Lagrangian Mechanics

4.1: Non-Conservative Forces

4.2 Forces of Constraints and Generalized Coordinates

Suppose that a particle is free to move in all directions. Three coordinates are needed to specify its location.

- Cartesian: (x, y, z)
- Cylinderical: (ρ, ψ, z)
- Spherical: (r, θ, ϕ)

The presence of constrains mean that some coordinates might be less than three positions. A constraint that reduces the number of position of a particle is called a **holonomic constraint**. The minimal set of required independent coordinates are called **generalized coordinates**, denoted by q_k.

4.3: Hamilton's Mechanics

The Lagrangian is defined as

$$L = T - U \tag{1}$$

where T is kinetic energy and U is potential energy. Having chosen a set of generalized coordinates, the Lagrangian can be written as:

$$L = T - U = L(t, q_1, q_1, \cdots, \dot{q}_1, \dot{q}_2, \cdots)$$

We define its action $S[q_k(t)]$ as the functional of the time integrand over the Lagrange L, from a starting time t_a to an ending time t_b.

$$S[q_k(t)] = \int_{t_a}^{t_b} dt L(t, q_1, q_1, \cdots, \dot{q}_1, \dot{q}_2, \cdots) = \int_{t_a}^{t_b} dt L(t, q_k, \dot{q}_k)$$

When S is stationary, δS(the change in position) is zero, giving us

$$0 = \delta \int_{t_a}^{t_b} dt L(t, q_k, \dot{q}_k)$$

Using knowledge from the previous chapter, the Lagrange is equal to (The integral represents the area under curve, where the curve/function is defined by the integrand. In this case, the integrand is L. When S is stationary, it has reached a minimum or a maximum. As we derived in chapter, when the value of a function is extremeized, by Euler's equation $0 = \partial L/\partial q_k - (d/dt)(\partial L/\partial \dot{q}_k)$):

$$0 = \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$$

4.32: Newton's Law and Hamilton Mechanics

$$F = ma = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$$

Hamilton Mechanics is consistent of Newton's law. The second term of the Lagrange equation essentially represents:

while the first term sums of force components. This can be seen clearly from below examples.

Example 4.1: A Simple Pendulum

An inertial observer sees that a small plumb bob of mass *m* is free to swing back and forth in a vertical *x*–*z* plane at the end of a string of length *R*. The position of the bob can be specified uniquely by its angle θ measured up from its equilibrium position at the bottom, so we choose θ as the generalized coordinate, as illustrated in Figure 4.2. Find the equation of motion and the second order differential equation.

▼ `Solution`

The origin of the system is defined at the bottom. Kinetic energy is $K = 1/2mv^2$, where velocity v requires three coordinates to specify its direction. By cylindrical coordinates, we know that $(x, y, z) = (r, r\theta, z)$. Speed v is equal to:

$$v = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2 + r^2\dot{\theta}^2 + \dot{z}^2$$

Kinetic energy is equal to:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(r^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

Given the constraint that the string length R is fixed, $\dot{r} = 0$ and initial speed $\dot{z} = 0$, the equation becomes:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(R^2\dot{\theta}^2)$$

The simple pendulum, with θ as the generalized coordinate.

$$L = T - U = \frac{1}{2}m(R^2\dot{\theta}^2) - mgR(1 - \cos\theta)$$

The Euler's equation for $L = L(t, \theta, \dot{\theta})$ is equal to:

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgR\sin\theta - \frac{d}{dt}(\frac{1}{2}mR^2 2\dot{\theta}\ddot{\theta})$$

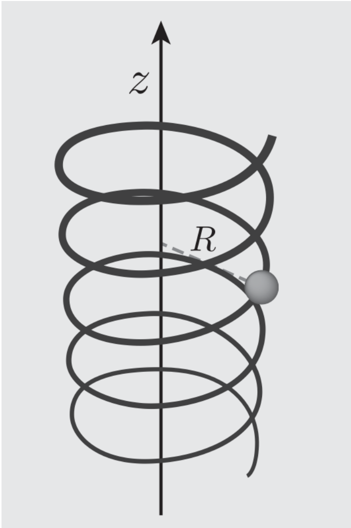
Simplifying the expression, we get

$$0 = \ddot{\theta} + \frac{g}{R}\sin\theta$$

The second order differential equation shows the pendulum is in simple harmonic motion with an angular frequency of $\omega = \sqrt{g/R}$. Torque is equal to $\tau = I\omega = mR^2 g/R = mgR$.

Example 4.2: A Bead Sliding on a Vertical Helix

A bead of mass *m* is slipped onto a *frictionless* wire wound in the shape of a helix of radius R, whose symmetry axis is oriented vertically in a uniform gravitational field, as shown in Figure 4.3. As always, we assume the description is from an inertial frame's perspective (unless explicitly stated otherwise). Using cylindrical coordinates to find equations of motion.



