Proof *[182 marks]*

1a. Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n\in\mathbb{Z}$. *[2 marks]*

Markscheme

attempting to expand the LHS ***(M1)***

LHS $=\left(4n^2-4n+1\right)+\left(4n^2+4n+1\right)$ ***A1***

$=8n^2+2$ (= RHS) ***AG***

[2 marks]

1b. Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. *[3 marks]*

Markscheme

METHOD 1

recognition that $2n-1$ and $2n+1$ represent two consecutive odd integers (for $n\in\mathbb{Z}$) ***R1***

$8n^2+2=2\left(4n^2+1\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

METHOD 2

recognition, *eg* that n and $n+2$ represent two consecutive odd integers (for $n\in\mathbb{Z}$) ***R1***

$n^2+(n+2)^2=2\left(n^2+2n+2\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

2a. Explain why any integer can be written in the form $4k$ or $4k+1$ or $4k+2$ or $4k+3$, where $k \in \mathbb{Z}$. *[2 marks]*

Markscheme

Upon division by 4 ***M1***

any integer leaves a remainder of 0, 1, 2 or 3. ***R1***

Hence, any integer can be written in the form $4k$ or $4k+1$ or $4k+2$ or $4k+3$, where $k \in \mathbb{Z}$ ***AG***

[2 marks]

2b. Hence prove that the square of any integer can be written in the form $4t$ or $4t+1$, where $t \in \mathbb{Z}^+$. *[6 marks]*

Markscheme

$(4k)^2 = 16k^2 = 4t$ ***M1A1***

$(4k+1)^2 = 16k^2 + 8k + 1 = 4t + 1$ ***M1A1***

$(4k+2)^2 = 16k^2 + 16k + 4 = 4t$ ***A1***

$(4k+3)^2 = 16k^2 + 24k + 9 = 4t + 1$ ***A1***

Hence, the square of any integer can be written in the form $4t$ or $4t+1$, where $t \in \mathbb{Z}^+$. ***AG***

[6 marks]

The function f is defined by $f(x) = \frac{ax+b}{cx+d}$, for $x \in \mathbb{R},\ x \neq -\frac{d}{c}$.

The function g is defined by $g(x) = \frac{2x-3}{x-2},\ x \in \mathbb{R},\ x \neq 2$

3. Express $g(x)$ in the form $A + \frac{B}{x-2}$ where A, B are constants. *[2 marks]*

Markscheme

$g(x) = 2 + \frac{1}{x-2}$ ***A1A1***

[2 marks]

4a. Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0,\ n \in \mathbb{Z}$. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ ***M1***

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ ***A1***

$= \sqrt{n+1} - \sqrt{n}$ ***AG***

[2 marks]

4b. Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$. *[2 marks]*

Markscheme

$\sqrt{2} - 1 = \frac{1}{\sqrt{2}+\sqrt{1}}$ ***A2***

$< \frac{1}{\sqrt{2}}$ ***AG***

[2 marks]

4c. *[9 marks]*

Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2,\ n \in \mathbb{Z}$.

Markscheme

consider the case $n = 2$: required to prove that $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ ***M1***

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2} - 1$

hence $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n = 2$ ***A1***

now assume true for $n = k : \sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ ***M1***

$\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n = k + 1 : \frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

(***M1)***

from assumption, we have that $\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$ ***M1***

so attempt to show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

EITHER

$\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$ ***A1***

$\frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k}+\sqrt{k+1}}$, (from part a), which is true ***A1***

OR

$\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}\sqrt{k}+1}{\sqrt{k+1}}$ ***A1***

$> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \sqrt{k+1}$ ***A1***

THEN

so true for $n = 2$ and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \geq 2$
R1

Note: Award ***R1*** only if all previous ***M*** marks have been awarded.

