The Variational Principle

3.1: Fermat’s Principle

Fermat’s principles of least time: light travels between two points alone the path that requires the least time, as compared to other nearby paths.

Snell’s Law:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{1}$$

The refraction index of a medium n is defined as $n(r) = C/v$, where C is the speed of light in vacuum and v is the speed of light in the new medium. Then, speed of light in a medium is equal to $v = C/n(r)$, where $n(x, r, z) = n(r)$ can be a continuous function. The time required to travel a distance ds at speed v is given by:

$$dt = \frac{\text{distance}}{\text{speed}} = \frac{ds}{v}$$

The total time required to travel from a to b is given by:

$$t = \int dt = t \int \frac{d}{C/n(r)} = \frac{1}{C} \int \frac{ds}{n(r)} \tag{2}$$

💡 **Fermat’s principles of least time:** light travels between two points alone the path that requires the least time, as compared to other nearby paths.

One question raises from Fermat’s principle of least time is: ***how do light rays know which path is closest ?*** The equation is answered by a method called the variation of calculus.

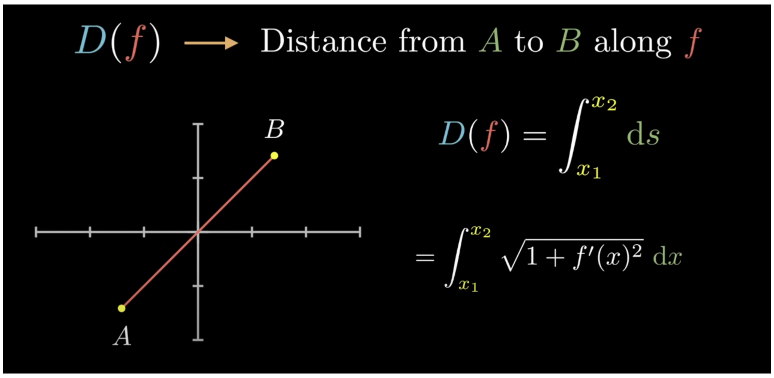
3.2: The Calculus of Variation

From the above section, we know that the time required to travel from points A to B is given by:

$$t = \frac{1}{C} \int n(r) ds$$

In the integral, ds can be expressed in terms of dx and dy, giving us $ds = \sqrt{dx^2 + dy^2}$. The refraction index $n(r) = n(x)$ because we are integrating alone that x coordinate. Then, the integral becomes:

$$t = \frac{1}{C} \int n(x) ds = \frac{1}{C} \int n(x) \sqrt{dx^2 + dy^2} = \frac{1}{C} \int n(x) \sqrt{1 + \frac{dy^2}{dx^2}} dx = \frac{1}{C} \int n(x) \sqrt{1 + y'^2} dx$$



The picture explains the integral $D(f)$, the distance from A to B alone f, where x coordinates varies from x_1 to x_2. Because we are integrating alone x coordinates (we use x coordinates to define/constrain the path), $n(r) = n(x)$

In this case, the integrand depends both on y' and dx, so the integral is rewritten as:

$$t = \frac{1}{C} \int n(x, y)\sqrt{1 + y'^2} dx = \int F(x, y(x), y'(x))dx$$

where

$$\frac{1}{C} n(x, y)\sqrt{1 + y'^2} = F(x, y(x), y'(x))$$

Example: Light Path Between Two Points in Glass

Consider first a very simple special case. Suppose the index of refraction *n* is *constant* throughout a sheet of glass, and that the endpoint of a light ray at $x = x_0$, $y = y_0$ is directly across the sheet from the beginning point at $x = 0$, $y = y_0$. Then the time to penetrate the sheet is what?

3.21: Euler's Method Derivation Part 1

More generally, Euler and Lagrange considered some arbitrary integrals in the form:

$$I = F(x, y(x), y'(x))$$

and wanted to solve paths $y(x)$ that maximum and minimized. In other words, they wanted to find the stationary points where $I' = 0$.