Example 4.3: Block on an Inclined Plane

A bead of mass *m* is slipped onto a *frictionless* wire wound in the shape of a helix of radius *R*, whose symmetry axis is oriented vertically in a uniform gravitational field, as shown in Figure 4.3

▼ `Solution`

The Lagrange L is equal to:

$$L = T - U = \frac{1}{2}mv^2 - mgX\sin\alpha = \frac{1}{2}m\dot{X}^2 - mgX\sin\alpha$$

Using Euler's equation, we let $L = L(t, \dot{X}, X)$, giving us:

$$0 = -mg\sin\alpha - \frac{d}{dt}\frac{1}{2}m(2\dot{X}) = -mg\sin\alpha - \frac{d}{dt}m\dot{X}$$

The equation is consistent with Newton's law, where $ma = -mg\sin\alpha$ in the horizontal (inclined) direction.

Summary for finding L and equations of motion

1. Use cartesian/polar/spherical/cylindrical coordinate to find positions of any arbitrary position $\vec{s}$.
2. Differentiate position $\vec{s}$ to find velocity v
3. Find the Lagrange $L = T - U$
    a. Find the expression for kinetic energy T

   b. Find the expression for potential energy U

4. Use Euler's equation to find equations of motion

4.4: Generalized Momenta and Cyclic Coordinates

In the last section, we has shown that Lagrange equation is consistent with Newton's second law. The first term represents force components, and the second term is ma. Because $F = dp/dt$, the term $\partial L/\partial \dot{x}$ can be seen as a component of momentum. Thus, it is natural to define the generalized momenta using generalized coordinates q_k

The generalized momenta is defined as:

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$$

By Newton's second law, $dp/dt = ma$. In terms of p_k, the Lagrange equation becomes:

$$\frac{dp_k}{dt} = \frac{\partial L}{\partial q_k}$$

Lagrange is expressed in the form of $L = L(t, q_k, \dot{q}_k)$. Sometimes, the coordinate q_k is missing while $\dot{q}_k$ is presents. A missing coordinate is said to be a **cyclic coordinate**. For such coordinate, the Lagrange equation tells us that the time derivative of corresponding generalized momentum is zero, so that the particular momentum is conserved.

$$\frac{dp_k}{dt} = \frac{\partial L}{\partial q_k} = 0$$

💡 One of the first thing to notice about a Lagrangian is whether there is a cyclic coordinate, because such coordinate leads to a conservation law that is also a first integral of motion. This means that the equation of motion is a first-order differential equation rather than the second-order differential equation for a noncyclic coordinate.

Explanation:

The explanation seems confusing. ***All it is trying to say is that: if there is a cyclic coordinate, there must be a conserved quantity.*** This is better explained by the below example. Consider the Lagrange $L = L(t, q_k, \dot{q}_k)$, where q_k is missing. Using Euler's equation, we get:

$$0 = \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$$

Because q_k is missing, the equation is independent of q_k and $\partial L/\partial q_k = 0$. Then,

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$$

so $\partial L/\partial \dot{q}_k = k$ is a constant. Thus, $\partial L/\partial \dot{q}_k$ is a conserved quantity.

Example: Particle on a Tabletop

Particle moving on a tabletop. There is a central for $F(r) = kr\hat{r}$ directed towards the origin. Derive the Lagrange equation and expressions for effective potential, and time.

▼ `Solution`

1. **Lagrange and Euler's Equations:**

For a particle moving in a plane (two dimensions) and subject to a central force, it is better to use polar coordinates (r, θ) about the origin. The kinetic energy is given by:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2)$$

The Lagrange $L = T - U$ is equal to:

$$L = T - U(r) = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) - U(r)$$

The generalized coordinates are r and θ. Notice that θ is the cyclic coordinate since it is missing and only $\dot{\theta}$ is present. Thus, there must be a conserved quantity. Using Euler's equation, we know that:

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

such that $\partial L/\partial\dot{\theta} = k$ is equal to a constant. Using chain rule for multivaraible functions, the conserved is:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

We recognize this as the angular momentum (The makes sense inutitively and theoretically. As shown in the previous section, the second term in Lagrange equation is equal to $d/dt(\partial L/\partial\dot{\theta}) = ma$ and $F = dp/dt$, so it is natural that $\partial L/\partial\dot{\theta}$ is some form of momentum).