[9 marks]

Total [13 marks]

5. Use mathematical induction to prove that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{px}) = p^{n-1}(px+n)\mathrm{e}^{px}$ *[7 marks]*
for $n \in \mathbb{Z}^+,\ p \in \mathbb{Q}$.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$n = 1:\ \text{LHS} = \frac{\mathrm{d}\,(x\mathrm{e}^{px})}{\mathrm{d}x} = xp\mathrm{e}^{px} + \mathrm{e}^{px} = (px+1)\mathrm{e}^{px},\ \text{RHS} = p^0(px+1)e^{px}$

$\text{LHS} = \text{RHS}$ so true for $n = 1$: ***A1***

Note: Award ***A1*** if $n = 0$ is proved.

assume proposition true for $n = k$, i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x\mathrm{e}^{px}) = p^{k-1}(px+k)\mathrm{e}^{px}$ ***M1***

Notes: Do not award ***M1*** if using n instead of k.
Assumption of truth must be present.
Subsequent marks are not dependent on this ***M1*** mark.

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(x\mathrm{e}^{px}) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x\mathrm{e}^{px})\right)$ ***(M1)***

$= \frac{\mathrm{d}}{\mathrm{d}x}\left(p^{k-1}(px+k)\mathrm{e}^{px}\right)$ ***M1***

$= p^{k-1}(px+k)p\mathrm{e}^{px} + \mathrm{e}^{px}\left(p^k\right)$

$= p^k(px+k)\mathrm{e}^{px} + \mathrm{e}^{px}\left(p^k\right)$ ***A1***

Note: Award ***A1*** for correct derivative.

$= p^k(px+k+1)\mathrm{e}^{px}$ ***A1***

$= p^{((k+1)-1)}(px+(k+1))\mathrm{e}^{px}$

Note: The final ***A1*** can be awarded for either of the two lines above.

hence true for $n = 1$ and $n = k$ true $\Rightarrow n = k+1$ true ***R1***

therefore true for all $n \in \mathbb{Z}^+$

Note: Only award the final ***R1*** if the three method marks have been awarded.

[7 marks]

6. Consider the function $f\left(x\right) = x\,\mathrm{e}^{2x}$, where $x \in \mathbb{R}$. The n^{th} derivative of $f\left(x\right)$ is denoted by $f^{(n)}\left(x\right)$. *[7 marks]*

Prove, by mathematical induction, that $f^{(n)}\left(x\right) = \left(2^n x + n2^{n-1}\right)\mathrm{e}^{2x}$, $n \in \mathbb{Z}^+$.

Markscheme

$f'\left(x\right) = \mathrm{e}^{2x} + 2x\mathrm{e}^{2x}$ ***A1***

Note: This must be obtained from the candidate differentiating $f\left(x\right)$.

$= \left(2^1 x + 1 \times 2^{1-1}\right)\mathrm{e}^{2x}$ ***A1***

(hence true for $n = 1$)

assume true for $n = k$: ***M1***

$f^{(k)}\left(x\right) = \left(2^k x + k2^{k-1}\right)\mathrm{e}^{2x}$

Note: Award ***M1*** if truth is assumed. Do not allow "let $n = k$".

consider $n = k + 1$:

$f^{(k+1)}\left(x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(2^k x + k2^{k-1}\right)\mathrm{e}^{2x}\right)$

attempt to differentiate $f^{(k)}\left(x\right)$ ***M1***

$f^{(k+1)}\left(x\right) = 2^k\mathrm{e}^{2x} + 2\left(2^k x + k2^{k-1}\right)\mathrm{e}^{2x}$ ***A1***

$f^{(k+1)}\left(x\right) = \left(2^k + 2^{k+1}x + k2^k\right)\mathrm{e}^{2x}$

$f^{(k+1)}\left(x\right) = \left(2^{k+1}x + \left(k+1\right)2^k\right)\mathrm{e}^{2x}$ ***A1***

$= \left(2^{k+1}x + \left(k+1\right)2^{(k+1)-1}\right)\mathrm{e}^{2x}$

True for $n = 1$ and $n = k$ true implies true for $n = k + 1$.

Therefore the statement is true for all $n \left(\in \mathbb{Z}^+\right)$ ***R1***

Note: Do not award final ***R1*** if the two previous ***M1s*** are not awarded. Allow full marks for candidates who use the base case $n = 0$.