Suppose we have a function $f(x_1, x_2, \cdots) = f(x_i)$ of independent variables $i = 1, 2, 3, \cdots, N$, and we are asked to find the stationary points. At stationary points, $f'(x_i) = 0$, so $f(x_i)$ must be a constant. (This might sounds confusing at the first point. Consider functions $f(x_i) = 3x$ and $f(x_i) = 3$, their derivatives $f'(x_i) = 3 \neq 0$ and $f'(x_i) = 0$ $f(x_i)$ must be a constant. Thus, $f(x_i)$ must be a constant for $f'(x_i) = 0$). We represent a horizontal shift in $f(x_i)$ as $f(x_i + \delta x_i)$. When reaching a stationary point, the function is a constant, so $f(x_i) \approx f(x_i + \delta x_i)$, where $\delta x_i \to 0$. Using Taylor expansion, $f(x_i + \delta x_i)$ is equal to:

$$f(x_i + \delta x_i) = f(x_i) + \frac{\partial f}{\partial x_j}\delta x_j + \frac{1}{2!}\frac{\partial f}{\partial^2 x_j x_k}\delta x_j \delta x_k + \cdots$$

In order to the constant for $f(x_i + \delta x_i) = f(x_i)$, all other terms need to be 0. This is only possible if $\partial f/\partial x_j = 0$. Thus, when $y(x)$ reaches a stationary point,

$$\boxed{\frac{\partial f}{\partial x_j} = 0 \quad \text{with} \quad j = 1, 2, \cdots, N} \tag{3}$$

3.22: Euler's Equation Derivation Part 2

In order to find such paths, we introduce some new functions ϵ and η. Suppose $\bar{y}(x)$ is a path for stationary points and $y(x)$ is a path nearby. The path $\bar{y}(x)$ can be expressed using $y(x)$ plus a small difference.

$$y(x) = \bar{y}(x) + \epsilon\eta$$

At stationary points, $I' = 0$. Applying chain rule for multivariable functions, we integrate I with respect to ϵ, giving us:

$$\begin{aligned} 0 = \frac{dI}{d\epsilon} &= \int \frac{d}{d\epsilon}(F(x, y(x), y'(x))dx \\ &= \int \frac{\partial F}{x}\frac{\partial x}{\partial \epsilon} + \frac{\partial F}{y(x)}\frac{\partial y(x)}{\partial \epsilon} + \frac{\partial F}{\partial y'(x)}\frac{\partial y'(x)}{\partial \epsilon}dx \end{aligned}$$

Because $\partial f/\partial x_j = 0$, the expression becomes:

$$0 = \int \frac{\partial F}{y(x)}\frac{\partial y(x)}{\partial \epsilon} + \frac{\partial F}{\partial y'(x)}\frac{\partial y'(x)}{\partial \epsilon}$$

Integrating $y(x)$ and $y'(x)$ with respect to ϵ, we get:

$$\frac{\partial y(x)}{\epsilon} = \eta \quad \frac{\partial y'(x)}{\partial \epsilon} = \eta'$$

Then, the integral becomes:

$$0 = \int \frac{\partial F}{\partial y(x)}\eta + \frac{\partial F}{\partial y'(x)}\eta' dx$$

Using integrating by parts, we get:

$$\int \frac{\partial F}{\partial y'(x)}\eta' dx = \frac{\partial F}{\partial y'(x)}\int \eta' dx - \int \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)dx = \frac{\partial F}{\partial y'(x)}\eta - \int \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)dx$$

Because $\eta(x) = 0$, the equation is equal to:

$$\int \frac{\partial F}{\partial y'(x)}\eta' dx = -\int \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)dx$$

Then, the integral becomes:

$$0 = \int \frac{\partial F}{\partial y(x)}\eta dx - \int \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)dx = \int \left(\frac{\partial F}{\partial y(x)}dx - \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)\right)\eta dx$$

giving us

$$\boxed{\frac{\partial F}{\partial y(x)} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)} \tag{4}$$

The integrand is known us the Euler's equation.

$$\boxed{0 = \frac{\partial F}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)} \tag{5}$$

💡 This might be nuance, but we are getting used to $y' = dy'dt$. Note that in Euler's equation, $y' = dy/dx$.

