the curve (Figure **2.63**).

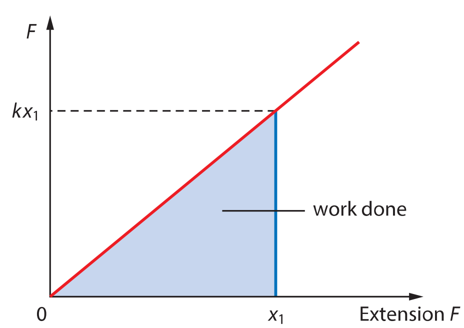


Figure 2.63 The force F stretches the spring. Notice that as the extension increases the force increases as well.

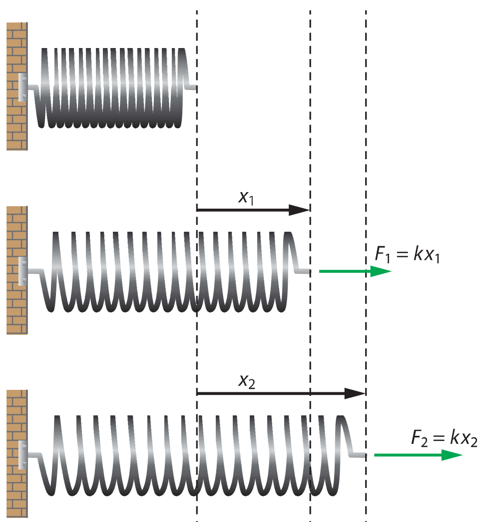


Figure 2.62 Stretching a spring requires work to be done.

To find the work done in extending the spring from its natural length ($x = 0$) to extension x_1, we need to calculate the area of the triangle of base x_1 and height kx_1. Thus:

$$\text{area} = \tfrac{1}{2}kx_1 \times x_1$$

$$\text{area} = \tfrac{1}{2}kx_1^{\,2}$$

The work to extend a spring from its natural length by an amount x_1 is thus:

$$W = \tfrac{1}{2}kx_1^{\,2}$$

It follows that the work done when extending a spring from an extension x_1 to an extension x_2 (so $x_2 > x_1$) is:

$$W = \tfrac{1}{2}k(x_2^{\,2} - x_1^{\,2})$$

The work done by the force extending the spring goes into elastic potential energy stored in the spring. The elastic potential energy of a spring whose extension is x is $E_{el} = \tfrac{1}{2}kx^2$.

Exam tip
In discussing work done it is always important to keep a clear picture of the force whose work we are calculating.

Worked example

2.31 A mass of 8.4 kg rests on top of a vertical spring whose base is attached to the floor. The spring compresses by 5.2 cm.

a Calculate the spring constant of the spring.

b Determine the energy stored in the spring.

a The mass is at equilibrium so $mg = kx$. So:

$$k = \frac{mg}{x}$$

$$k = \frac{8.4 \times 9.8}{5.2 \times 10^{-2}}$$

$$k = 1583 \approx 1600\ \text{N m}^{-1}$$

b The stored energy E_{el} is:

$$E_{el} = \tfrac{1}{2}kx^2$$

$$E_{el} = \tfrac{1}{2} \times 1583 \times (5.2 \times 10^{-2})^2$$

$$E_{el} = 2.1\ \text{J}$$

Work done by gravity

We will now concentrate on the work done by a very special force, namely the weight of a body. Remember that weight is mass times acceleration of free fall and is directed vertically down. Thus, if a body is displaced horizontally, the work done by mg is zero. In this case the angle between the force and the direction of motion is 90° (Figure **2.64**), so:

$$W = mgs\cos 90° = 0$$

Exam tip
When a body is displaced such that its final position is at the same vertical height as the original position, the work done by the weight is zero.

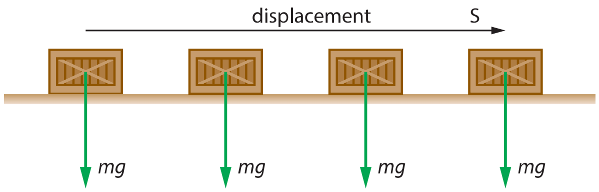


Figure 2.64 The force of gravity is normal to this horizontal displacement, so no work is being done.

We are not implying that it is the weight that is forcing the body to move along the table. We are calculating the work done by a particular force (the weight) if the body (somehow) moves in a particular way.

If the body falls a vertical distance h, then the work done by the weight is $+mgh$. The force of gravity is parallel to the displacement, as in Figure **2.65a**.

If the body moves vertically upwards to a height h from the launch point, then the work done by the weight is $-mgh$ since now the angle between direction of force (vertically down) and displacement (vertically up) is 180°. The force of gravity is parallel to the displacement but opposite in direction, as in Figure **2.65b**.

Suppose now that instead of just letting the body fall or throwing it upwards, we use a rope to either lower it or raise it, at constant speed, by a height h (Figure **2.66**). The work done by the weight is the same as before, so nothing changes. But we now ask about the work done by the force F that lowers or raises the body. Since F is equal and opposite to the weight, the work done by F is $-mgh$ as the body is lowered and $+mgh$ as it is being raised.

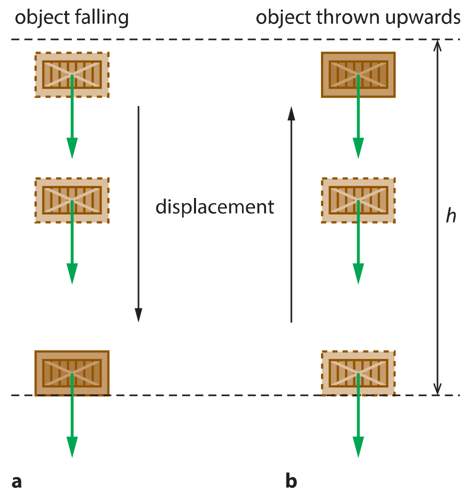


Figure 2.65 The force of gravity (green arrows) is parallel to the displacement in **a** and opposite in **b**.

You should be able to see how this is similar to the work done by the stretching and tension forces in a spring.

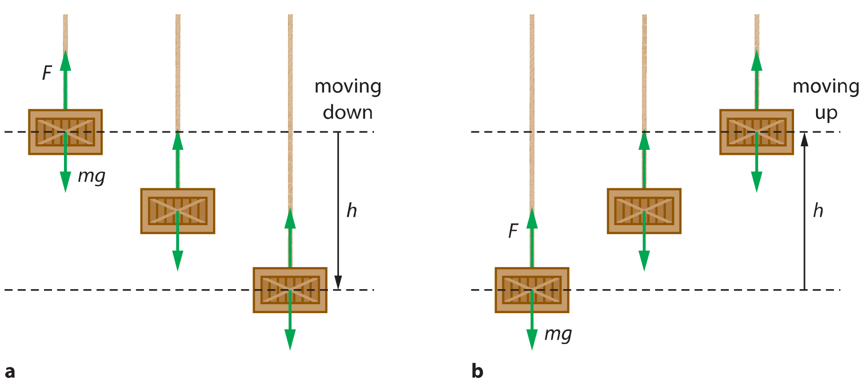


Figure 2.66 Lowering and raising an object at constant speed using a rope.

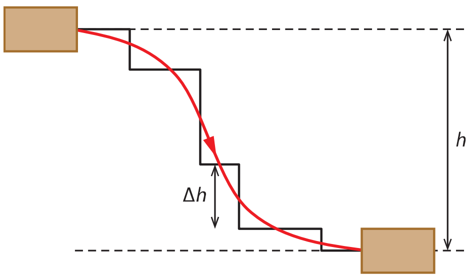


Figure 2.67 The work done by gravity is independent of the path followed.

Consider now the case where a body moves along some arbitrary path, as shown by the red line in Figure **2.67**. The work done by the weight of the body as the body descends along the curve is still mgh. You can prove this amazing result easily by approximating the curved path with a 'staircase' of vertical and horizontal steps. Along the horizontal steps the work done is zero, $\cos 90° = 0$. Along the vertical steps the work is $mg\Delta h$, where Δh is the step height. Adding up all the vertical steps gives mgh. This means that:

The work done by gravity is independent of the path followed and depends only on the vertical distance separating the initial and final positions.

The independence of the work done on the path followed is a property of a class of forces (of which weight is a prominent member) called **conservative forces**.

Mechanical energy

In the previous two sections we discussed the work done when a body is moved when attached to a spring and in a gravitational field. We derived two main results.

In the case of the spring, we showed that the work done by the stretching force in extending the spring a distance x away from the natural length of the spring is $W = \frac{1}{2}x^2$.

In the case of motion within a gravitational field the work done by the force moving the body, is $W = mgh$ to raise the body a height h from its initial position.

We use these results to define two different kinds of **potential energy**, E_P.

For the mass–spring system we define the **elastic potential energy** to be the work done by the pulling force in stretching the spring by an amount x, that is:

$$E_P = \frac{1}{2}kx^2$$

For the Earth–mass system we define the **gravitational potential energy** to be the work done by the moving force in placing a body a height h above its initial position, that is:

$$E_P = mgh$$

Notice that potential energy is the property of a system, not of an individual particle.

So we are now in a position to go back to the first part of Subtopic **2.3** and answer some of the questions posed there. We said that:

$$\Delta E = W + Q$$

Exam tip
Potential energy is the energy of a system due to its position or shape and represents the work done by an external agent in bringing the system to that position or shape.

Exam tip
Notice that in the data booklet the formula uses Δx in place of our x.

If the system is in contact with surroundings at a different temperature there will be a transfer of heat, Q. If there is no contact and no temperature difference, then $Q = 0$.

If no work is done on the system from outside, then $W = 0$. When $Q + W = 0$, the system is called **isolated** and in that case $\Delta E = 0$. The total energy of the system does not change. We have **conservation of the total energy** of the system.

What does the total energy E consist of? It includes chemical energy, **internal energy** (due to the translational, rotational energy and vibrational energy of the molecules of the substance), nuclear energy, kinetic energy, elastic potential energy, gravitational potential energy and any other form of potential energy such as electrical potential energy.

But in this section, dealing with mechanics, the total energy E will be just the sum of the kinetic, the elastic and the gravitational potential energies.