[7 marks]

7a. Solve the inequality $x^2 > 2x + 1$. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$x < -0.414,\ x > 2.41$ ***A1A1***

$\left(x < 1 - \sqrt{2},\ x > 1 + \sqrt{2}\right)$

Note: Award ***A1*** for −0.414, 2.41 and ***A1*** for correct inequalities.

[2 marks]

7b. Use mathematical induction to prove that $2^{n+1} > n^2$ for $n \in \mathbb{Z}$, $n \geqslant 3$. *[7 marks]*

Markscheme

check for $n = 3$,

16 > 9 so true when $n = 3$ ***A1***

assume true for $n = k$

$2^{k+1} > k^2$ ***M1***

Note: Award ***M0*** for statements such as “let $n = k$”.

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

prove true for $n = k + 1$

$2^{k+2} = 2 \times 2^{k+1}$

$> 2k^2$ ***M1***

$= k^2 + k^2$ ***(M1)***

$> k^2 + 2k + 1$ (from part (a)) ***A1***

which is true for k ≥ 3 ***R1***

Note: Only award the ***A1*** or the ***R1*** if it is clear why. Alternate methods are possible.

$= (K + 1)^2$

hence if true for $n = k$ true for $n = k + 1$, true for $n = 3$ so true for all n ≥ 3 ***R1***

Note: Only award the final ***R1*** provided at least three of the previous marks are awarded.

[7 marks]

8. *[6 marks]*

Use mathematical induction to prove that $\sum_{r=1}^{n} r\left(r!\right)=\left(n+1\right)!-1$, for $n\in\mathbb{Z}^{+}$.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

consider $n=1$. $1\left(1!\right)=1$ and $2!-1=1$ therefore true for $n=1$ ***R1***

Note: There must be evidence that $n=1$ has been substituted into both expressions, or an expression such LHS=RHS=1 is used. "therefore true for $n=1$" or an equivalent statement must be seen.

assume true for $n=k$, (so that $\sum_{r=1}^{k} r\left(r!\right)=\left(k+1\right)!-1$) ***M1***

Note: Assumption of truth must be present.

consider $n=k+1$

$\sum_{r=1}^{k+1} r\left(r!\right)=\sum_{r=1}^{k} r\left(r!\right)+\left(k+1\right)\left(k+1\right)!$ ***(M1)***

$=\left(k+1\right)!-1+\left(k+1\right)\left(k+1\right)!$ ***A1***

$=\left(k+2\right)\left(k+1\right)!-1$ ***M1***

Note: ***M1*** is for factorising $\left(k+1\right)!$

$=\left(k+2\right)!-1$

$=\left(\left(k+1\right)+1\right)!-1$

so if true for $n=k$, then also true for $n=k+1$, and as true for $n=1$ then true for all $n\left(\in\mathbb{Z}^{+}\right)$ ***R1***

Note: Only award final ***R1*** if all three method marks have been awarded. Award ***R0*** if the proof is developed from both LHS and RHS.

[6 marks]

9. Use the principle of mathematical induction to prove that *[7 marks]*

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}$, where $n\in\mathbb{Z}^+$.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

if $n=1$

$\text{LHS}=1\,;\,\text{RHS}=4-\frac{3}{2^0}=4-3=1$ ***M1***

hence true for $n=1$

assume true for $n=k$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on the first two ***M1*** marks.

so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

if $n=k+1$

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

finding a common denominator for the two fractions ***M1***

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$

$=4-\frac{2(k+2)-(k+1)}{2^k}=4-\frac{k+3}{2^k}\left(=4-\frac{(k+1)+2}{2^{(k+1)-1}}\right)$ ***A1***

hence if true for $n=k$ then also true for $n=k+1$, as true for $n=1$, so true (for all $n\in\mathbb{Z}^+$) ***R1***

Note: Award the final ***R1*** only if the first four marks have been awarded.