Example: The Straight Wire

Minimize the equation

$$s = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + y'^2} dx$$

▼ `Solution`

In the problem $F = \sqrt{1 + y'^2}$. Because the integrand is independent of $y(x)$, $\partial F / \partial y(x) = 0$. The Euler's equation becomes:

$$0 = \frac{d}{dx} \left(\frac{\partial F}{\partial y'(x)} \right)$$

Then

$$\frac{\partial F}{\partial y'(x)} = \frac{2y'}{2\sqrt{1 + y'^2}} = \frac{y'}{\sqrt{1 + y'^2}} = \text{A constant}$$

The shortest distance on a plane between two points is a straight line (!),

3.3: Geodesics

One application of the calculus of variation is to find geodesics, which are the stationary (usually shortest) paths between two points on a given surface. Before diving into details, we will talk about cylindrical and spherical coordinates in Cartesian and polar forms.

3.31: Coordinates

Cylindrical Coordinates

$$\text{Cartesin} \quad (r\cos\theta, r\sin\theta, z)$$

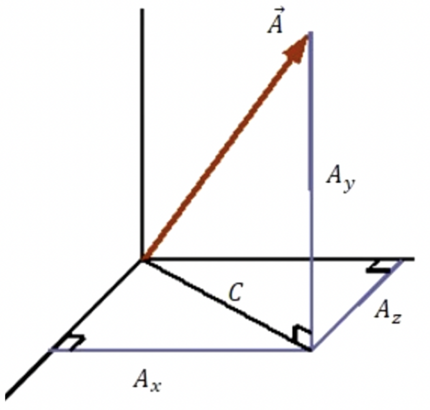
In polar form

$$\text{Polar} \quad s = (r\theta, z)$$

$$\text{Polar} \quad d\vec{s} = (r d\theta, z)$$

$$\text{Polar} \quad ds = \sqrt{r^2 d\theta^2 + z^2}$$

Spherical coordinates



By pythagorean theorem, the length of the red line is given by:

$$s = \sqrt{x^2 + y^2 + z^2}$$

where:

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\phi$$

For spherical coordinates, we divide a sphere into infinitesimal number of cubes.

$$s = rd\theta$$

Example: Geodesics on a Sphere

Consider the problem of finding the shortest distance between two points on the surface of a sphere. The sphere has a radius of R. Find the geodesics on a sphere

▼ `Solution`

There are two methods for representing the coordinates on the surface of a sphere. Both approaches give rise to the correct/same integral. The first approach is to use spherical coordinates, giving us:

$$x = R\cos\theta\sin\phi, y = R\sin\theta\sin\phi, z = R\cos\phi$$

Because we want to find ds, we differentiate x, y and z to find dx, dy, and dz using the chain rule for multivariable functions.

$$dx = -R\sin\theta\sin\phi d\theta + R\cos\theta\cos\phi d\phi$$

$$dy = R\cos\theta\sin\phi d\theta + R\sin\theta\cos\phi d\phi$$

$$dz = -R\sin\theta d\theta$$

Then, ds is equal to:

$$\begin{aligned} ds^2 &= R^2(\cos\theta\cos\phi d\theta - \sin\theta\sin\phi d\phi)^2 + R^2(\cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi)^2 + R^2\sin^2\phi d\theta^2 \\ &= R^2(\cos^2\theta\cos^2\phi d\theta^2 - 2\cos\theta\sin\theta\cos\phi\sin\phi d\theta d\phi + \sin^2\theta\sin^2\phi d\phi^2) + R^2(\cos^2\theta\sin^2\phi d\theta^2 + 2\cos\theta\sin (\\ &= R^2(\cos^2\theta(\cos^2\phi + \sin^2\phi)d\theta^2 + \sin^2\theta(\sin^2\theta + \cos^2\theta)d\phi + \sin^2\theta d\theta^2) \\ &= R^2(\cos^2\theta(\cos^2\phi + \sin^2\phi)d\theta^2 + \sin^2\theta(\sin^2\theta + \cos^2\theta)d\phi^2 + \sin^2\theta d\theta^2) \\ &= R^2(\cos^2\theta d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta d\theta^2) \\ &= R^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Then, ds is equal to:

$$ds = R\sqrt{(d\theta^2 + \sin^2\theta d\phi^2)} = R\sqrt{1 + \sin^2\theta(\frac{d\phi}{d\theta})^2}d\theta = R\sqrt{1 + \phi'^2\sin^2\theta}d\theta$$