So for a single particle of mass m, the energy is:

$$E = \frac{1}{2}mv^2 + mgh + \frac{1}{2}kx^2$$

This is also called the **total mechanical energy** of the system consisting of the particle, the spring and the Earth. W stands for work done by forces outside the system. So this does not include work due to spring tension forces or the weight since the work of these forces is already included as potential energy in E.

Exam tip

You must make sure that you do not confuse the work–kinetic energy relation $W_{net} = \Delta E_K$ with $\Delta E = W$. The work–kinetic energy relation relates the net work on a system to the change in the system's kinetic energy. The other relates the work done by outside forces to the change of the total energy.

Worked examples

2.32 You hold a ball of mass 0.25 kg in your hand and throw it so that it leaves your hand with a speed of $12\,\text{m s}^{-1}$. Calculate the work done by your hand on the ball.

The question asks for work done but here we do not know the forces that acted on the ball nor the distance by which we moved it before releasing it. But using $\Delta E = W$, we find:

$$W = \frac{1}{2}mv^2$$

$$W = \frac{1}{2} \times 0.25 \times 12^2 = 36\,\text{J}$$

Notice that here we have no springs and we may take $h = 0$.

2.33 Suppose that in the previous example your hand moved a distance of 0.90 m in throwing the ball. Estimate the average net force that acted on the ball.

The work done was 36 J and so $Fs = 36\,\text{J}$ with $s = 0.90\,\text{m}$. This gives $F = 40\,\text{N}$.

2.34 A body of mass 4.2 kg with initial speed $5.6\,\mathrm{m\,s^{-1}}$ begins to move up an incline, as shown in Figure **2.68**.

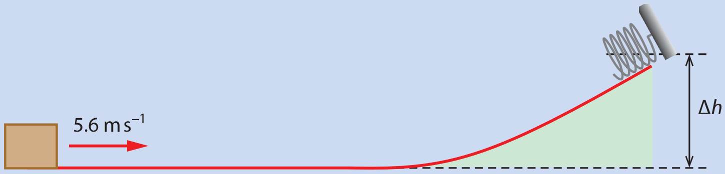


Figure 2.68

The body will be momentarily brought to rest after colliding with a spring of spring constant $220\,\mathrm{N\,m^{-1}}$. The body stops a vertical distance 0.85 m above its initial position. Determine the amount by which the spring has been compressed. There are no frictional forces.

There are no external forces doing work and so $W=0$. The system is isolated and we have conservation of total energy.

Initially we have just kinetic energy, so:

$$E_{\text{initial}} = \tfrac{1}{2}mv^2 + mgh + \tfrac{1}{2}kx^2 = \tfrac{1}{2}\times 4.2\times 5.6^2 + 0 + 0 = 65.856\,\mathrm{J}$$

When the body stops we have:

$$E_{\text{initial}} = \tfrac{1}{2}mv^2 + mgh + \tfrac{1}{2}kx^2 = 0 + 4.2\times 9.8\times 0.85 + \tfrac{1}{2}\times 220\times x^2 = 34.99 + 110x^2$$

Thus, equating E_{initial} to E_{final} we find:

$$34.99 + 110x^2 = 65.856$$
$$110x^2 = 30.866$$
$$x^2 = 0.2806$$
$$x = 0.53\,\mathrm{m}$$

2.35 We repeat the previous example question but now there is constant frictional force opposing the motion along the uphill part of the path. The length of this path is 1.2 m and the frictional force is 15 N.

We have $\Delta E = W$. The work done is:

$$Fs\cos\theta = 15\times 1.2\times(-1) = -18\,\mathrm{J}$$

As in the previous example, we have:

$$E_{\text{initial}} = 65.856\,\mathrm{J}$$
$$E_{\text{final}} = 34.99 + 110x^2$$

leading to:

$$110x^2 = 12.866$$
$$x^2 = \frac{12.866}{110}$$
$$x = 0.34\,\mathrm{m}$$

The 'work done by friction' of −18 J is energy that is dissipated as thermal energy inside the body *and* its surroundings. It is in general very difficult to estimate how much of this thermal energy stays within the body and how much goes into the surroundings.

2.36 A mass of 5.00 kg moving with an initial velocity of $2.0\,\mathrm{m\,s^{-1}}$ is acted upon by a force 55 N in the direction of the velocity. The motion is opposed by a frictional force. After travelling a distance of 12 m the velocity of the body becomes $15\,\mathrm{m\,s^{-1}}$. Determine the magnitude of the frictional force.

Here $Q=0$ so that $\Delta E=W$.

The change in total energy ΔE is the change in kinetic energy (we have no springs and no change of height):

$$\Delta E=\tfrac{1}{2}\times 5.00\times 15^2-\tfrac{1}{2}\times 5.00\times 2.0^2=552.5\,\mathrm{J}$$

Let the frictional force be f. The work done on the mass is $(55-f)\times 12$, and so:

$$(55-f)\times 12=552.5$$

$$55-f=\frac{552.5}{12}$$

$$55-f=46.0$$

$$f=9.0\,\mathrm{N}$$

The 'work done by friction' of $-9.0\times 12=-108\,\mathrm{J}$ is energy that is dissipated as thermal energy inside the body *and* its surroundings.

2.37 A mass m hangs from two strings attached to the ceiling such that they make the same angle with the vertical (as shown in Figure **2.69**). The strings are shortened very slowly so that the mass is raised a distance Δh above its original position. Determine the work done by the tension in each string as the mass is raised.

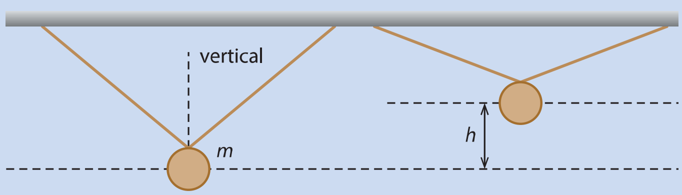


Figure 2.69

The net work done is zero since the net force on the mass is zero. The work done by gravity is $-mg\Delta h$ and thus the work done by the two equal tension forces is $+mg\Delta h$. The work done by each is thus $\frac{mg\Delta h}{2}$.

2.38 A pendulum of length 1.0 m is released from rest with the string at an angle of 10° to the vertical. Find the speed of the mass on the end of the pendulum when it passes through its lowest position.

Let us take as the reference level the lowest point of the pendulum (Figure **2.70**). The total energy at that point is just kinetic, $E_K = \frac{1}{2}mv^2$, where v is the unknown speed.

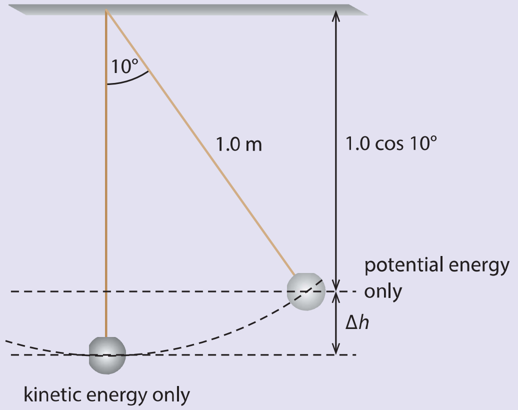


Figure 2.70

At the initial point, the total energy is just potential, $E_P = mg\Delta h$, where Δh is the vertical difference in height between the two positions. From the diagram:

$$\Delta h = 1.00 - 1.00 \cos 10°$$

$$\Delta h = 0.015\,\text{m}$$

Equating the expressions for the total energy at the lowest point and at the start:

$$\tfrac{1}{2}mv^2 = mg\Delta h$$

$$v = \sqrt{2g\Delta h}$$

$$v = 0.55\,\text{m s}^{-1}$$

Note how the mass has dropped out of the problem. (At positions other than the two shown, the mass has both kinetic and potential energy.)

2.39 Determine the minimum speed of the mass in Figure **2.71** at the initial point such that the mass makes it over the barrier of height h.

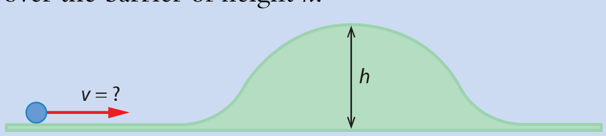


Figure 2.71

To make it over the barrier the mass must be able to reach the highest point. Any speed it has at the top will mean it can carry on to the other side. Therefore, at the very least, we must be able to get the ball to the highest point with zero speed.

With zero speed at the top, the total energy at the top of the barrier is $E = mgh$.

The total energy at the starting position is $\frac{1}{2}mv^2$.

Equating the initial and final energy:

$$\frac{1}{2}mv^2 = mgh$$

$$\Rightarrow \quad v = \sqrt{2gh}$$

Thus, the initial speed must be bigger than $v = \sqrt{2gh}$.

Note that if the initial speed u of the mass is larger than $v = \sqrt{2gh}$, then when the mass makes it to the original level on the other side of the barrier, its speed will be the same as the starting speed u.

2.40 A ball rolls off a 1.0 m high table with a speed of $4.0\,\text{m}\,\text{s}^{-1}$, as shown in Figure **2.72**. Calculate the speed as the ball strikes the floor.

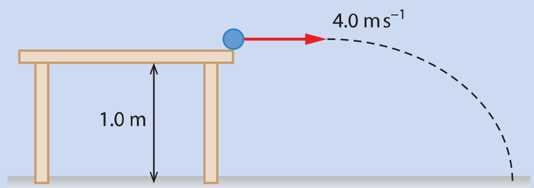


Figure 2.72

The total energy of the mass is conserved. As it leaves the table with speed u it has total energy given by $E_{\text{initial}} = \frac{1}{2}mu^2 + mgh$ and as it lands with speed v the total energy is $E_{\text{final}} = \frac{1}{2}mv^2$ (v is the speed we are looking for).

Equating the two energies gives:

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh$$

$$\Rightarrow \quad v^2 = u^2 + 2gh$$

$$v^2 = 16 + 20 = 36$$

$$\Rightarrow \quad v = 6.0\,\text{m}\,\text{s}^{-1}$$

2.41 Two identical balls are launched from a table with the same speed u (Figure **2.73**). One ball is thrown vertically up and the other vertically down. The height of the table from the floor is h. Predict which of the two balls will hit the floor with the greater speed.