[7 marks]

10. Use mathematical induction to prove that $(1-a)^n>1-na$ for $\{n:n\in\mathbb{Z}^+,\ n\geqslant 2\}$ where $0<a<1$. *[7 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

Let P_n be the statement: $(1-a)^n > 1 - na$ for some $n \in \mathbb{Z}^+$, $n \geqslant 2$ where $0 < a < 1$ consider the case $n = 2$: $(1-a)^2 = 1 - 2a + a^2$ ***M1***

$> 1 - 2a$ because $a^2 < 0$. Therefore P_2 is true ***R1***

assume P_n is true for some $n = k$

$(1-a)^k > 1 - ka$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on this ***M1***.

EITHER

consider $(1-a)^{k+1} = (1-a)(1-a)^k$ ***M1***

$> 1 - (k+1)a + ka^2$ ***A1***

$> 1 - (k+1)a \Rightarrow \mathrm{P}_{k+1}$ is true (**as** $ka^2 > 0$) ***R1***

OR

multiply both sides by $(1-a)$ (which is positive) ***M1***

$(1-a)^{k+1} > (1-ka)(1-a)$

$(1-a)^{k+1} > 1 - (k+1)a + ka^2$ ***A1***

$(1-a)^{k+1} > 1 - (k+1)a \Rightarrow \mathrm{P}_{k+1}$ is true (**as** $ka^2 > 0$) ***R1***

THEN

P_2 is true P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true so P_n true for all $n > 2$ (or equivalent) ***R1***

Note: Only award the last ***R1*** if at least four of the previous marks are gained including the ***A1***.

[7 marks]

Consider the function $f_n(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^n x), n \in \mathbb{Z}^+$.

11a. Determine whether f_n is an odd or even function, justifying your answer. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

even function ***A1***

since $\cos kx = \cos(-kx)$ **and** $f_n(x)$ is a product of even functions ***R1***

OR

even function ***A1***

since $(\cos 2x)(\cos 4x)\ldots = (\cos(-2x))\,(\cos(-4x))\ldots$ ***R1***

Note: Do not award ***A0R1***.

[2 marks]

11b. By using mathematical induction, prove that *[8 marks]*

$f_n(x) = \frac{\sin 2^{n+1}x}{2^n \sin 2x},\ x \neq \frac{m\pi}{2}$ where $m \in \mathbb{Z}$.

Markscheme

consider the case $n = 1$

$\frac{\sin 4x}{2 \sin 2x} = \frac{2 \sin 2x \cos 2x}{2 \sin 2x} = \cos 2x$ ***M1***

hence true for $n = 1$ ***R1***

assume true for $n = k$, *ie*, $(\cos 2x)(\cos 4x) \ldots (\cos 2^k x) = \frac{\sin 2^{k+1} x}{2^k \sin 2x}$ ***M1***

Note: Do not award ***M1*** for "let $n = k$" or "assume $n = k$" or equivalent.

consider $n = k + 1$:

$f_{k+1}(x) = f_k(x)(\cos 2^{k+1} x)$ ***(M1)***

$= \frac{\sin 2^{k+1} x}{2^k \sin 2x} \cos 2^{k+1} x$ ***A1***

$= \frac{2 \sin 2^{k+1} x \cos 2^{k+1} x}{2^{k+1} \sin 2x}$ ***A1***

$= \frac{\sin 2^{k+2} x}{2^{k+1} \sin 2x}$ ***A1***

so $n = 1$ true and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \in \mathbb{Z}^+$
R1

Note: To obtain the final ***R1***, all the previous ***M*** marks must have been awarded.

[8 marks]

11c. Hence or otherwise, find an expression for the derivative of $f_n(x)$ with respect to x. *[3 marks]*

Markscheme

attempt to use $f' = \frac{vu' - uv'}{v^2}$ (or correct product rule) ***M1***

$f_n'(x) = \frac{(2^n \sin 2x)(2^{n+1} \cos 2^{n+1} x) - (\sin 2^{n+1} x)(2^{n+1} \cos 2x)}{(2^n \sin 2x)^2}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[3 marks]

11d. Show that, for $n > 1$, the equation of the tangent to the curve $y = f_n(x)$ at $x = \frac{\pi}{4}$ is $4x - 2y - \pi = 0$. *[8 marks]*