From the equation, we can see that $F = F(\theta, \phi', \phi) = R\sqrt{1 + \phi'^2\sin^2\theta}$. Using Euler's equation, we know that:

$$0 = \frac{\partial F}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right)$$

Because F is independent of θ, $\partial F/\partial\phi = 0$, giving us

$$0 = \frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right)$$

Then, $\partial F/\partial\theta$' $= \mathrm{A\ constant} = k$. Differentiating F with respect to ϕ', we get:

$$k = \frac{1}{\sqrt{1+\phi'^2\sin^2\theta}}\sin^2\theta$$

Solving for ϕ'

Using separable integration, ϕ is equal to:

$$\phi = \alpha\sin^{-1} q + \beta$$

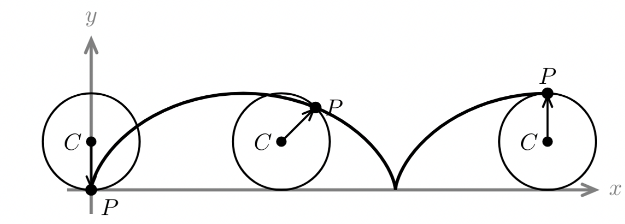
3.4: Brachistochrone

3.41: A Cycloid

The section is not included in the textbook, but it will be beneficial to know some math on cycloids for our understanding of the brachistochrone problem.

Cycloids

Cups:



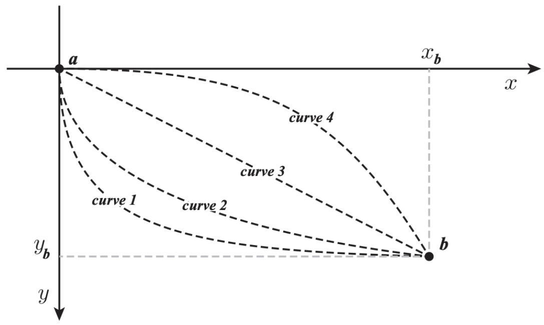
A graph of a cycloid.

In polar forms, a cycloid has equations of:

$$x = r(\theta - \sin\theta) \quad y = r(1-\cos\theta)$$

3.4.2: Time and Paths Taken for a Brachistochrone

The problem is to find the shape of a track between two given points, such that a small ball starting at rest at the upper point-and sliding without friction alone the track under the influence of gravity-arrives at the lower point in the shortest time.



Brachistochrone: The path between two points A and B which minimizes the time taking by a particle falling from A to B under the influence of gravity.

$$t = \int \frac{ds}{v}$$

Let point A be the origin of the coordinate system we define. Because the particle is dropped, it has an initial velocity and kinetic energy of zero. As the point is zero, gravitational potential energy is also zero. Thus, the initial, which is equal to the total, energy is zero. By energy conservation, we get:

$$E = \frac{1}{2}mv^2 - mgy = 0$$

The distance ds is given by:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx$$

Then, time t is equal to:

$$t = \int \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}dx \quad \text{or} \quad t = \int \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}}dy \tag{6}$$

Using Euler's equation, $t = t(y, x', x)$ and $F() = (1 + x'^2)/(2gy)$. Then,

$$0 = \frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial x'}$$

Because F is independent of x, $\partial F/\partial x = 0$, giving us:

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{2gy}}\frac{x'}{\sqrt{1 + x'^2}} = k$$

Solving for x', we get:

$$x' = \pm\sqrt{\frac{2gk^2y}{1 - 2k^2gy}} = \pm\sqrt{\frac{y}{1/2gk^2 - y}} \equiv \sqrt{\frac{y}{a - y}}$$