The Lagrange equations are (Notice that there are two independent variables/generalized coordinates, so we use variational calculus for several variables in chapter three The Variational Principle):

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \qquad \frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

which becomes:

$$m(r\dot{\theta}^2) - \frac{\partial U(r)}{\partial r} = \frac{d}{dt}(m\dot{r}), \quad 0 = \frac{d}{dt}mr^2\dot{\theta}$$

$$-\frac{\partial U(r)}{\partial r} = m\ddot{r} - mr\dot{\theta}^2, \quad p_\theta = mr^2\dot{\theta}$$

This is equal to:

$$F(r) = ma_r = m(\ddot{r} - r\dot{\theta}^2), \quad F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

2. **Newton's Law and Lagrange Equations: 2nd Order DE Derivations**

Notice this is consistent with Newton's second law, where the acceleration for an object in circular motion is equal to $a = a_r + a_\theta = (\ddot{r} - r\dot{\theta})\hat{r} + (r + \ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$. Because $p_\theta = mr\dot{\theta}^2$ is a conserved quantity, we can solved for $\dot{\theta}$, giving us:

$$\dot{\theta} = \frac{p_\theta}{mr^2}$$

Substituting the expression for $\dot{\theta}$ into the Lagrange equation for $-\partial U(r)/\partial r$, we get:

$$-\frac{\partial U(r)}{\partial r} = m\ddot{r} - mr(\frac{p_\theta}{mr^2})^2$$

where the central force $F(r) = \partial U(r)/\partial r = -kr$ (This is shown in example 1.6 in chapter one Newtonian Mechanics). Then, the expression becomes:

$$-kr = m\ddot{r} - \frac{mp_\theta^2}{m^2r^3}$$

Rearranging the equation, we get:

$$0 = m\ddot{r} - \frac{p_\theta^2}{mr^3} + kr$$

$$0 = \ddot{r} - \frac{p_\theta^2}{m^2r^3} + \frac{k}{m}r$$

$$\boxed{0 = \ddot{r} - \frac{p_\theta^2}{m^2r^3} + \omega^2 r}$$

3. **Effective Potential/Energy Conservation Derivation**

Rearranging the second order differential equation, $m\ddot{r}$ is equal to:

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} - m\omega^2 r = F(r)$$

where $F(r) = m\ddot{r}$ is the central force directed towards the origin. Recall that effective potential U_{eff} is defined as (Note that effective potential is NOT same as potential energy. It is defined to simplifies two dimensional motion into one dimension. For more details, please refer to the :

$$U_{eff} = -\int F(r)dr$$

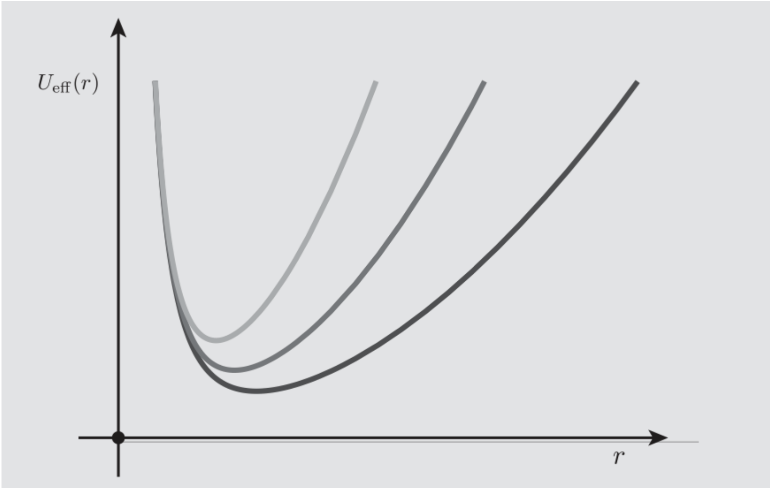
Integrate $F(r)$ alone the path for r, U_{eff} is equal to:

$$U_{eff} = -\int (\frac{p_\theta^2}{mr^3} - m\frac{k}{m}r)dr = \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2 + C$$

where C is a constant. Then, the second first integral of motion is given by energy conservation in this one-dimensional system.

$$\boxed{E = K + U_{eff}(r) = \frac{1}{2}m\dot{r}^2 + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2}$$

Note that $K \neq T$ in the Lagrange equation. The kinetic energy K includes the radial component only. This is because kinetic energy in tangential component in $\hat{\theta}$ direction is a part of the effective potential U_{eff}, so that the motion is simplified from two dimensional into one dimensional.



4. **Time**

   Rearranging the equation for energy E, the expression becomes:

$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$

   Solving for $\dot{r}$, we get:

$$\dot{r} = \sqrt{\frac{2E}{m} - U_{eff}(r)}$$

   Expressing $\dot{r}$ as a differential, we get:

$$\dot{r} = \frac{dr}{dt}$$

   Using separable integration, time t is equal to:

$$\boxed{t = \int dt = \int \sqrt{\frac{2E}{m} - U_{eff}(r)} dr}$$

Example 4.5: Spherical Pendulum

A ball of mass *m* swings on the end of an unstretchable string of length *R* in the presence of a uniform gravitational field *g*. This is often called the “spherical pendulum,” because the ball moves as though it were sliding on the frictionless surface of a spherical bowl. We aim to find its equations of motion.

▼ `Solution`

Before solving the problem, we recognize the constraint that the pendulum has a fixed radius of R. Then, the ball has two degrees of freedom:

1. It can move horizontally around a vertical axis, corresponding to the azimuthal angle ϕ.
2. It can also move in the polar direction, described by the angle θ.