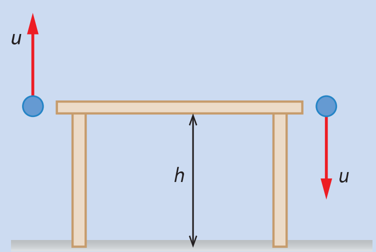


Figure 2.73

At launch both balls have the same kinetic energy and the same potential energy. When they hit the floor their energy will be only kinetic. Hence the speeds will be identical and equal to v, where:

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh$$

$$\Rightarrow \quad v^2 = u^2 + 2gh$$

$$\Rightarrow \quad v = \sqrt{u^2 + 2gh}$$

2.42 A body of mass 2.0 kg (initially at rest) slides down a curved path of total length 22 m, as shown in Figure **2.74**. The body starts from a vertical height of 5.0 m from the bottom. When it reaches the bottom, its speed is measured and found to equal $6.0\,\mathrm{m\,s^{-1}}$.

a Show that there is a force resisting the motion.

b Assuming the force to have constant magnitude, determine the magnitude of the force.

Figure 2.74

a The only external force that could do work is a frictional force.

At the top: $E_{\text{initial}} = \frac{1}{2}mv^2 + mgh = 0 + 2.0 \times 9.8 \times 5.0 = 98\,\mathrm{J}$

At the bottom: $E_{\text{final}} = \frac{1}{2}mv^2 + mgh = \frac{1}{2} \times 2.0 \times 6.0^2 + 0 = 36\,\mathrm{J}$

The total energy has reduced, which shows the presence of a frictional force resisting the motion.

b From $\Delta E = W$ we deduce that $W = -62\,\mathrm{J}$. This is the work done by the frictional force, magnitude f.

The force acts in the opposite direction to the motion, so:

$$fs \times (-1) = -62\,\mathrm{J}$$

$$\Rightarrow f = \frac{62}{22}$$

$$f = 2.8\,\mathrm{N}$$

Power

When a machine performs work, it is important to know not only how much work is being done but also how much work is performed within a given time interval. A cyclist will perform a lot of work in a lifetime of cycling, but the same work can be performed by a powerful car engine in a much shorter time. **Power** is the rate at which work is being performed or the rate at which energy is being transferred.

When a quantity of work ΔW is performed within a time interval Δt the power developed is given by the ratio:

$$P = \frac{\Delta W}{\Delta t}$$

is called the power developed. Its unit is joule per second and this is given the name watt (W): $1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}}$.

Consider a constant force F, which acts on a body of mass m. The force does an amount of work $F\Delta x$ in moving the body a small distance Δx along its direction. If this work is performed in time Δt, then:

$$P = \frac{\Delta W}{\Delta t}$$

$$P = F\frac{\Delta x}{\Delta t}$$

$$P = Fv$$

where v is the instantaneous speed of the body. This is the power produced in making the body move at speed v. As the speed increases, the power necessarily increases as well.

Consider an aeroplane moving at constant speed on a straight-line path. If the power produced by its engines is P, and the force pushing it forward is F, then P, F and v are related by the equation above. But since the plane moves with no acceleration, the total force of air resistance must equal F. Hence the force of air resistance can be found simply from the power of the plane's engines and the constant speed with which it coasts.

Worked example

2.43 Estimate the minimum power required to lift a mass of 50.0 kg up a vertical distance of 12 m in 5.0 s.

The work done in lifting the mass is mgh:

$$W = mgh = 50.0 \times 10 \times 12$$

$$W = 6.0 \times 10^3\,\text{J}$$

The power is therefore:

$$P = \frac{W}{\Delta t}$$

$$P = \frac{6.0 \times 10^3}{5.0} = 1200\,\text{W}$$

This is the minimum power required. In practice, the mass has to be accelerated from rest, which will require additional work and hence more power. There will also be frictional forces to overcome.

Efficiency

If a machine, such as an electric motor, is used to raise a load, electrical energy must be provided to the motor. This is the input energy to the motor. The motor uses some of this energy to do the useful work of raising the load. But some of the input energy is used to overcome frictional forces and therefore gets converted to thermal energy. So the ratio:

$$\frac{\text{useful energy out}}{\text{actual energy in}} \quad \text{or} \quad \frac{\text{useful power out}}{\text{actual power in}}$$

is less than one. We call this ratio the **efficiency** of the machine.

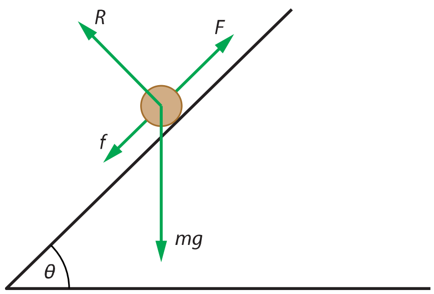


Figure 2.75 Forces on a body on an inclined plane: pulling force *F*, frictional force *f*, reaction *R* and weight *mg*.

Suppose that a body is being pulled up along a rough inclined plane with constant speed. The mass is 15 kg and the angle of the incline is 45°. There is a constant frictional force of 42 N opposing the motion.

The forces on the body are shown in Figure **2.75**. Since the body has no acceleration, we know that:

$$R = mg\cos\theta = 106.1\,\text{N}$$

$$F = mg\sin\theta + f = 106.1 + 42 = 148.1\,\text{N} \approx 150\,\text{N}$$

Let the force raise the mass a distance of 20 m along the plane. The work done by the force F is:

$$W = 148.1 \times 20$$

$$W = 2960\,\text{J} \approx 3.0 \times 10^3\,\text{J}$$

The force effectively raised the 15 kg a vertical height of 14.1 m and so increased the potential energy of the mass by $mgh = 2121$ J. The efficiency with which the force raised the mass is thus:

$$\text{efficiency} = \frac{2121}{2960}$$

$$\text{efficiency} = 0.72$$

Worked example

2.44 A 0.50 kg battery-operated toy train moves with constant velocity 0.30 m s^{-1} along a level track. The power of the motor in the train is 2.0 W and the total force opposing the motion of the train is 5.0 N.

a Determine the efficiency of the train's motor.

b Assuming the efficiency and the opposing force stay the same, calculate the speed of the train as it climbs an incline of 10.0° to the horizontal.

a The power delivered by the motor is 2.0 W. Since the speed is constant, the force developed by the motor is also 5.0 N.

The power used in moving the train is $Fv = 5.0 \times 0.30 = 1.5$ W.

Hence the efficiency is:

$$\frac{\text{total power out}}{\text{total power in}} = \frac{1.5\,\text{W}}{2.0\,\text{W}}$$

$$\frac{\text{total power out}}{\text{total power in}} = 0.75$$

The efficiency of the train's motor is 0.75 (or 75%).

b The component of the train's weight acting down the plane is $mg\sin\theta$ and the force opposing motion is 5.0 N. Since there is no acceleration (constant velocity), the net force F pushing the train up the incline is:

$$F = mg\sin\theta + 5.0$$

$$F = 0.50 \times 10 \times \sin 10° + 5.0$$

$$F = 5.89\,\text{N} \approx 5.9\,\text{N}$$

Thus:

$$\text{efficiency} = \frac{5.89 \times v}{2.0}$$

But from part **a** the efficiency is 0.75, so:

$$0.75 = \frac{5.89 \times v}{2.0}$$

$$\Rightarrow \quad v = \frac{2.0 \times 0.75}{5.89}$$

$$v = 0.26\,\text{m}\,\text{s}^{-1}$$

Nature of science

The origin of conservation principles

Understanding of what energy is has evolved over time, with Einstein showing that there is a direct relationship between mass and energy in his famous equation $E = mc^2$. In this section we have seen how the principle of conservation of energy can be applied to different situations to predict and explain what will happen. Scientists have been able to use the theory to predict the outcome of previously unknown interactions in particle physics.

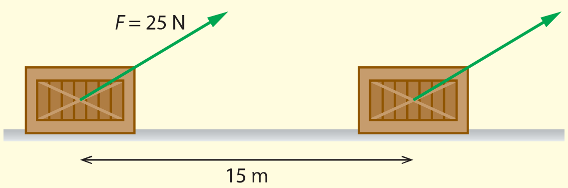
The principle of conservation of energy is perhaps the best known example of a conservation principle. But where does it come from? It turns out that all conservation principles are consequences of symmetry. In the case of energy, the symmetry is that of 'time translation invariance'. This means that when describing motion (or anything else) it does not matter when you started the stopwatch. So a block of mass 1 kg on a table 1 m above the floor will have a potential energy of 10 J according to both an observer who starts his stopwatch 'now' and another who started it 10 seconds ago. The principle of conservation of momentum, which is discussed in Subtopic **2.4**, is also the result of a symmetry. The symmetry this time is 'space translation invariance', which means that in measuring the position of events it does not matter where you place the origin of your ruler.

Test yourself

55 A horizontal force of 24 N pulls a body a distance of 5.0 m along its direction. Calculate the work done by the force.

56 A block slides along a rough table and is brought to rest after travelling a distance of 2.4 m. A force of 3.2 N opposes the motion. Calculate the work done by the opposing force.

57 A block is pulled as shown in the diagram by a force making an angle of 20° to the horizontal. Find the work done by the pulling force when its point of application has moved 15 m.



58 A block of mass 2.0 kg and an initial speed of 5.4 m s^{-1} slides on a rough horizontal surface and is eventually brought to rest after travelling a distance of 4.0 m. Calculate the frictional force between the block and the surface.

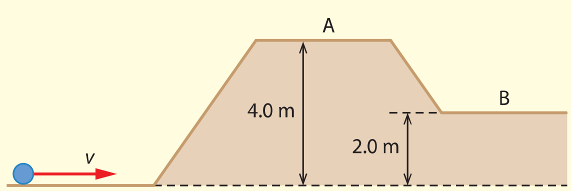
59 A spring of spring constant $k = 200$ N m^{-1} is slowly extended from an extension of 3.0 cm to an extension of 5.0 cm. Calculate the work done by the extending force.

60 Look at the diagram.

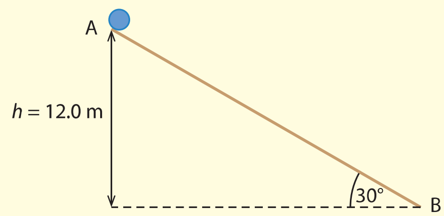
a **i** Calculate the minimum speed v the ball must have in order to make it to position **B**.

ii What speed will the mass have at **B**?

b Given that $v = 12.0$ m s^{-1}, calculate the speed at **A** and **B**.