We choose the plus square root. Integrating over y, we get:

$$x = \int dx = \int \sqrt{\frac{y}{a - y}}dy$$

Note that we integrate with respect to y because $x' = x'(y)$ is a function of y, where y is the dependent variable. The integral is evaluated using substitution. Let $y = a\sin^2(\theta/2) = a/2(1 - \cos\theta)$, the integral is equal to:

$$x = \frac{a}{2}(\theta - \sin\theta)$$

Differentiating x with respect to θ, we get

$$x' = \frac{a}{2}(1 - \cos\theta)$$

Expressing dy in term of θ, we get:

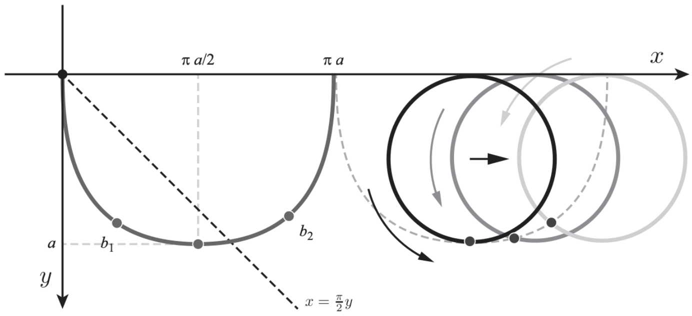
$$\frac{dy}{d\theta} = \frac{a}{2}(\sin\theta)$$

$$dy = \frac{a}{2}(\sin\theta)d\theta$$

Using separable integration, y is equal to:

$$y = \frac{a}{2}(1 - \cos\theta)$$

Notice that both x and y represent equations for a cycloid, the curve traced by a point on a circle as it rolls alone a straight line without slipping. The quantity a and final angle θ_b can be determined from coordinates (x_b, y_b) of the final position.



A graph of a cycloid. Note that top points of a cycloid represents a half of a complete revolution, where $\theta = \pi$. In order to see this, imaging a point on ground initially, when the points is at top, the circle has completed one half of a revolution, so $\theta = \pi$.

Substitute x or y into (6), the time t required to fall to the final position is:

$$t = \int \sqrt{\frac{1 + x'^2}{2gy}} dy = \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1 + a^2(1 - \cos\theta)^2/4}{a(1 - \cos\theta)/2}} \frac{a}{2}(1 + \sin\theta)d\theta$$

$$\boxed{t = \sqrt{\frac{a}{2g}} \int_0^{\theta_f} d\theta = \sqrt{\frac{a}{2g}}\theta_f} \tag{7}$$

Notice that if $(x_b, y_b) = (a\pi/2, a)$, time t is equal to:

$$t = \pi\sqrt{\frac{a}{2g}}$$

Example: Path Traveled by Lights in Atmosphere

We return to where we began the chapter, with Fermat's principle of stationary time, illustrated in Figure 3.9(a). Bringing to bear the calculus of variations, we can now find the path of a light ray in a medium like earth's atmosphere, where the index of refraction *n* is a continuous function of position. If a ray of light from a star descends through the atmosphere, it encounters an increasing density and an increasing index of refraction. We might therefore expect the ray to bend continuously, entering the atmosphere at some angle θ_a and reaching the ground at a steeper angle θ_b . For simplicity, take the earth to be essentially flat over the horizontal range of the ray and assume the index of refraction $n = n(y)$ only, where y is the vertical direction. The light travel time is then equal to what?