When finding equations of motion, we follow the below steps:

1. Specify the position of an arbitrary point using Cartesian/Spherical/Cylindrical coordinates.
2. Differentiate position $\vec{s}$ to find velocity $\vec{v}$.
3. Find kinetic energy T and potential energy U to get the Lagrange $L = T - U$.
4. Apply Euler's equation to L to find equations of motion.
5. First integrals of motion.

STEP 1 Find $\vec{s}$:

Using spherical coordinates, the position of a point is specified by:

$$\vec{s} = (r, r\theta, r\sin\theta\phi)$$

Given that radius $r = R$, the expression becomes:

$$\vec{s} = (R, R\theta, R\sin\theta\phi)$$

STEP 2: Velocity

Differentiating $\vec{s}$ with respect to time, velocity $\vec{v}$ is equal to:

$$\vec{v} = (0, R\dot{\theta}, R\sin\theta\dot{\phi})$$

where $\dot{R} = 0$ since R is fixed (a constant).

STEP 3: Lagrangian

Using $\vec{v}$, kinetic energy T is equal to:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

Potential energy U is equal to:

$$U = mgR\cos\theta$$

The Lagrange $L = T - U$ is equal to:

$$L = T - U = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = mgR\cos\theta$$

STEP 4: Euler's equation to find equations of motion

Using Euler's equation, we get:

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}, \quad 0 = \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$$

STEP 5: First Integral of motion

Because ϕ is cyclic, we know that p_ϕ, the angular momentum, is conserved. Note that the expression for angular momentum p_ϕ is a first integral of motion..

$$p_\phi = mR^2\sin^2\theta\dot{\phi} = \text{constant}$$

Rearranging the Lagrange equation for θ, we get:

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{g}{R}\sin\theta = 0$$

Solving for $\dot{\phi}$ using p_ϕ, the expression becomes:

$$\ddot{\theta} - (\frac{p_\theta}{mR^2})^2 \sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0$$

The second first integral of motion is the expression for conservation of energy, where $E = T + U$.

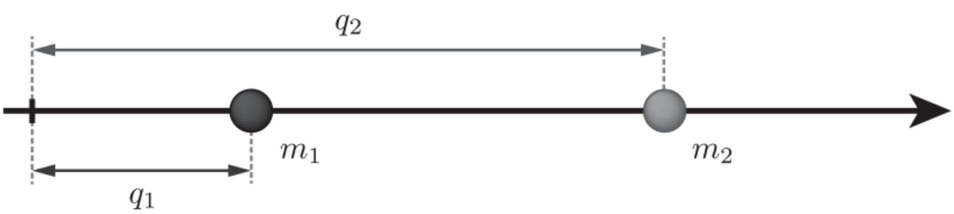
4.5: Systems of Particles

The last section considers the motion for a single particle only. The section studies the motion of a system of particles

Recall that the action for the system is defined as:

$$S[q_k(t)] = \int_{t_a}^{t_b} dtL(t, q_1, q_1, \cdots, \dot{q}_1, \dot{q}_2, \cdots) = \int_{t_a}^{t_b} dtL(t, q_k, \dot{q}_k)$$

Consider a system of two particles confined in the horizontal direction and is moving alone a frictionless rail.



The Lagrange of the system is equal to:

$$L = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - U(x_2 - x_1)$$

where $U(x_2 - x_1)$ is the potential energy between masses, describing the interaction between two particles. The action of the system is equal to:

$$S = \int dt(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - U(x_2 - x_1))$$

Notice that there are two generalized coordinates, x_1 and x_2, for the system. Using Euler's equation, Lagrange equations are:

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = 0$$

$$\frac{\partial L}{\partial x_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = 0$$

Using chain rule for multivariable functions, we get:

$$-\frac{\partial U(x_2 - x_1)}{\partial x_1} - (m_1 \ddot{x}_1) = -\frac{\partial U(x)}{\partial x_1} - (m_1 \ddot{x}_1) = 0$$

giving us:

$$m_1 \ddot{x}_1 = -\frac{\partial U}{\partial x_1}$$

The expression for m_2 is:

$$-\frac{\partial U(x_2 - x_1)}{\partial x_2} - (m_2 \ddot{x}_2) = -\frac{\partial U}{\partial x_2} - (m_2 \ddot{x}_2) = 0$$

giving us:

$$m_2 \ddot{x}_2 = -\frac{\partial U}{\partial x_2}$$

According to Newton's third law, the force exerted by m_1 on m_2 is equal to the magnitude of the force exerted by m_2 on m_1.

💡 In multi-particle systems, we consider the total kinetic energy T minus the total potential energy U. Sometimes. particles give action-reaction pairs $\partial/\partial x_1 = \partial/\partial x_2$, come out free.