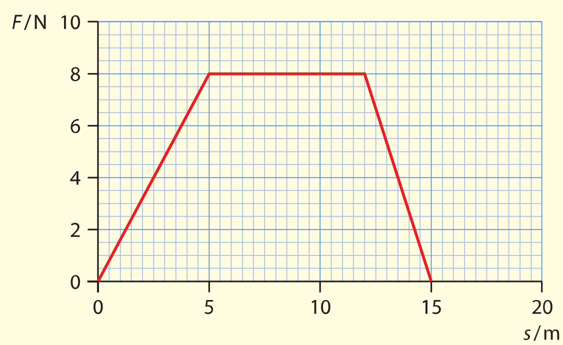


61 The speed of the 8.0 kg mass in position **A** in the diagram is 6.0 m s^{-1}. By the time it gets to **B** the speed is measured to be 12.0 m s^{-1}.



Estimate the frictional force opposing the motion. (The frictional force is acting along the plane.)

62 A force F acts on a body of mass $m = 2.0$ kg initially at rest. The graph shows how the force varies with distance travelled (along a straight line).



a Find the work done by this force.

b Calculate the final speed of the body.

63 A body of mass 12 kg is dropped vertically from rest from a height of 80 m.

a Ignoring any resistance forces during the motion of this body, draw graphs to represent the variation with distance fallen of:

i the potential energy

ii the kinetic energy.

b For the same motion draw graphs to represent the variation with time of:

i the potential energy

ii the kinetic energy.

c Describe qualitatively the effect of a constant resistance force on each of the four graphs you drew.

64 The engine of a car is developing a power of 90 kW when it is moving on a horizontal road at a constant speed of 100 km h^{-1}. Estimate the total horizontal force opposing the motion of the car.

65 The motor of an elevator develops power at a rate of 2500 W.

a Calculate the speed that a 1200 kg load is being raised at.

b In practice it is found that the load is lifted more slowly than indicated by your answer to **a**. Suggest reasons why this is so.

66 A load of 50 kg is raised a vertical distance of 15 m in 125 s by a motor.

a Estimate the power necessary for this.

b The power supplied by the motor is in fact 80 W. Calculate the efficiency of the motor.

c The same motor is now used to raise a load of 100 kg the same distance. The efficiency remains the same. Estimate how long this would take.

67 The top speed of a car whose engine is delivering 250 kW of power is 240 $\mathrm{km\,h^{-1}}$. Calculate the value of the resistance force on the car when it is travelling at its top speed on a level road.

68 An elevator starts on the ground floor and stops on the 10th floor of a high-rise building. The elevator reaches a constant speed by the time it reaches the 1st floor and decelerates to rest between the 9th and 10th floors. Describe the energy transformations taking place between the 1st and 9th floors.

69 A mass m of 4.0 kg slides down a frictionless incline of $\theta = 30°$ to the horizontal. The mass starts from rest from a height of 20 m.

a Sketch a graph of the kinetic and potential energies of the mass as a function of time.

b Sketch a graph of the kinetic and potential energies of the mass as a function of distance travelled along the incline.

c On each graph, sketch the sum of the potential and kinetic energies.

70 A mass m is being pulled up an inclined plane of angle θ by a rope along the plane.

a Find is the tension in the rope if the mass moves up at constant speed v.

b Calculate is the work done by the tension when the mass moves up a distance of d m along the plane.

c Find is the work done by the weight of the mass.

d Find is the work done by the normal reaction force on the mass.

e What is the net work done on the mass?

71 A battery toy car of mass 0.250 kg is made to move up an inclined plane that makes an angle of 30° with the horizontal. The car starts from rest and its motor provides a constant acceleration of 4.0 $\mathrm{m\,s^{-2}}$ for 5.0 s. The motor is then turned off.

a Find the distance travelled in the first 5 s.

b Find the furthest the car gets on the inclined plane.

c Calculate when the car returns to its starting position.

d Sketch a graph of the velocity as a function of time.

e On the same axes, sketch a graph of the kinetic energy and potential energy of the car as a function of the distance travelled.

f State the periods in the car's motion in which its mechanical energy is conserved.

g Estimate the average power developed by the car's motor.

h Determine the maximum power developed by the motor.

Learning objectives

- Be able to re-formulate Newton's second law when the mass is variable.
- Understand the concept of impulse and be able to analyse force–time graphs.
- Be able to derive and apply the law of conservation of momentum.
- Analyse elastic and inelastic collisions and explosions.

2.4 Momentum and impulse

This section introduces the concept of linear momentum, which is a very useful and powerful concept in physics. Newton's second law is expressed in terms of momentum. The law of conservation of linear momentum makes it possible to predict the outcomes in very many physical situations.

Newton's second law in terms of momentum

We saw earlier that Newton's second law was expressed as $F_{net} = ma$. In fact, this equation is only valid when the mass of the system remains constant. But there are plenty of situations where the mass does *not* remain constant. In cases where the mass changes, a different version of the second law must be used. Examples include:

- the motion of a rocket, where the mass decreases due to burnt fuel ejected away from the rocket
- sand falling on a conveyor belt so the mass increases
- a droplet of water falling through mist and increasing in mass as more water condenses.

We define a new concept, **linear momentum**, p, to be the product of the mass of a body times its velocity:

$$p = mv$$

Momentum is a vector and has the direction of the velocity. Its unit is kg m s^{-1} or the equivalent N s.

In terms of momentum, Newton's second law is:

$$F_{net} = \frac{\Delta p}{\Delta t}$$

The average net force on a system is equal to the rate of change of the momentum of the system.

It is easy to see that if the mass stays constant, then this version reduces to the usual ma:

$$F_{net} = \frac{\Delta p}{\Delta t} = \frac{p_{final} - p_{initial}}{\Delta t}$$

$$= \frac{mv_{final} - mv_{initial}}{\Delta t}$$

$$= m\left(\frac{v_{final} - v_{initial}}{\Delta t}\right)$$

$$= \frac{m\Delta v}{\Delta t}$$

$$F_{net} = ma$$

Worked examples

2.45 A cart moves in a horizontal line with constant speed v. Rain starts to fall and the cart fills with water at a rate of $\sigma\,\mathrm{kg\,s^{-1}}$. (This means that in one second, σ kg have fallen on the cart.) The cart must keep moving at constant speed. Determine the force that must be applied on the cart.

Exam tip
Worked example **2.45** should alert you right away that you must be careful when mass changes. Zero acceleration does not imply zero net force in this case.

Notice right away that if $F_{net} = ma$ (we drop the bold italic of the vector notation) were valid, the force would have to be zero since the car is not accelerating. But we do need a force to act on the cart because the momentum of the cart is increasing (because the mass is increasing). This force is:

$$F_{net} = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{v\Delta m}{\Delta t} = v\sigma$$

Putting some real values in, if $\sigma = 0.20\,\mathrm{kg\,s^{-1}}$ and $v = 3.5\,\mathrm{m\,s^{-1}}$, the force would have to be 0.70 N.

2.46 Gravel falls vertically on a conveyor belt at a rate of $\sigma\,\mathrm{kg\,s^{-1}}$, as shown in Figure **2.76**.

This very popular exam question is similar to Worked example **2.45**, but is worth doing again.

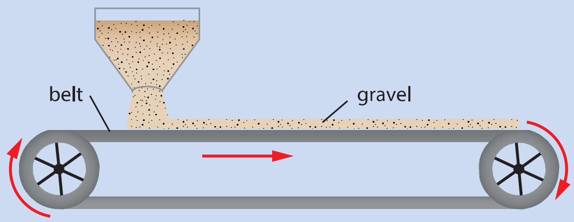


Figure 2.76

a Determine:
- **i** the force that must be applied on the belt to keep it moving at constant speed v
- **ii** the power that must be supplied by the motor turning the belt
- **iii** the rate at which the kinetic energy of the gravel is changing.

b Explain why the answers to **a ii** and **a iii** are different.

a **i** The force is:

$$F_{net} = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{v\Delta m}{\Delta t} = v\sigma$$

ii The power is found from $P = Fv$. Substituting for F:

$$P = (v\sigma)v = \sigma v^2$$

iii In 1 second the mass on the belt increases by σ kg. The kinetic energy of this mass is:

$$E_K = \tfrac{1}{2}\sigma v^2$$

This is the increase in kinetic energy in a time of 1 s, so the rate of kinetic energy increase is $\frac{1}{2}\sigma v^2$.

b The rate of increase in kinetic energy is less than the power supplied. This is because the power supplied by the motor goes to increase the kinetic energy of the gravel and also to provide the energy needed to accelerate the gravel from 0 to speed v in the short interval of time when the gravel slides on the belt before achieving the constant final speed v.

2.47 A 0.50 kg ball is dropped from rest above a hard floor. When it reaches the floor it has a velocity of $4.0\,\mathrm{m\,s^{-1}}$. The ball then bounces vertically upwards. Figure **2.77** is the graph of velocity against time for the ball. The positive direction for velocity is upwards.

a Find the magnitude of the momentum change of the ball during the bounce.

b The ball stayed in contact with the floor for 0.15 s. What average force did the floor exert on the ball?

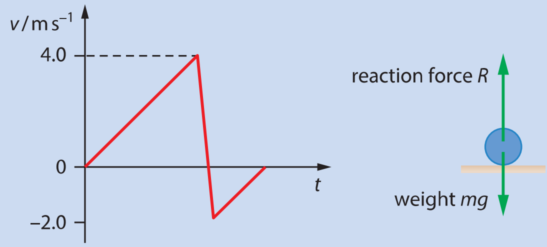


Figure 2.77

a The momentum when the ball hits the floor is: $0.50 \times 4.0 = 2.0\,\mathrm{N\,s}$

The momentum when the ball rebounds from the floor is: $0.50 \times (-2.0) = -1.0\,\mathrm{N\,s}$

The magnitude of the momentum change is therefore 3.0 N s.

b The forces on the ball are its weight and the reaction from the floor, R.