▼ `Solution`

Recall that refraction index of a medium $n(r) = C/v$. Then, velocity of light in air is equal to $v = C/n(r)$ and the infinitesimal distance $ds = \sqrt{1+x'^2}dy$. Time t is equal to:

$$t = \int \frac{ds}{v} = \frac{1}{C}\int n(y)\sqrt{1+x'^2}dy$$

In order to minimized the integral, we use Euler's equation, where $F = F(y, x', x) = n(y)\sqrt{1+y'^2}$. Because F is independent of x, we get:

$$k = \frac{\partial F}{\partial x'} = n(y)\frac{x'}{\sqrt{1+x'^2}}$$

The expression can be simplified using trigonometry. We know that $x' = dx/dy = \tan\theta$, and $\sqrt{1+\tan\theta} = \sec\theta$.(Note that $dx = dx/dy$. This is obtained when calculating $ds = \sqrt{dx^2+dy^2}$, where we divide the expression under the square root by dy^2 to get $ds = \sqrt{1+(dx/dy)^2}dy = \sqrt{1+(x')^2}dy$) Then, k is equal to:

$$k = n(x)\frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta}\cos\theta = n(y)\sin(\theta)$$

Doesn't this looks similar? We have derived Snell's law!

$$\boxed{k = n(y)\sin\theta}$$

3.5 Several Dependent Variables

$$\boxed{\frac{\partial F}{\partial y_i(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'(x)}\right) = 0}$$

Example: Geodesics in Three Dimension

Using Euler's equation for several dependent variable, find geodesics in three dimensions.

▼ Solution

Distance ds is equal to:

$$ds = \sqrt{dx^2+dy^2+dx^2} = \sqrt{1+(\frac{dy}{dx})^2+(\frac{dz}{dx})^2} = \sqrt{1+y'^2+z'^2}$$

Let $F = F(x, y', y, z', z)$. We choose x to be the independent variable and y and z as dependent variables. Using Euler's equation, we get:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \quad \frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} = 0$$

Because F is independent of y and z, equations are reduced to:

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \quad \frac{d}{dx}\frac{\partial F}{\partial z'} = 0$$

Differentiating F with respect of x' and y', we get:

$$k_1 = \frac{y'}{\sqrt{1+y'^2+z'^2}}, \quad k_2 = \frac{z'}{\sqrt{1+y'^2+z'^2}}$$

where k_1 and k_2 are constants. The equation can be coupled by taking the sum of the squares of the two equations to show that the denominator of each equation or constant, or equivalently $y'^2 + z'^2 = k$, where k is a constnat.

$$k_1^2 + k_2^2 = \frac{y'^2 + z'^2}{1 + y'^2 + z'^2}$$
$$k_1^2 + k_2^2 + (k_1^2 + k_2^2)(y'^2 + z'^2) = y'^2 + z'^2)$$
$$k_1^2 + k_2^2 = (1 - k_1^2 - k_2^2)(y'^2 + z'^2)$$

Then,

$$k = \frac{k_1^2 + k_2^2}{1 - k_1^2 - k_2^2} = y'^2 + z'^2$$

The minimum path has a constant and positive slope in both and $x - y$ and $x - z$ planets, showing a stright line.

3.6: Mechanics From a Variation Principle

Recall from relativity that proper time (the reading in the frame moving with the clock) is equal to $\tau = \Delta t \gamma$, where Δt is the reading in another frame (For instance, earth's frame).

$$\Delta t = \frac{\tau}{\gamma}$$

We want to find a general formulation of mechanics that is based entirely on a variational principle. Consider a relativistic particle, the proper time in the particle's frame is τ, and the reading in earth's frame is Δt.

$$I = \int d\tau = \int \frac{dt}{\gamma}$$

Using space-time four dimensional coordinates (c, x, y, z), we know that

$$I = \int d\tau = \int \frac{dt}{\gamma} = \int dt \sqrt{1 - \frac{\dot{x}}{c} - \frac{\dot{y}}{c} - \frac{\dot{z}}{c}}$$

Using variation of calculus for several calculus we derived in section 3.6, we get:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'} = 0, \quad \frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial y'} = 0, \quad \frac{\partial F}{\partial z} - \frac{d}{dt}\frac{\partial F}{\partial z'} = 0$$

Because the F is independent of x, y and z, expression becomes:

$$\frac{d}{dt}\frac{\partial F}{\partial x'} = 0, \quad \frac{d}{dt}\frac{\partial F}{\partial y'} = 0, \quad \frac{d}{dt}\frac{\partial F}{\partial z'} = 0$$

$$\frac{d}{dt}(\gamma \dot{x}) = 0, \quad \frac{d}{dt}(\gamma \dot{y}) = 0, \quad \frac{d}{dt}(\gamma \dot{z}) = 0$$

Will come back later after finishing chapter 2

3.7: Motion in a Uniform Field

3.7.1: Doppler Effect

Dropper effect: the change in frequency of a wave when a source and an observer are moving relativity to each other. Frequency increases when the observer and the source approach each other and decreases when moving away.