Markscheme

$f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n \sin \frac{\pi}{2}\right)\left(2^{n+1} \cos 2^{n+1} \frac{\pi}{4}\right) - \left(\sin 2^{n+1} \frac{\pi}{4}\right)\left(2^{n+1} \cos \frac{\pi}{2}\right)}{\left(2^n \sin \frac{\pi}{2}\right)^2}$ ***(M1)(A1)***

$f_n'\left(\frac{\pi}{4}\right) = \frac{(2^n)\left(2^{n+1} \cos 2^{n+1} \frac{\pi}{4}\right)}{(2^n)^2}$ ***(A1)***

$= 2 \cos 2^{n+1} \frac{\pi}{4} \ (= 2 \cos 2^{n-1} \pi)$ ***A1***

$f_n'\left(\frac{\pi}{4}\right) = 2$ ***A1***

$f_n\left(\frac{\pi}{4}\right) = 0$ ***A1***

Note: This ***A*** mark is independent from the previous marks.

$y = 2\left(x - \frac{\pi}{4}\right)$ ***M1A1***

$4x - 2y - \pi = 0$ ***AG***

[8 marks]

12. Use the method of mathematical induction to prove that $4^n + 15n - 1$ is divisible by 9 for $n \in \mathbb{Z}^+$. *[6 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

let $P(n)$ be the proposition that $4^n + 15n - 1$ is divisible by 9

showing true for $n = 1$ ***A1***

*ie*for $n = 1,\ 4^1 + 15 \times 1 - 1 = 18$

which is divisible by 9, therefore $P(1)$ is true

assume $P(k)$ is true so $4^k + 15k - 1 = 9A,\ (A \in \mathbb{Z}^+)$ ***M1***

Note: Only award ***M1*** if "truth assumed" or equivalent.

consider $4^{k+1} + 15(k+1) - 1$

$= 4 \times 4^k + 15k + 14$

$= 4(9A - 15k + 1) + 15k + 14$ ***M1***

$= 4 \times 9A - 45k + 18$ ***A1***

$= 9(4A - 5k + 2)$ which is divisible by 9 ***R1***

Note: Award ***R1*** for either the expression or the statement above.

since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, therefore (by the principle of mathematical induction) $P(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if the 2 ***M1***s have been awarded.

[6 marks]

13. Prove by mathematical induction that *[9 marks]*

$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n-1}{2} = \binom{n}{3}$, where $n \in \mathbb{Z}, n \geqslant 3$

.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$$

show true for $n=3$ ***(M1)***

$\mathrm{LHS}=\binom{2}{2}=1\ \mathrm{RHS}=\binom{3}{3}=1$ ***A1***

hence true for $n=3$

assume true for $n=k:\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}=\binom{k}{3}$

M1

consider for $n=k+1:\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}$

(M1)

$=\binom{k}{3}+\binom{k}{2}$ ***A1***

$=\frac{k!}{(k-3)!3!}+\frac{k!}{(k-2)!2!}\ \left(=\frac{k!}{3!}\left[\frac{1}{(k-3)!}+\frac{3}{(k-2)!}\right]\right)$ or any correct expression with a visible common factor ***(A1)***

$=\frac{k!}{3!}\left[\frac{k-2+3}{(k-2)!}\right]$ or any correct expression with a common denominator ***(A1)***

$=\frac{k!}{3!}\left[\frac{k+1}{(k-2)!}\right]$

Note: At least one of the above three lines or equivalent must be seen.

$=\frac{(k+1)!}{3!(k-2)!}$ or equivalent ***A1***

$=\binom{k+1}{3}$

Result is true for $k=3$. If result is true for k it is true for $k+1$. Hence result is true for all $k\geqslant 3$. Hence proved by induction. ***R1***

Note: In order to award the ***R1*** at least ***[5 marks]*** must have been awarded.

[9 marks]

14a. Find the value of $\sin\frac{\pi}{4}+\sin\frac{3\pi}{4}+\sin\frac{5\pi}{4}+\sin\frac{7\pi}{4}+\sin\frac{9\pi}{4}$. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\sin\frac{\pi}{4}+\sin\frac{3\pi}{4}+\sin\frac{5\pi}{4}+\sin\frac{7\pi}{4}+\sin\frac{9\pi}{4}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$ ***(M1)A1***

Note: Award ***M1*** for 5 equal terms with \) + \) or $-$ signs.