Because the total momentum of the system is conserved, the change in momentum $\Delta p = dp/dt = 0$, giving us:

$$\frac{d}{dt}(m_1 \dot{x}_1 + m_2 \dot{x}_2) = 0$$

💡 In a multiple-particle system, the total momentum must be conserved.

The new coordinates are center of mass, giving by:

$$X \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

Central Force, Reduced Mass, Energy

Let $x = x_2 - x_1$. The Lagrange of the system is:

$$\boxed{L = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} \mu \dot{x}^2 - U(x)} \tag{2}$$

where μ is the reduced mass

$$\mu \equiv \frac{m_1 m_2}{M} \tag{3}$$

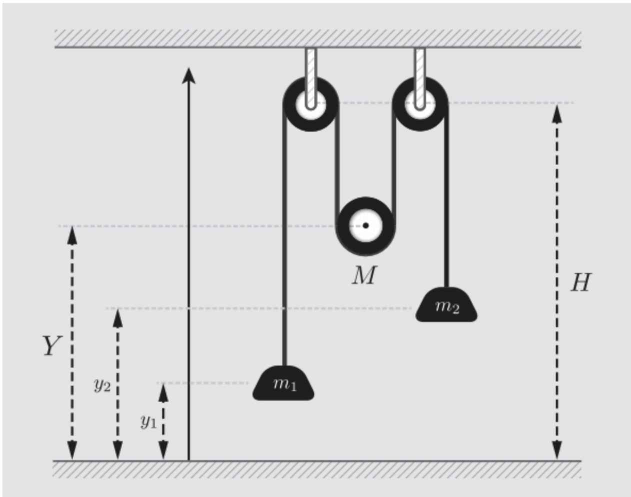
The corresponding momentum P is equal to:

$$P = \frac{\partial L}{\partial \dot{X}} = M\dot{=}m_1\dot{x}_1 + m_2\dot{x}_2$$

and is conserved.

Example: Pulley

A contraption of pulleys. We want to find the accelerations of all three weights. We assume that the three pulleys have negligible mass so they have negligible kinetic and potential energies.



▼ Solution

1. **Lagrangian**

In order to write the Lagrange, we need to know the system's kinetic and potential energies. Kinetic energy of the system is equal to:

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}M\dot{Y}^2$$

2. **Length Constraint**

Because there is only two degrees of freedom, we need to eliminate one variable in $\dot{y}_1$, $\dot{y}_2$ and $\dot{Y}$. When solving problems involving pulley, one common technique is to use length constraint to relate length with acceleration to eliminate a variable. Because the total length of the string is fixed, we know that:

$$(H - y_1) + 2(H - Y) + (H - y_2) = 4H - 2Y - y_2 - y_1 = \text{constant} = L$$

Differentiating the equation with respect to time t, the expression is equal to:

$$-2\dot{Y} - \dot{y}_2 - \dot{y}_1 = 0$$

giving us:

$$\dot{Y} = -\frac{1}{2}(\dot{y}_1 + \dot{y}_2)$$

3. **Substitution**

Substituting the expression for $\dot{Y}$ into T, we get:

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}M(\dot{\frac{1}{2}}(\dot{y}_1 + \dot{y}_2))2$$

Knowing the kinetic energy, we need to find potential energy U, giving us:

$$U = m_1 g y_1 + m_2 g y_2 + MgY$$

$$U = m_1 g y_1 + m_2 g y_2 - Mg(\frac{y_1 + y_2}{2}) + \text{constant}$$

4. **Euler's equation**

The Lagrange is equal to:

$$L = T - U = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}M(\dot{\frac{1}{2}}(\dot{y}_1 + \dot{y}_2))2 - m_1 g y_1 - m_2 g y_2 + Mg(\frac{y_1 + y_2}{2}) + \text{constant}$$

Using Euler's equation, we get:

$$0 = \frac{\partial L}{\partial y_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1}, \quad 0 = \frac{\partial L}{\partial y_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2}$$

Using chain rule, we get:

$$m_1\ddot{y}_1 + \frac{M}{4}(\ddot{y}_1 + \ddot{y}_2) = -m_1 g + \frac{Mg}{2}$$

$$m_2\ddot{y}_2 + \frac{M}{4}(\ddot{y}_1 + \ddot{y}_2) = -m_2 g + \frac{Mg}{2}$$

5. **Solving the system of equations**

Solving the system of equations, we get:

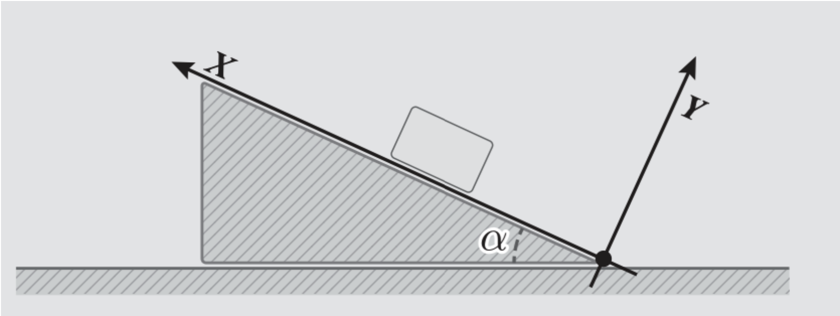
$$\dot{y}_1 = -g + \frac{-4m_2 g}{m_1 + m_4 m_1 m_2/M}$$

$$\dot{y}_2 = -g + \frac{-4m_1 g}{m_1 + m_4 m_1 m_2/M}$$

$$\dot{Y} = g - \frac{2(m_1 + m_2)g}{m_1 + m_4 m_1 m_2/M}$$

Example: A block on a Inclined Plane

Let us return to the classic problem of a block sliding down a frictionless inclined plane, as in Example 4.2, except that we will make things a bit more interesting: now the inclined plane itself is allowed to *move*. Figure 4.11(a) shows the system. A block of mass m rests on an inclined plane of mass M: both the block and the inclined plane are free to move without friction. The plane's angle is denoted by α. The problem is to find the acceleration of the block

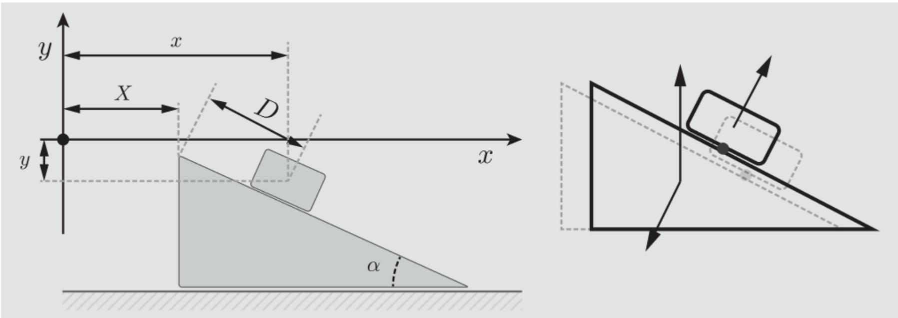


▼ `Solution`

This is a classic problem. In the approach of Newton's law, we would draw force diagrams and use the constraint of a fixed angle to solve the system of equations to find acceleration. In Hamilton's mechanics, we will find the Lagrange $L = T - U$ first.

The first observation we need to make is that the plane is not fixed. As the block slides down, it exerts a force on the plane, causing the object to move leftward (the ground is frictionless or the force exerted by the block exceeds stative friction). Kinetic energy of the system the includes two components: the block and the plane.

Define the coordinate system: Another thing we need to do before solving the problem is to define the coordinate system. Let X be the horizontal distance slided by the plane, and x and y be the horizontal and vertical distances slide by block. The origin is shown in the figure.



Kinetic energy of the system is:

$$T = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Because $\dot{Y}$ is fixed, $\dot{Y} = 0$. Then, T us equal to (This makes sense and can be seen esaily as the plane moves in the horizontal only. The problem uses $v^2 = \dot{X}^2 + \dot{Y}^2$ to show a more comprehensive approach):

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Potential energy is equal to:

$$U = U_{plane} - mgy$$

where U_{plane} is the plane's gravitational potential energy, and the quantity remains unchanged.

There are three variables $\dot{X}$, $\dot{x}$ and $\dot{y}$ in T. Because the degree of freedom is two (The degree of freedom of two refer to X and Y. We start with four coordinates, $\dot{X}$, $\dot{Y}$, $\dot{x}$, and $\dot{y}$. After identifying two constraints: Y is fixed and $\dot{x}$ and $\dot{y}$ can be represented using D and α, the degree of freedom is 2), we need to replace one of the variables. Using trigonometry

$$x = \dot{X} + D\cos\alpha, \quad y = -D\sin\alpha$$

Differentiating expressions with respect to t, we get

$$\dot{x} = \dot{X} + \dot{D}\cos\alpha, \quad \dot{y} = -\dot{D}\sin\alpha$$

Substituting, we get:

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m((\dot{X} + \dot{D}\cos\alpha)^2 + (-\dot{D}\sin\alpha)^2)$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{X}^2 + m\dot{X}\dot{D}\cos\alpha + \frac{1}{2}m\dot{D}^2\cos^2\alpha + \frac{1}{2}m\dot{D}^2\sin^2\alpha$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{X}^2 + m\dot{X}\dot{D}\cos\alpha + \frac{1}{2}m\dot{D}^2$$

The Lagrangian is equal to:

$$L = T - U = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{X}^2 + m\dot{X}\dot{D}\cos\alpha + \frac{1}{2}m\dot{D}^2 - U_{plane} + mgD\sin\alpha$$

Using Euler's equation, we get:

$$0 = \frac{\partial L}{\partial X} - \frac{d}{dt}\frac{\partial L}{\partial \dot{X}}, \quad 0 = \frac{\partial L}{\partial D} - \frac{d}{dt}\frac{\partial L}{\partial \dot{D}}$$