$F_{net} = R - mg$

This is also the force that produces the change in momentum:

$$F_{net} = \frac{\Delta p}{\Delta t}$$

Substituting in this equation:

$$F_{net} = \frac{3.0}{0.15} = 20\,\mathrm{N}$$

We need to find R, so:

$R = 20 + 5.0 = 25\,\mathrm{N}$.

The average force exerted on the ball by the floor is 25 N.

Exam tip

This is a very tricky problem with lots of possibilities for error. A lot of people forget to include the minus sign in the rebound velocity and also forget the weight, so they answer incorrectly that $R = 20\,\mathrm{N}$.

Impulse and force–time graphs

We may rearrange the equation:

$$F_{net} = \frac{\Delta p}{\Delta t}$$

to get:

$$\Delta p = F_{net}\Delta t$$

The quantity $F_{net}\Delta t$ is called the **impulse** of the force, and is usually denoted by J. It is the product of the average force times the time for which the force acts. The impulse is also equal to the change in momentum. Notice that impulse is a vector whose direction is the same as that of the force (or the change in momentum).

When you jump from a height of, say, 1 m, you will land on the ground with a speed of about $4.5\,\mathrm{m\,s^{-1}}$. Assuming your mass is 60 kg, your momentum just before landing will be 270 N s and will become zero after you land. From $F_{net} = \frac{\Delta p}{\Delta t}$, this can be achieved with a small force acting for a long time or large force acting for a short time. You will experience the large force if you do not bend your knees upon landing – keeping your knees stiff means that you will come to rest in a short time. This means Δt will be very small and the force large (which may damage your knees).

The three graphs of Figure **2.78** show three different force–time graphs. Figure **2.78a** shows a (non-constant) force that increases from zero, reaches a maximum value and then drops to zero again. The force acted for a time interval of about 2 ms. The impulse is the area under the curve. Without calculus we can only estimate this area by tediously counting squares: each small square has area $0.1\,\mathrm{ms} \times 0.2\,\mathrm{N} = 2 \times 10^{-5}\,\mathrm{N\,s}$. There are about 160 full squares under the curve and so the impulse is $3 \times 10^{-3}\,\mathrm{N\,s}$. (In this case it is not a bad approximation to consider the shape under the curve to be a triangle but with a base of 1.3 ms so that the area is then $\frac{1}{2} \times 1.3 \times 10^{-3} \times 4 \approx 3 \times 10^{-3}\,\mathrm{N\,s}$.)

In the second graph, the force is constant (Figure **2.78b**). The impulse of the force is $6.0 \times (8.0 - 2.0) = 36\,\mathrm{N\,s}$. Suppose this force acts on a body of mass 12 kg, initially at rest. Then the speed v of the body after the force stops acting can be found from:

$$\Delta p = 36\,\mathrm{N\,s}$$

$$mv - 0 = 36\,\mathrm{N\,s}$$

$$v = \frac{36}{12} = 3.0\,\mathrm{m\,s^{-1}}$$

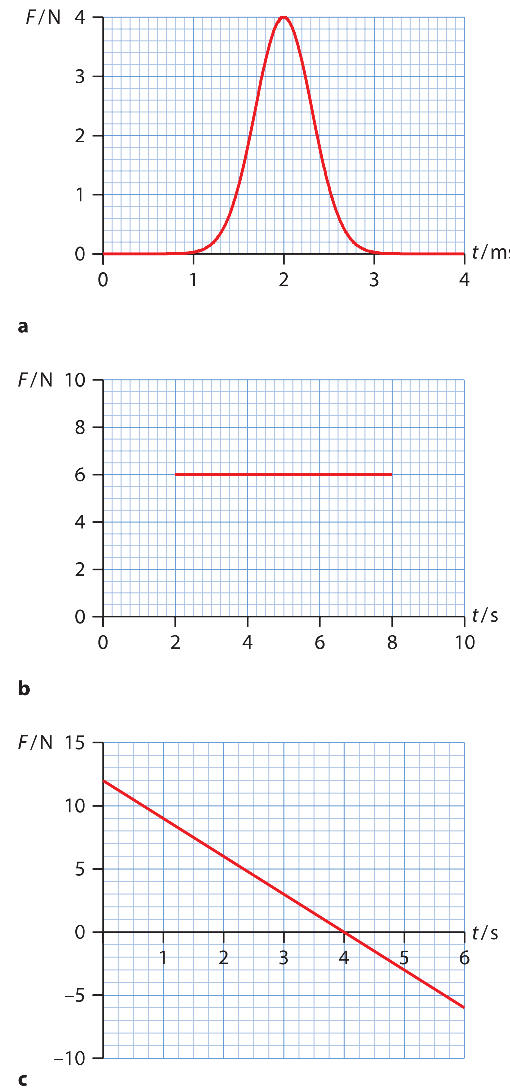


Figure 2.78 Three different force–time graphs: **a** non-constant force, **b** constant force; **c** force that varies linearly with time.

Worked examples

2.48 Consider the graph of Figure **2.78c**. The force acts on a body of mass 3.0 kg initially at rest. Calculate:

a the initial acceleration of the body

b the speed at 4.0 s

c the speed at 6.0 s.

a The initial acceleration a is at $t=0$, when $F=12$ N.

$$a=\frac{F}{m}=\frac{12}{3.0}=4.0\,\text{m s}^{-2}$$

b The impulse from 0 s to 4.0 s is the area under this part of the graph:

$\frac{1}{2}\times 4.0\times 12=24$ N s

This is equal to the change in momentum.

Let v be the speed at 4.0 s. As the body is initially at rest, the momentum change is:

$mv-0=24$

So $v=\frac{24}{m}=\frac{24}{3.0}=8.0\,\text{m s}^{-1}$

c The impulse from 0 s to 6.0 s is the area under the graph, which includes part above the axis and part below the axis. The part under the axis is negative, as the force is negative here, so the impulse is:

$\frac{1}{2}\times 4.0\times 12-\frac{1}{2}\times 2.0\times 6.0=18$ N s

Hence the speed at 6.0 s is $v=\frac{18}{3.0}=6.0\,\text{m s}^{-1}$.

2.49 A ball of mass 0.20 kg moving at $3.6\,\text{m s}^{-1}$ on a horizontal floor collides with a vertical wall. The ball rebounds with a speed of $3.2\,\text{m s}^{-1}$. The ball was in contact with the wall for 12 ms. Determine the maximum force exerted on the ball, assuming that the force depends on time according to Figure **2.79**.

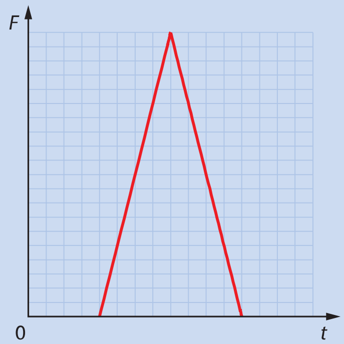


Figure 2.79

Let the initial velocity be positive. The rebound velocity is then negative.

Initial momentum: $0.20 \times 3.6 = 0.72\,\text{N s}$

Final momentum: $0.20 \times (-3.2) = -0.64\,\text{N s}$

The change in momentum of the ball is:

$$-0.64 - 0.72 = -1.36\,\text{N s}$$

The magnitude of the change in momentum is equal to the area under the force–time graph.

The area is $\frac{1}{2} \times 12 \times 10^{-3} \times F_{max}$ and so:

$$\frac{1}{2} \times 12 \times 10^{-3} \times F_{max} = 1.36\,\text{N s}$$

$$\Rightarrow F_{max} = 0.227 \times 10^3 \approx 2.3 \times 10^2\,\text{N}$$

Conservation of momentum

Consider a system with momentum p. The net force on the system is:

$$F_{net} = \frac{\Delta p}{\Delta t}$$

and so if $F_{net} = 0$ it follows that $\Delta p = 0$. There is no change in momentum. This is expressed as the law of **conservation of momentum**:

When the net force on a system is zero the momentum does not change, i.e. it stays the same. We say it is conserved.

Notice that 'system' may refer to a single body or a collection of many different bodies.

Let us consider the blue block of mass 4.0 kg moving at speed $6.0\,\text{m s}^{-1}$ to the right shown in Figure **2.80**. The blue block collides with the red block of mass 8.0 kg that is initially at rest. After the collision the two blocks move off together.

As the blocks collide, each will exert a force on the other. By Newton's third law, the magnitude of the force on each block is the same. There are no forces that come from outside the system, i.e. no external forces. You might say that the weights of the blocks are forces that come from the outside. That is correct, but the weights are cancelled by the normal reaction forces from the table. So the net external force on the system is zero. Hence we expect that the total momentum will stay the same.

The total momentum before the collision is:

$$4.0 \times 6.0 + 8.0 \times 0 = 24\,\text{N s}$$

The total momentum after the collision is:

$$(4.0 + 8.0) \times v = 12v$$

where v is the common speed of the two blocks.

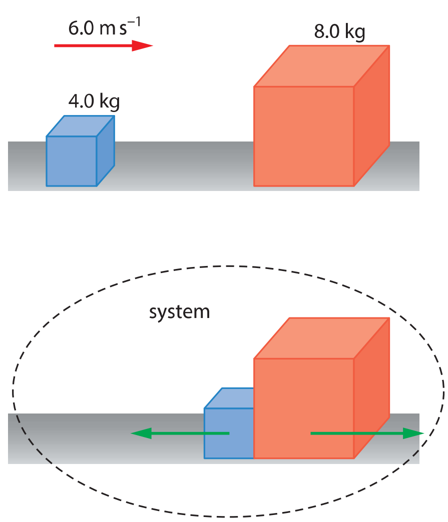


Figure 2.80 In a collision with no external forces acting, the total momentum of the system stays the same.

Equating the momentum after the collision and the momentum before the collision:

$$12v = 24$$

$$\Rightarrow \quad v = 2.0\,\mathrm{m\,s^{-1}}$$

The kinetic energy before the collision is:

$$\tfrac{1}{2} \times 4.0 \times 6.0^2 = 72\,\mathrm{J}$$

After the collision the kinetic energy is:

$$\tfrac{1}{2} \times 12 \times 2.0^2 = 24\,\mathrm{J}$$

It appears that 48 J has been 'lost' (into other forms of energy, e.g. thermal energy in the blocks themselves and the surrounding air or energy to deform the bodies during the collision and some to sound generated in the collision).