[2 marks]

14b. Show that $\frac{1-\cos 2x}{2\sin x}\equiv\sin x,\ x\neq k\pi$ where $k\in\mathbb{Z}$. *[2 marks]*

Markscheme

$\frac{1-\cos 2x}{2\sin x}\equiv\frac{1-(1-2\sin^2 x)}{2\sin x}$ ***M1***

$\equiv\frac{2\sin^2 x}{2\sin x}$ ***A1***

$\equiv\sin x$ ***AG***

[2 marks]

14c. Use the principle of mathematical induction to prove that *[9 marks]*

$\sin x+\sin 3x+\ldots+\sin(2n-1)x=\frac{1-\cos 2nx}{2\sin x},\ n\in\mathbb{Z}^+,\ x\neq k\pi$ where $k\in\mathbb{Z}$.

Markscheme

let $\mathrm{P}(n)$: $\sin x + \sin 3x + \ldots + \sin(2n-1)x \equiv \frac{1-\cos 2nx}{2\sin x}$

if $n = 1$

$\mathrm{P}(1)$: $\frac{1-\cos 2x}{2\sin x} \equiv \sin x$ which is true (as proved in part (b)) ***R1***

assume $\mathrm{P}(k)$ true, $\sin x + \sin 3x + \ldots + \sin(2k-1)x \equiv \frac{1-\cos 2kx}{2\sin x}$ ***M1***

Notes: Only award ***M1*** if the words "assume" and "true" appear. Do not award ***M1*** for "let $n = k$" only. Subsequent marks are independent of this ***M1***.

consider $\mathrm{P}(k+1)$:

$\mathrm{P}(k+1)$: $\sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x \equiv \frac{1-\cos 2(k+1)x}{2\sin x}$

$LHS = \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x$ ***M1***

$\equiv \frac{1-\cos 2kx}{2\sin x} + \sin(2k+1)x$ ***A1***

$\equiv \frac{1-\cos 2kx + 2\sin x \sin(2k+1)x}{2\sin x}$

$\equiv \frac{1-\cos 2kx + 2\sin x \cos x \sin 2kx + 2\sin^2 x \cos 2kx}{2\sin x}$ ***M1***

$\equiv \frac{1-\left((1-2\sin^2 x)\cos 2kx - \sin 2x \sin 2kx\right)}{2\sin x}$ ***M1***

$\equiv \frac{1-(\cos 2x \cos 2kx - \sin 2x \sin 2kx)}{2\sin x}$ ***A1***

$\equiv \frac{1-\cos(2kx+2x)}{2\sin x}$ ***A1***

$\equiv \frac{1-\cos 2(k+1)x}{2\sin x}$

so if true for $n = k$, then also true for $n = k+1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Accept answers using transformation formula for product of sines if steps are shown clearly.

Note: Award ***R1*** only if candidate is awarded at least 5 marks in the previous steps.

[9 marks]

14d. Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$. *[6 marks]*

Markscheme

EITHER

$\sin x + \sin 3x = \cos x \Rightarrow \frac{1 - \cos 4x}{2 \sin x} = \cos x$ ***M1***

$\Rightarrow 1 - \cos 4x = 2 \sin x \cos x, \ (\sin x \neq 0)$ ***A1***

$\Rightarrow 1 - (1 - 2 \sin^2 2x) = \sin 2x$ ***M1***

$\Rightarrow \sin 2x(2 \sin 2x - 1) = 0$ ***M1***

$\Rightarrow \sin 2x = 0$ or $\sin 2x = \frac{1}{2}$ ***A1***

$2x = \pi, \ 2x = \frac{\pi}{6}$ and $2x = \frac{5\pi}{6}$

OR

$\sin x + \sin 3x = \cos x \Rightarrow 2 \sin 2x \cos x = \cos x$ ***M1A1***

$\Rightarrow (2 \sin 2x - 1) \cos x = 0, \ (\sin x \neq 0)$ ***M1A1***

$\Rightarrow \sin 2x = \frac{1}{2}$ of $\cos x = 0$ ***A1***

$2x = \frac{\pi}{6}, \ 2x = \frac{5\pi}{6}$ and $x = \frac{\pi}{2}$

THEN

$\therefore x = \frac{\pi}{2}, \ x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$ ***A1***

Note: Do not award the final ***A1*** if extra solutions are seen.