Using Chain Rule, we get

$$0 = \frac{d}{dt}(M\dot{X} + m\dot{X} + m\dot{D}\cos\alpha) = M\ddot{X} + m\ddot{X} + m\ddot{D}\cos\alpha$$

$$0 = mg\sin\alpha - \frac{d}{dt}(m\dot{X}\cos\alpha + m\dot{D}) = mg\sin\alpha - m\ddot{X}\cos\alpha + m\ddot{D}$$

This gives us a system of equations

$$0 = (M + m)\ddot{X} + m\ddot{D}\cos\alpha$$

$$mg\sin\alpha = m\ddot{X}\cos\alpha + m\ddot{D}$$

Solving the system of equations, we get:

$$\boxed{\ddot{X} = \frac{-mg\cos\alpha\sin\alpha}{M + m\sin^2\alpha}}$$

$$\boxed{\ddot{D} = \frac{(M + m)g\sin\alpha}{M + m\sin^2\alpha}}$$

Then, $\ddot{x}$ and $\ddot{y}$ are equal to:

$$\ddot{x} = \frac{-mg\cos\alpha\sin\alpha}{M + m\sin^2\alpha} + \frac{(M + m)g\sin\alpha}{M + m\sin^2\alpha}\cos\alpha$$

$$\boxed{\ddot{x} = \frac{Mg\cos\alpha\sin\alpha}{M + m\sin^2\alpha}}$$

$$\ddot{y} = -\ddot{D}\cos\alpha = -\frac{(M+m)g\sin\alpha}{M+m\sin^2\alpha}\sin\alpha$$

$$\boxed{\ddot{y} = -\frac{(M+m)g\sin^2\alpha}{M+m\sin^2\alpha}}$$

4.6: Hamiltonian

Let the Lagrange be $L = L(t, q_k, \dot{q}_k)$. Using chain rule to multi-variable functions, we differentiate L with respect to time, giving us:

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k$$

Using product rule for multi-variable functions, we recognize that:

$$\frac{d}{dt}(\dot{q}_k\frac{\partial L}{\partial \dot{q}_k}) = \ddot{q}_k\frac{\partial L}{\partial \dot{q}_k} + \dot{q}_k\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k})$$

We notice that:

$$\frac{\partial L}{\partial t} - \frac{d}{dt}(L - \dot{q}_k\frac{\partial L}{\partial \dot{q}_k}) = 0$$

The Hamiltonian H of a particle is defined as:

$$\boxed{H \equiv \dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L = \dot{q}_k p_k - L} \quad (4)$$

where $p_k = \partial L/\partial \dot{q}_k$. From the equation, we can see that:

$$\boxed{\frac{\partial L}{\partial t} = -\frac{dH}{dt}}$$

The result is interesting if L is not an explicit function of time. For example, if $\partial L/\partial t = 0$, H is equal to a constant and thus H is conserved.

💡 If L is not an explicit function of time, $\partial L/\partial t = 0$, so the Hamiltonian H is conserved, giving us a first integral of motion.

Though the Hamiltonian seems to be an abstract quantity, it is closely related to energy. Suppose that a particle is free to move in three dimensions in a potential $U(x, y, z)$ without constraints, and that we are using Cartesian coordinates. Momentums are $p_x = m\dot{x}, p_y = m\dot{y}$, and $p_z = m\dot{z}$. Then, $\sum \dot{q}_k p_k = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The Hamiltonian is equal to:

4.6.1: Interpretation of the Hamiltonian

$$H = \dot{q}_k p_k - L = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U$$

This is equal to:

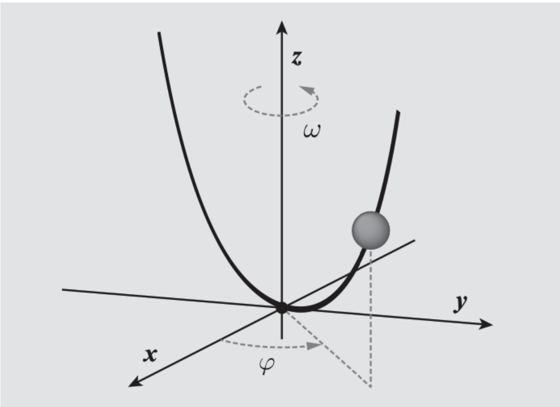
$$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U = E$$

where E is the total energy of the system. In this example, the Hamiltonian H represents the conserved quantity of $E = T + U$. However, be aware that there are cases in which $H \neq E$, although often it is equal to E. In the next section, we will explain the precise conditions for which $H \neq E$.

💡 Is H always equal to $E = T + U$? The answer is no, although very often it is. The precise conditions for which $H \neq E$ are worked out in Section 4.7.

