Proof By Contradiction MS (From Edexcel Sample Papers)

Q1	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. 'Assumption: given a rational number *a* and an irrational number *b*, assume that $a - b$ is rational.'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Sets up the proof by defining the different rational and irrational numbers. The choice of variables does not matter. Let $a = \frac{m}{n}$ As we are assuming $a - b$ is rational, let $a - b = \frac{p}{q}$ So $a - b = \frac{p}{q} \Rightarrow \frac{m}{n} - b = \frac{p}{q}$	**M1**	2.2a	
	Solves $\frac{m}{n} - b = \frac{p}{q}$ to make *b* the subject and rewrites the resulting expression as a single fraction: $\frac{m}{n} - b = \frac{p}{q} \Rightarrow b = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}$	**M1**	1.1b	
	Makes a valid conclusion. $b = \frac{mq - pn}{nq}$, which is rational, contradicts the assumption *b* is an irrational number. Therefore the difference of a rational number and an irrational number is irrational.	**B1**	2.4	
				(4 marks)

Q2	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. ‘Assumption: there exists a product of two odd numbers that is even.’	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Defines two odd numbers. Can choose any two different variables. ‘Let 2*m* + 1 and 2*n* + 1 be our two odd numbers.’	**B1**	2.2a	
	Successfully multiplies the two odd numbers together: $(2m+1)(2n+1) \equiv 4mn+2m+2n+1$	**M1**	1.1b	
	Factors the expression and concludes that this number must be odd. $4mn+2m+2n+1 \equiv 2(2mn+m+n)+1$ $2(2mn+m+n)$ is even, so $2(2mn+m+n)+1$ must be odd.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption that the product of two odd numbers is even, therefore the product of two odd numbers is odd.	**B1**	2.4	
				(5 marks)

Notes

Alternative method

Assume the opposite is true: there exists a product of two odd numbers that is even. (**B1)**

If the product is even then 2 is a factor. (**B1)**

So 2 is a factor of at least one of the two numbers. (**M1)**

So at least one of the two numbers is even. (**M1)**

This contradicts the statement that both numbers are odd. (**B1)**

Q3	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. ‘Assumption: there exists a number n such that n is odd and $n^3 + 1$ is also odd.’	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Defines an odd number. ‘Let $2k + 1$ be an odd number.’	**B1**	2.2a	
	Successfully calculates $(2k+1)^3+1$ $(2k+1)^3+1 \equiv \left(8k^3+12k^2+6k+1\right)+1 \equiv 8k^3+12k^2+6k+2$	**M1**	1.1b	
	Factors the expression and concludes that this number must be even. $8k^3+12k^2+6k+2 \equiv 2\left(4k^3+6k^2+3k+1\right)$ $2\left(4k^3+6k^2+3k+1\right)$ is even.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption that there exists a number n such that n is odd and $n^3 + 1$ is also odd, so if n is odd, then $n^3 + 1$ is even.	**B1**	2.4	

(5 marks)

Notes

Alternative method

Assume the opposite is true: there exists a number n such that n is odd and $n^3 + 1$ is also odd. (**B1**)

If $n^3 + 1$ is odd, then n^3 is even. (**B1**)

So 2 is a factor of n^3. (**M1**)

This implies 2 is a factor of n. (**M1**)

This contradicts the statement n is odd. (**B1**)

Q4	Scheme	Marks	AOs	
(a)	Begins the proof by assuming the opposite is true. ‘Assumption: there exists a number n such that n^2 is even and n is odd.’	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Defines an odd number (choice of variable is not important) and successfully calculates n^2 Let $2k + 1$ be an odd number. $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$	**M1**	2.2a	
	Factors the expression and concludes that this number must be odd. $4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1$, so n^2 is odd.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption n^2 is even. Therefore if n^2 is even, n must be even.	**B1**	2.4	
		(4)		
(b)	Begins the proof by assuming the opposite is true. ‘Assumption: $\sqrt{2}$ is a rational number.’	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Defines the rational number: $\sqrt{2} = \frac{a}{b}$ for some integers a and b, where a and b have no common factors.	**M1**	2.2a	
	Squares both sides and concludes that a is even: $\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$ From part **a**: a^2 is even implies that a is even.	**M1**	1.1b	
	Further states that if a is even, then $a = 2c$. Choice of variable is not important.	**M1**	1.1b	
	Makes a substitution and works through to find $b^2 = 2c^2$, concluding that b is also even. $a^2 = 2b^2 \Rightarrow (2c)^2 = 2b^2 \Rightarrow 4c^2 = 2b^2 \Rightarrow b^2 = 2c^2$ From part **a**: b^2 is even implies that b is even.	**M1**	1.1b	
	Makes a valid conclusion.	**B1**	2.4	

	If a and b are even, then they have a common factor of 2, which contradicts the statement that a and b have no common factors. Therefore $\sqrt{2}$ is an irrational number.			
		(6)		
				(10 marks)
Q5	**Scheme**	**Marks**	**AOs**	**Pearson Progression Step and Progress descriptor**
	Begins the proof by assuming the opposite is true. 'Assumption: there exists a rational number $\frac{a}{b}$ such that $\frac{a}{b}$ is the greatest positive rational number.'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Makes an attempt to consider a number that is clearly greater than $\frac{a}{b}$: 'Consider the number $\frac{a}{b}+1$, which must be greater than $\frac{a}{b}$ '	**M1**	2.2a	
	Simplifies $\frac{a}{b}+1$ and concludes that this is a rational number. $\frac{a}{b}+1 \equiv \frac{a}{b}+\frac{b}{b} \equiv \frac{a+b}{b}$ By definition, $\frac{a+b}{b}$ is a rational number.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption that there exists a greatest positive rational number, so we can conclude that there is not a greatest positive rational number.	**B1**	2.4	
				(4 marks)

Q6	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. 'Assumption: there do exist integers *a* and *b* such that $25a + 15b = 1$'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Understands that $25a + 15b = 1 \Rightarrow 5a + 3b = \frac{1}{5}$ 'As both 25 and 15 are multiples of 5, divide both sides by 5 to leave $5a + 3b = \frac{1}{5}$'	**M1**	2.2a	
	Understands that if *a* and *b* are integers, then $5a$ is an integer, $3b$ is an integer and $5a + 3b$ is also an integer.	**M1**	1.1b	
	Recognises that this contradicts the statement that $5a + 3b = \frac{1}{5}$, as $\frac{1}{5}$ is not an integer. Therefore there do not exist integers *a* and *b* such that $25a + 15b = 1$'	**B1**	2.4	
				(4 marks)