But consider now the outcome of the collision of these two blocks in which the blue block rebounds with speed $2.0\,\mathrm{m\,s^{-1}}$, as shown in Figure **2.81**. The red block moves off in the original direction with speed v.

What is the speed of the red block? As before, the total momentum before the collision is 24 N s. The total momentum after the collision is (watch the minus sign):

$$\underbrace{(4.0 \times -2.0)}_{\text{blue block}} + \underbrace{(8.0 \times v)}_{\text{red block}}$$

Equating the total momentum before and after the collision we find:

$$-8.0 + 8.0 \times v = 24$$

This gives $v = 4.0\,\mathrm{m\,s^{-1}}$.

The total kinetic energy after the collision is then:

$$\underbrace{\tfrac{1}{2} \times 4.0 \times (-2.0)^2}_{\text{blue block}} + \underbrace{\tfrac{1}{2} \times 8.0 \times 4.0^2}_{\text{red block}} = 72\,\mathrm{J}$$

This is the same as the initial kinetic energy.

So, in a collision the momentum is always conserved but kinetic energy may or may not be conserved. You will find out more about this in the next section.

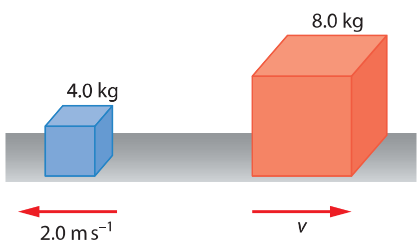


Figure 2.81 An outcome of the collision in which total kinetic energy stays the same.

Predicting outcomes

Physics is supposed to be able to predict outcomes. So why is there more than one outcome in the collision of Figure **2.80**? Physics does predict what happens, but more information about the nature of the colliding bodies is needed. We need to know if they are soft or hard, deformable or not, sticky or breakable, etc. If this information is given physics will uniquely predict what will happen.

Kinetic energy and momentum

We have seen that, in a collision or explosion where no external forces are present, the total momentum of the system is conserved. You can easily convince yourself that in the three collisions illustrated in Figure **2.82** momentum is conserved. The incoming body has mass 8.0 kg and the other a mass of 12 kg.

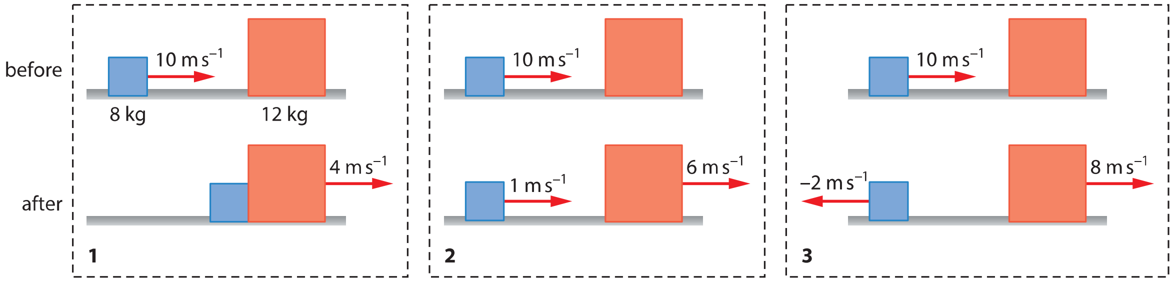


Figure 2.82 Momentum is conserved in these three collisions.

Let us examine these collisions from the point of view of energy.

In all cases the total kinetic energy before the collision is:

$$E_K = \frac{1}{2} \times 8.0 \times 10^2 = 400\,\text{J}$$

The total kinetic energy after the collision in each case is:

case 1: $E_K = \frac{1}{2} \times 20 \times 4^2 = 160\,\text{J}$

case 2: $E_K = \frac{1}{2} \times 8.0 \times 1^2 + \frac{1}{2} \times 12 \times 6^2 = 220\,\text{J}$

case 3: $E_K = \frac{1}{2} \times 8.0 \times 2^2 + \frac{1}{2} \times 12 \times 8^2 = 400\,\text{J}$

We thus observe that whereas momentum is conserved in all cases, kinetic energy is not. When kinetic energy is conserved (case 3), the collision is said to be **elastic**. When it is not (cases 1 and 2), the collision is *inelastic*. In an inelastic collision, kinetic energy is lost. When the bodies stick together after a collision (case 1), the collision is said to be **totally inelastic** (or **plastic**), and in this case the maximum possible kinetic energy is lost.

The lost kinetic energy is transformed into other forms of energy, such as thermal energy, deformation energy (if the bodies are permanently deformed as a result of the collision) and sound energy.

Notice that using momentum, we can obtain a useful additional formula for kinetic energy:

$$E_K = \frac{1}{2}mv^2 = \frac{m^2v^2}{2m}$$

$$E_K = \frac{p^2}{2m}$$

Worked examples

2.50 A moving body of mass m collides with a stationary body of double the mass and sticks to it. Calculate the fraction of the original kinetic energy that is lost.

The original kinetic energy is $\frac{1}{2}mv^2$ where v is the speed of the incoming mass. After the collision the two bodies move as one with speed u that can be found from momentum conservation:

$$mv = (m + 2m)u$$

$$\Rightarrow \quad u = \frac{v}{3}$$

The total kinetic energy after the collision is therefore:

$$\frac{1}{2}(3m) \times \left(\frac{v}{3}\right)^2 = \frac{mv^2}{6}$$

and so the lost kinetic energy is

$$\frac{mv^2}{2} - \frac{mv^2}{6} = \frac{mv^2}{3}$$

The fraction of the original energy that is lost is thus

$$\frac{mv^2/3}{mv^2/2} = \frac{2}{3}$$

2.51 A body at rest of mass M explodes into two pieces of masses $M/4$ and $3M/4$. Calculate the ratio of the kinetic energies of the two fragments.

Here it pays to use the formula for kinetic energy in terms of momentum: $E_K = \frac{p^2}{2m}$. The total momentum before the explosion is zero, so it is zero after as well. Thus, the two fragments must have equal and opposite momenta. Hence:

$$\frac{E_{light}}{E_{heavy}} = \frac{p^2/(2M_{light})}{(-p)^2/(2M_{heavy})}$$

$$\frac{E_{light}}{E_{heavy}} = \frac{M_{heavy}}{M_{light}}$$

$$\frac{E_{light}}{E_{heavy}} = \frac{3M/4}{M/4}$$

$$\frac{E_{light}}{E_{heavy}} = 3$$

It all depends on the system!

Consider a ball that you drop from rest from a certain height. As the ball falls, its speed and hence its momentum increases so momentum does not stay the same (Figure 2.83).

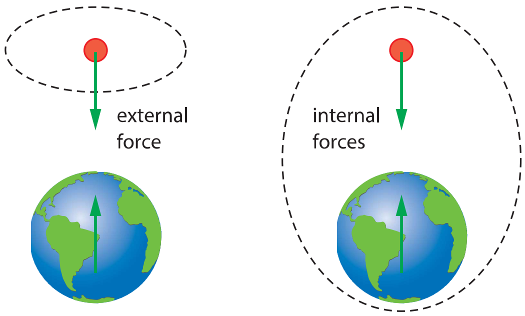


Figure 2.83 As the ball falls, an external force acts on it (its weight), increasing its momentum.

This is to be expected – there is an external force on the ball, namely its weight. So the momentum of the system that consists of just the falling ball is not conserved. If we include the Earth as part of the system then there are no external forces and the total momentum will be conserved. This means that the Earth moves up a bit as the ball falls!

The rocket equation

The best example of motion with varying mass is, of course, the rocket (Figure 2.84).

This is quite a complex topic and is included here only as supplementary material. The rocket moves with speed v. The engine is turned on and gases leave the rocket with speed u relative to the rocket. The initial mass of the rocket including the fuel is M. After a short time δt the rocket has ejected fuel of mass δm. The mass of the rocket is therefore reduced to $M - \delta m$ and its speed increased to $v + \delta v$ (Figure 2.85).

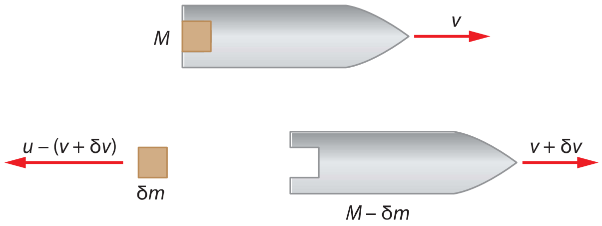


Figure 2.85 Diagram for deriving the rocket equation. The velocities are relative to an observer 'at rest on the ground'.

Figure 2.84 Exhaust gases from the booster rockets propel this space shuttle during its launch.

Applying the law of conservation of momentum gives (in the equation below terms shaded the same colour cancel out):

$$Mv = (M - \delta m)(v + \delta v) - \delta m \underbrace{(u - v - \delta v)}_{\text{speed relative to ground}}$$

$$Mv = Mv + M\delta v - v\delta m - \delta m\delta v - u\delta m + v\delta m + \delta m\delta v$$

$$M\delta v = u\delta m$$

$$\delta v = \frac{\delta m}{Mu}$$

This gives the change in speed of the rocket as a result of gases leaving with speed u relative to the rocket. At time t the mass of the rocket is M. Dividing by δt and taking the limit as δt goes to zero gives the rocket differential equation:

$$M\frac{dv}{dt} = \mu u$$

where μ is the rate at which mass is being ejected.

Nature of science

General principles such as the conservation of momentum allow for simple and quick solutions to problems that may otherwise look complex. Consider, for example, a man of mass m who stands on a plank also of mass m. There is no friction between the floor and the plank. A man starts walking on the plank until he get gets to the other end, at which point he stops. What happens to the plank?

The centre of mass must remain in the same place since there is no external force. So the final position of the plank will be as shown in Figure 2.86: the plank moves half its length to the left and stops.

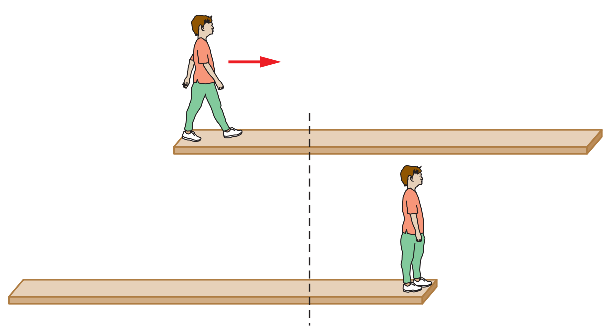


Figure 2.86 Conservation of momentum.