Example: Bead on a Rotating Parabolic Wire

Suppose we bend a wire into the shape of a vertically oriented parabola defined in cylindrical coordinates by $z = \alpha\rho^2$, as illustrated in Figure 4.12: here z is the vertical coordinate, and ρ is the distance of a point on the wire from the vertical axis of symmetry. Using a synchronous motor, we can force the wire to spin at constant angular velocity ω about its symmetry axis. Then we slip a bead of mass *m* onto the wire and we want to determine its equation of motion – assuming that it slides without friction along the wire.



▼ Solution

Given that we are using cylindrical coordinates, we know that positions are described by:

$$\text{position} = s = (\rho, \rho\varphi, z)$$

Differentiating each components with respect to t, v^2 is equal to:

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2 + (2\alpha\rho\dot{\rho})^2$$

Because the wire spins at a constant angular velocity ω, angle φ is equal to:

$$\varphi = \omega t, \quad \text{and} \quad \dot{\varphi} = \omega$$

Then, kinetic energy T is equal to:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\omega^2 + (2\alpha\rho\dot{\rho})^2) = \frac{1}{2}m(\dot{\rho}^2(1 + 4\alpha^2\rho^2) + \rho^2\omega^2)$$

The gravitational potential energy of the bead is equal to

$$U = mgz = mg\alpha\rho^2$$

The Lagrange $L = T - U$ is equal to:

$$L = T - U = \frac{1}{2}m(\dot{\rho}^2(1 + 4\alpha^2\rho^2) + \rho^2\omega^2) - mg\alpha\rho^2$$

From the above expression, we can see that $L = L(t, \dot{\rho}, \rho)$. Using Euler's equation, the Lagrange equation is equal to:

$$0 = \frac{\partial L}{\partial \rho} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} = \frac{1}{2}m(\dot{\rho}^2(8\alpha^2\rho) + 2\rho\omega) - 2mg\alpha\rho - \frac{d}{dt}(m(1 + 4\alpha^2\rho^2)\dot{\rho})$$

Rearranging the equation, we get:

$$0 = (1 + 4\alpha^2\rho^2)\ddot{\rho} + 4\alpha^2\rho\dot{\rho}^2 + (2g\alpha - \omega^2)\rho$$

4.7: When is $H \neq e$

In the example from 4.6, the Hamiltonian H is conserved and E is not. The question becomes why are they different and why was H conserved while E was not?

Recall that the Hamiltonian is defined as

$$H = \dot{q}_k p_k - L$$

where L is the Lagrange and is equal to $L = T - U$. The expression becomes:

$$H = \dot{q}_k p_k - T + U$$

Let $r(q_k, t)$ be the position vector is the particle from some arbitrary origin fixed in an inertial frame, represented using generalized coordinates q_k and time. The velocity of the partcile is:

$$v = \frac{r(q_k, t)}{dt} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial q_k}\dot{q}_k$$

The kinetic energy of the particle is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mv \cdot v$$

The dot product of $v \cdot v$ is equal to:

$$v \cdot v = (\frac{\partial r}{\partial t} + \frac{\partial r}{\partial q_l}\dot{q}_l) \cdot (\frac{\partial r}{\partial t} + \frac{\partial r}{\partial q_m}\dot{q}_m)$$

$$H = E - m\frac{dr}{dt} \cdot \frac{\partial r}{\partial t} = E - p \cdot \frac{\partial r}{\partial t} \quad (5)$$

where p is the momentum of the particle in the chosen inertial frame. From the equation, we can see that the Hamiltonian is equal to energy $H = E$ if $\partial r/\partial t = 0$. This occurs when there are no constraints of when any constraints are fixed in place (the particle is not moving/stationary). When constraints are moving, they generally depend on time, causing $H \neq E$.

4.9 Small Oscillations about Equilibrium

In general, many if not most mechanical systems can be accorded an energy of the form

$$\text{constant} \times \dot{q}_k^2 + U_{eff}(q_k) = E$$

Using Taylor expansion, the effective potential is equal to:

$$U_{eff}(q) = U_eff(q_0) + \frac{dU_{eff}}{dq}|_{q_0}(q - q_0) + \frac{dU_{eff}^2}{dq^2}|_{q_0}(q - q_0)^2 + \cdots$$

At the equilibrium point, $dU_{eff}/dq = 0$, and the third term is in the form of $1/2k_{eff}(q - q_0)^2$. We define the second derivative of the effective potential the effective spring constant.

$$k_{eff} = \frac{dU_{eff}^2}{dq^2} = U''_{eff}$$

Angular frequency ω is equal to:

$$\omega = \sqrt{\frac{k_{eff}}{m}} = \sqrt{U''_{eff}}$$

Example:

In Example 4.4 we considered a particle moving on a frictionless tabletop subject to a central Hooke's-law spring force. There is an equilibrium radius for given energy and angular momentum for which the particle orbits in a circle of some radius *r*0. We now want to find the oscillation frequency ω for the mass about the equilibrium radius if it were perturbed slightly from this circular orbit.