[6 marks]

15a. Use de Moivre's theorem to find the value of $\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i} \sin\left(\frac{\pi}{3}\right)\right)^3$. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3 = \cos\pi + \mathrm{i}\sin\pi$ ***M1***

$= -1$ ***A1***

[2 marks]

15b. Use mathematical induction to prove that *[6 marks]*

$$(\cos\theta - \mathrm{i}\sin\theta)^n = \cos n\theta - \mathrm{i}\sin n\theta \text{ for } n \in \mathbb{Z}^+.$$

Markscheme

show the expression is true for $n = 1$ ***R1***

assume true for $n = k,\ (\cos\theta - \mathrm{i}\sin\theta)^k = \cos k\theta - \mathrm{i}\sin k\theta$ ***M1***

Note: Do not accept "let $n = k$" or "assume $n = k$", assumption of truth must be present.

$(\cos\theta - \mathrm{i}\sin\theta)^{k+1} = (\cos\theta - \mathrm{i}\sin\theta)^k(\cos\theta - \mathrm{i}\sin\theta)$

$= (\cos k\theta - \mathrm{i}\sin k\theta)(\cos\theta - \mathrm{i}\sin\theta)$ ***M1***

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta - \mathrm{i}(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$ ***A1***

Note: Award ***A1*** for any correct expansion.

$= \cos((k+1)\theta) - \mathrm{i}\sin((k+1)\theta)$ ***A1***

therefore if true for $n = k$ true for $n = k + 1$, true for $n = 1$, so true for all $n(\in \mathbb{Z}^+)$ ***R1***

Note: To award the final ***R*** mark the first 4 marks must be awarded.

[6 marks]

Let $z = \cos\theta + i\sin\theta$.

15c. Find an expression in terms of θ for $(z)^n + (z^*)^n,\ n \in \mathbb{Z}^+$ where z^* is the complex conjugate of z. *[2 marks]*

Markscheme

$(z)^n + (z^*)^n = (\cos\theta + \mathrm{i}\sin\theta)^n + (\cos\theta - \mathrm{i}\sin\theta)^n$

$= \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta = 2\cos(n\theta)$ ***(M1)A1***

[2 marks]

15d. (i) Show that $zz^* = 1$. *[5 marks]*

(ii) Write down the binomial expansion of $(z + z^*)^3$ in terms of z and z^*.

(iii) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Markscheme

(i) $zz* = (\cos\theta + \mathrm{i}\sin\theta)(\cos\theta - \mathrm{i}\sin\theta)$

$= \cos^2\theta + \sin^2\theta$ ***A1***

$= 1$ ***AG***

Note: Allow justification starting with $|z| = 1$.

(ii) $(z + z^*)^3 = z^3 + 3z^2z^* + 3z(z^*)^2 + (z^*)3 \left(= z^3 + 3z + 3z^* + (z^*)^3\right)$

A1

(iii) $(z + z^*)^3 = (2\cos\theta)^3$ ***A1***

$z^3 + 3z + 3z^* + (z^*)^3 = 2\cos 3\theta + 6\cos\theta$ ***M1A1***

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ ***AG***

Note: ***M1*** is for using $zz^* = 1$, this might be seen in d(ii).

[5 marks]

15e. Hence solve $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$ for $0 \leqslant \theta < \pi$. *[6 marks]*

Markscheme

$4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = 2\cos^2\theta - 1$

$\cos(3\theta) = \cos(2\theta)$ ***A1A1***

Note: ***A1*** for $\cos(3\theta)$ and ***A1*** for $\cos(2\theta)$.