The same principles can be extended to analyse and predict the outcomes of a wide range of physical interactions, from large-scale motion to microscopic collisions.

Test yourself

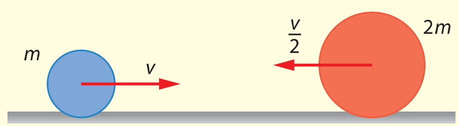
72 The momentum of a ball increased by 12.0 N s as a result of a force that acted on the ball for 2.00 s. Find the average force on the ball.

73 A 0.150 kg ball moving horizontally at 3.00 m s^{-1} collides normally with a vertical wall and bounces back with the same speed.

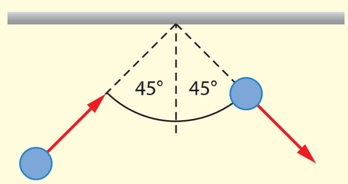
a Calculate the impulse delivered to the ball.

b The ball was in contact with the wall for 0.125 s. Find the average force exerted by the ball on the wall.

74 The bodies in the diagram suffer a head-on collision and stick to each other afterwards. Find their common velocity.



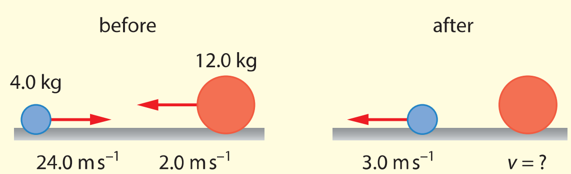
75 A ball of mass 250 g rolling on a horizontal floor with a speed 4.00 m s^{-1} hits a wall and bounces with the same speed, as shown in the diagram.



a What is the magnitude and direction of the momentum change of the ball?

b Is momentum conserved here? Why or why not?

76 Two masses moving in a straight line towards each other collide as shown in the diagram. Find the velocity (magnitude and direction) of the heavier mass after the collision.

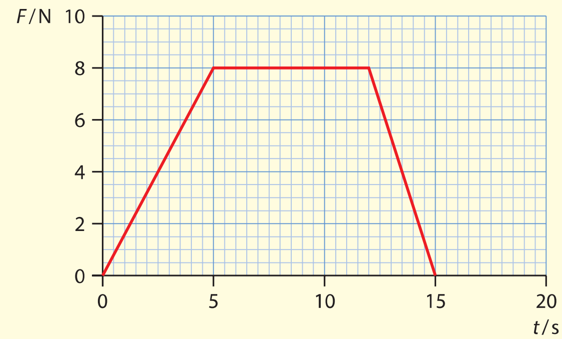


77 A time-varying force varies with time as shown in the graph. The force acts on a body of mass 4.0 kg.

a Find the impulse of the force from $t = 0$ to $t = 15$ s.

b Find the speed of the mass at 15 s, assuming the initial velocity was zero.

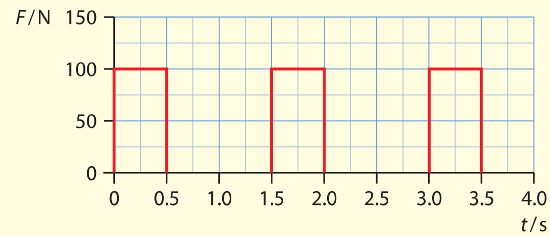
c State the initial velocity of the body such it is brought to rest at 15 s.



78 A boy rides on a scooter pushing on the road with one foot with a horizontal force that depends on time, as shown in the graph. While the scooter rolls, a constant force of 25 N opposes the motion. The combined mass of the boy and scooter is 25 kg.

a Find the speed of the boy after 4.0 s, assuming he started from rest.

b Draw a graph to represent the variation of the boy's speed with time.



79 A ball of mass m is dropped from a height of h_1 and rebounds to a height of h_2. The ball is in contact with the floor for a time interval of t.

a Show that the average net force on the ball is given by:

$$F = m\frac{\sqrt{2gh_1} + \sqrt{2gh_2}}{2}$$

b If $h_1 = 8.0$ m, $h_2 = 6.0$ m, $t = 0.125$ s and $m = 0.250$ kg, calculate the average force exerted by the ball on the floor.

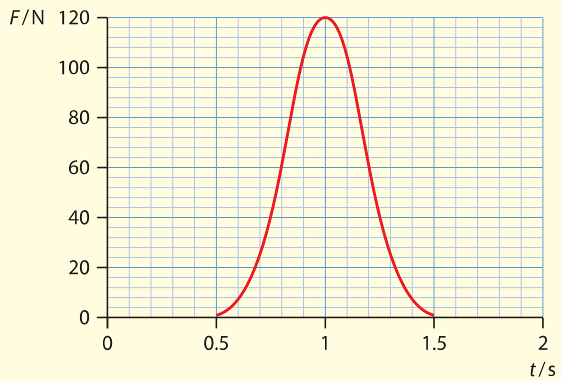
80 A ball of mass m moving vertically, hits a horizontal floor normally with speed v_1 and rebounds with speed v_2. The ball was in contact with the floor for a time t.

a Show that the average force F on the ball from the floor during the collision is given by:

$$F = \frac{m(v_1 + v_2)}{t} + mg$$

b Find an expression for the average net force on the ball.

81 The diagram shows the variation with time of the force exerted on a ball as the ball came into contact with a spring.



a For how long was the spring in contact with the ball?

b Estimate the magnitude of the change in momentum of the ball.

c What was the average force that was exerted on the ball?

82 Two masses of 2.0 kg and 4.0 kg are held in place, compressing a spring between them. When they are released, the 2.0 kg moves away with a speed of $3.0\,\mathrm{m\,s^{-1}}$. What was the energy stored in the spring?

83 A rocket in space where gravity is negligible has a mass (including fuel) of 5000 kg. It is desired to give the rocket an average acceleration of $15.0\,\mathrm{m\,s^{-2}}$ during the first second of firing the engine. The gases leave the rocket at a speed of $1500\,\mathrm{m\,s^{-1}}$ (relative to the rocket). Estimate how much fuel must be burnt in that second.

Exam-style questions

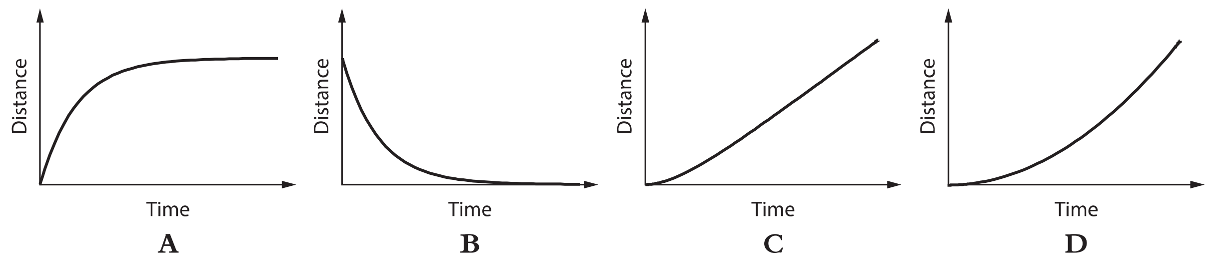
1 Four cars race along a given race track starting at the same time. The car that will reach the finishing line first is the one with the largest

A maximum speed
B acceleration
C power
D average speed

2 A body that started from rest moves with constant acceleration in a straight line. After travelling a distance d the speed of the car is v. What is the distance travelled when the speed of the car was $\frac{v}{2}$?

A $\frac{d}{2}$ **B** $\frac{d}{\sqrt{2}}$ **C** $\frac{d}{4}$ **D** $\frac{d}{2\sqrt{2}}$

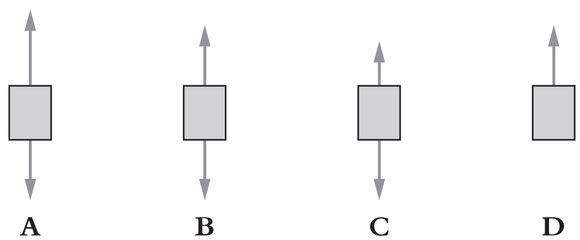
3 A sphere falls trough a liquid and eventually reaches terminal speed. Which graph shows the variation with time of the distance travelled by the sphere?



4 A steel ball of mass m is thrown vertically downwards with initial speed u near the Earth's surface. The rate of change of the momentum of the ball as it falls is:

A 0 **B** mu **C** $m(u+gt)$ **D** mg

5 A lunar module is descending vertically above the lunar surface. The speed of the module is decreasing. Which is a free-body diagram of the forces on the landing module?



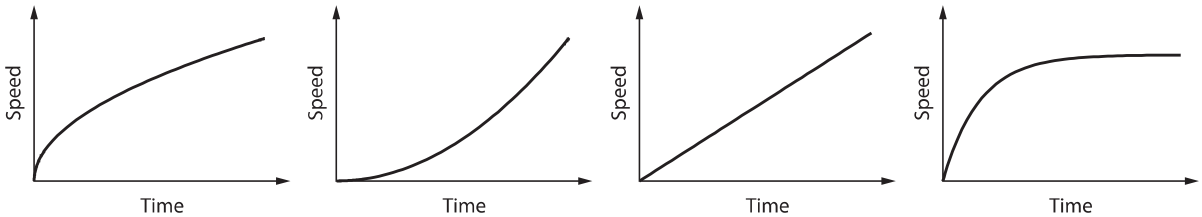
6 A person of mass m stands on weighing scales in an elevator. The elevator is accelerating upwards with acceleration a. The reaction force from the scales on the person is R. What is the reading on the scales?

A mg **B** $R+ma$ **C** $R-ma$ **D** R

7 A body of mass $3M$ at rest explodes into two pieces of mass M and $2M$. What is the ratio of the kinetic energy of M to that of $2M$?

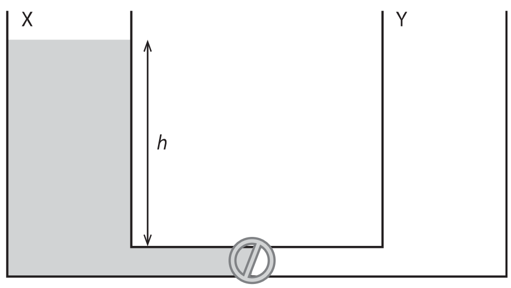
A $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** 4 **D** 2

8 The power delivered by a car engine is constant. A car starts from rest. Resistance forces are negligible. Which graph shows the variation with time of the speed of the car?



A **B** **C** **D**

9 The diagram shows two identical containers, X and Y, that are connected by a thin tube of negligible volume. Initially container X is filled with water of mass m up to a height h and Y is empty.



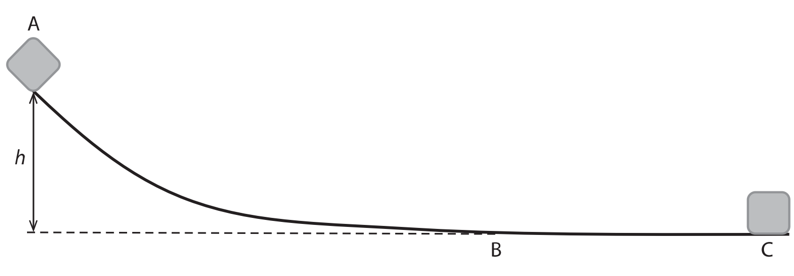
The valve is then opened and both containers contain equal quantities of water. The loss of gravitational potential energy of the water is:

A 0 **B** $\frac{mgh}{8}$ **C** $\frac{mgh}{4}$ **D** $\frac{mgh}{2}$

10 A person of mass m stands on roller skates facing a wall. After pushing against the wall with a constant force F he moves away, reaching speed v after a distance d. What is the work done by F?

A zero **B** mv^2 **C** $\frac{1}{2}mv^2$ **D** Fd

11 In a factory blocks of ice slide down a smooth curved path AB and then on to a rough horizontal path starting at B.



The length of the curved path AB is s; the block of ice takes time t to move from A to B.

a Explain why, for the motion of the block from A to B:

i the formula $s = \frac{1}{2}gt^2$ does not apply. **[1]**

ii the formula $v = \sqrt{2gh}$ does apply. **[1]**

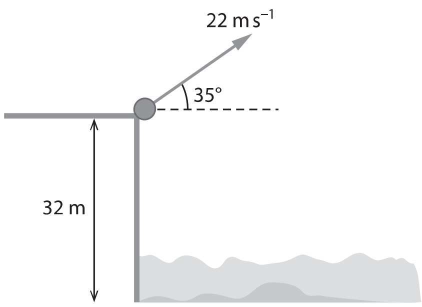
b A block of ice of mass 25 kg slides from A to B. The speed of the block at B is $v_B = 4.8\,\text{m s}^{-1}$. Calculate the height h. **[3]**

c **i** The coefficient of dynamic friction between the block of ice and the rough surface BC is 0.45. Show that the distance BC at which the block of ice is brought to rest is 2.7 m. **[2]**

ii Calculate the time it takes the block of ice to cover the distance BC. **[2]**

d The factory also produces blocks of ice of mass 50 kg that slide down the same path starting at A. Predict, for this heavier block of ice, the speed at B and the stopping distance BC. (The coefficient of friction stays the same.) **[3]**

12 A stone of mass 0.20 kg is thrown with speed $22\,\text{m s}^{-1}$ from the edge of a cliff that is 32 m above the sea. The initial velocity of the stone makes an angle of 35° with the horizontal. Air resistance is neglected.



a **i** Determine the horizontal and vertical components of the initial velocity. **[2]**

ii Sketch graphs showing the variation with time of the horizontal and vertical components of velocity. **[2]**

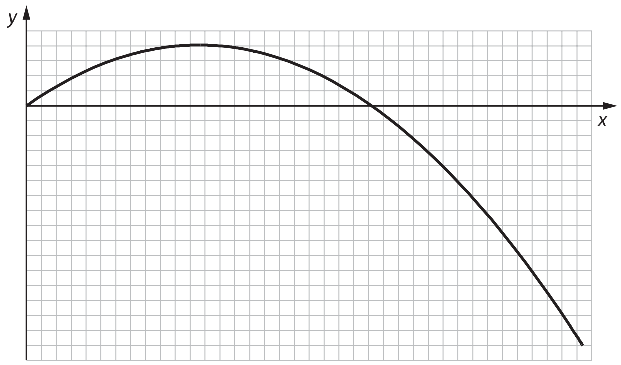
b **i** Calculate the maximum height above the cliff reached by the stone. **[3]**

ii State the net force on the stone at the highest point in its path. **[1]**

c **i** Using conservation of energy, determine the speed of the stone as it hits the sea. **[2]**

ii Hence or otherwise, determine the time it took the stone to reach the surface of the sea. **[2]**

The graph shows the path followed by this stone, until just before hitting the sea, in the absence of air resistance.



d **i** On a copy of the axes above, draw the path of the stone in the presence of an air resistance force opposite to the velocity and proportional to the speed. **[3]**

ii State and explain one difference between your graph and the graph above. **[2]**

13 A toy helicopter has mass $m = 0.30$ kg and blade rotors of radius $R = 0.25$ m. It may be assumed that as the blades turn, the air exactly under the blades is pushed downwards with speed v. The density of air is $\rho = 1.2\,\text{kg}\,\text{m}^{-3}$.

a **i** Show that the force that the rotor blades exert on the air is $\rho\pi R^2 v^2$. **[3]**

ii Hence estimate the speed v when the helicopter just hovers. **[2]**

b Determine the power generated by the helicopter's motor when it just hovers as in **a**. **[2]**

c The rotor blades now move faster pushing air downwards at a speed double that found in **a**. The helicopter is raised vertically a distance of 12 m.

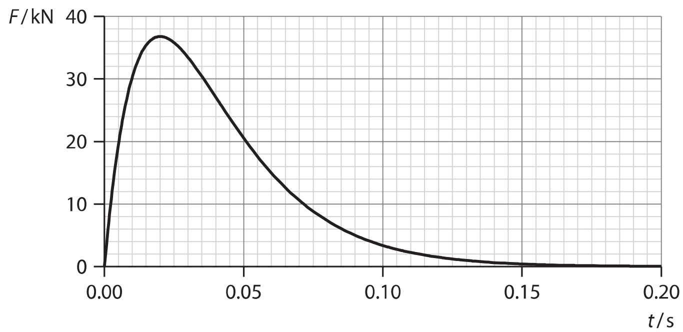
Estimate:

i the time needed to raise the helicopter. **[2]**

ii the speed of the helicopter after it is raised 12 m. **[2]**

iii the work done by the rotor in raising the helicopter. **[1]**

14 It is proposed to launch projectiles of mass 8.0 kg from satellites in space in order to destroy incoming ballistic missiles. The launcher exerts a force on the projectile that varies with time according to the graph.



The impulse delivered to the projectile is 2.0×10^3 N s. The projectile leaves the launcher in 0.20 s.

a Estimate:

i the area under the curve **[1]**

ii the average acceleration of the projectile **[3]**

iii the average speed of the projectile **[2]**

iv the length of the launcher. **[2]**

b Calculate, for the projectile as it leaves the launcher:

i the speed **[2]**

ii the kinetic energy. **[2]**

c Estimate the power delivered to the projectile by the launcher. **[2]**

15 A car of weight 1.4×10^4 N is moving up an incline at a constant speed of $6.2\,\text{m}\,\text{s}^{-1}$. The incline makes an angle of 5.0° to the horizontal. A frictional force of 600 N acts on the car in a direction opposite to the velocity.

a **i** State the net force on the car. **[1]**

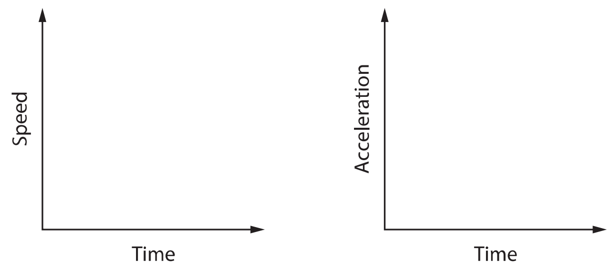
ii Calculate the force F pushing the car up the incline. **[3]**

b The power supplied by the car is 15 kW. Determine the efficiency of the car engine in pushing the car uphill. **[3]**

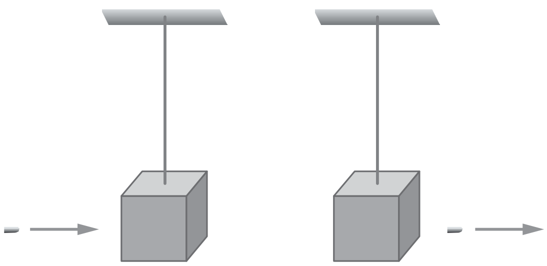
c The car is now allowed to roll down the incline from rest with the engine off. The only resistance force on the car is assumed to be proportional to speed. On a copy of the axes below, draw sketch graphs to show the variation with time of:

i the speed of the car **[2]**

ii the acceleration of the car. **[2]**



16 A bullet of mass 0.090 kg is shot at a wooden block of mass 1.20 kg that is hanging vertically at the end of a string.



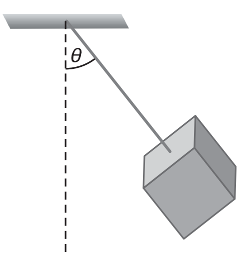
The bullet enters the block with speed 130 m s^{-1} and leaves it with speed 90 m s^{-1}. The mass of the block does not change appreciably as a result of the hole made by the bullet.

a **i** Calculate the change in the momentum of the bullet. **[2]**

ii Show that the initial velocity of the block is 3.0 m s^{-1}. **[1]**

iii Estimate the loss of kinetic energy in the bullet–block system. **[2]**

As a result of the impact, the block is displaced. The maximum angle that the string makes with the vertical is θ. The length of the string is 0.80 m.



b Show that $\theta \approx 65°$. **[3]**

c **i** State and explain whether the block in **b** is in equilibrium. **[2]**

ii Calculate the tension in the string in **b**. **[3]**