$\theta = 0$ ***A1***

or $3\theta = 2\pi - 2\theta$ $(\text{or } 3\theta = 4\pi - 2\theta)$ ***M1***

$\theta = \frac{2\pi}{5}, \frac{4\pi}{5}$ ***A1A1***

Note: Do not accept solutions via factor theorem or other methods that do not follow “hence”.

[6 marks]

16. Use mathematical induction to prove that $n(n^2+5)$ is divisible by 6 for $n \in \mathbb{Z}^+$. *[8 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

let $\mathrm{P}(n)$ be the proposition that $n(n^2+5)$ is divisible by 6 for $n \in \mathbb{Z}^+$

consider P(1):

when $n = 1,\ n(n^2+5) = 1 \times (1^2+5) = 6$ and so P(1) is true ***R1***

assume $\mathrm{P}(k)$ is true *ie*, $k(k^2+5) = 6m$ where $k,\ m \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1)$:

$(k+1)\left((k+1)^2+5\right)$ ***M1***

$= (k+1)(k^2+2k+6)$

$= k^3+3k^2+8k+6$ ***(A1)***

$= (k^3+5k)+(3k^2+3k+6)$ ***A1***

$= k(k^2+5)+3k(k+1)+6$ ***A1***

$k(k+1)$ is even hence all three terms are divisible by 6 ***R1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and P(1) is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

[8 marks]

17a. Show that $\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta$. *[1 mark]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$ ***M1***

$= \cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

17b. Consider $f(x) = \sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$ where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. *[7 marks]*

Markscheme

consider $n = 1,\ f'(x) = a\cos(ax)$ ***M1***

since $\sin\left(ax + \frac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$ ***R1***

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x) = \frac{\mathrm{d}\left(f^{(k)}(x)\right)}{\mathrm{d}x} \quad \left(= a\left(a^k \cos\left(ax + \frac{k\pi}{2}\right)\right)\right)$ ***M1***

$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ (using part (a)) ***A1***

$= a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior ***M*** and ***R*** marks have been awarded.

[7 marks]

Total [8 marks]

Let $y(x) = xe^{3x}, x \in \mathbb{R}$.

18a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = \left(\mathrm{e}^{3x} + 3x\mathrm{e}^{3x}\right)$ ***M1A1***

[2 marks]

18b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. *[7 marks]*

Markscheme

let $P(n)$ be the statement $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ which is $1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x}$ and RHS is $3^0\mathrm{e}^{3x} + x3^1\mathrm{e}^{3x}$ ***R1***

as $\mathrm{LHS} = \mathrm{RHS}$, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = k3^{k-1}\mathrm{e}^{3x} + x3^k\mathrm{e}^{3x}$

$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)$ ***(M1)***

$= k3^{k-1} \times 3\mathrm{e}^{3x} + 1 \times 3^k\mathrm{e}^{3x} + x3^k \times 3\mathrm{e}^{3x}$ ***A1***

$= (k+1)3^k\mathrm{e}^{3x} + x3^{k+1}\mathrm{e}^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true $(\forall n \in \mathbb{Z}^+)$ ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

18c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. *[5 marks]*

Justify whether any such point is a maximum or a minimum.

Markscheme

$e^{3x} + x \times 3e^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ ***M1A1***

point is $\left(-\frac{1}{3},\ -\frac{1}{3e}\right)$ ***A1***

EITHER

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2e^{3x}$

when $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} > 0$ therefore the point is a minimum ***M1A1***

OR

x	$-\frac{1}{3}$
$\frac{dy}{dx}$	$-ve$ 0 $+ve$

nature table shows point is a minimum ***M1A1***

[5 marks]

18d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. *[5 marks]*

Markscheme

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2e^{3x}$ ***A1***

$2 \times 3e^{3x} + x \times 3^2e^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ ***M1A1***

point is $\left(-\frac{2}{3},\ -\frac{2}{3e^2}\right)$ ***A1***

x	$-\frac{2}{3}$
$\frac{d^2y}{dx^2}$	$-ve$ 0 $+ve$

since the curvature does change (concave down to concave up) it is a point of inflection ***R1***

Note: Allow 3^{rd} derivative is not zero at $-\frac{2}{3}$

[5 marks]