Relativistic Dropper Shift

For a source directly approaching or moving away from an observer, the doppler effect gives:

$$f_{ob} = \frac{v}{v \pm v_{\mathrm{em}}} f_{\mathrm{em}}, \quad f_{ob} = f_{em} \frac{1}{1 \pm \frac{v_{em}}{v_{wave}}}$$

$$f_{ob} = f_{em} \sqrt{\frac{1 + v/c}{1 - v/c}}$$

Redshift: As an object moves away from us, the sound or light waves emitted the source are stretched out, making them have a lower pitch and move towards the red end of the electromagnetic spectrum (which has a longer wavelength and a smaller frequency). The lights from the object is known as redshift.

Blueshift: As an object moves toward from us, the sound or light waves emitted the source shrink out, making them have a higher pitch and move towards the blue end of the electromagnetic spectrum (which has a shorted wavelength and a larger frequency). The lights from the object is known as blueshift.

3.7.2: Principle of Equivalence

Consider two spaceships. One accelerating uniformly at a in gravity-free empty space and one standing at rest in a uniform gravity field. Let acceleration a of the first ship is adjusted to be equal, but opposite in direction, to the gravitational field g on the second ship.

An observer in the bow of the accelerating ship shies a laser beam at another observer in the stern of the ship. The ship is initially at rest and the laser emits monochromatic light of frequency f_{em} in the laser's rest frame. We assume that the distance traveled by the ship is very small while the bean is traveling (the assumption can be understood by considering the ship's and the laser's speed. The last has a speed of light C, which we assume, is largely greater than the ship's speed. Thus, the distance traveled by the ship is negligible compare to the distance traveled by the laser).

Let the length of the ship be h. The time it takes for beam to reach the stern is $t = h/c$. During the time, the stern attains a velocity $v = at = ah/c$.

$$f_{ob} = f_{em} \sqrt{\frac{1 + v/c}{1 - v/c}} = \approx f_{ob} = f_{em}(1 + \frac{v}{c}) = f_{em}(1 + \frac{ah}{c^2})$$

Let $h = y$. Because $a = g$, the equation becomes:

$$f_{ob} = f_{em}(1 + \frac{gh}{c^2})$$

We can gain several insights from the equation. First, the frequency at the bottom (stern) is increased by a factor of gh/c^2. In a time of $t = 1s$, the bow emits $t/\text{period} = tf_{em}$ wavelengths of light. However, because $f_{ob} > f_{em}$, observers at the bottom must collect theses waves in less than 1 second according to their own clocks (number of wavelengths of light remains unchanged for observers at the bottom and at the top, frequency $f_{ob} > f_{em}$, so time decreases).

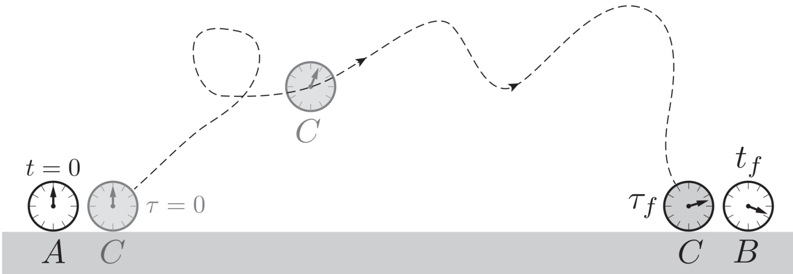
$$tf_{em} = n, \quad n = \text{number of wavelengths}$$

$$t_{em} = \frac{n}{f_{em}}, \quad t_{ob} = \frac{n}{f_{ob}} = \frac{n}{f_{em}(1 + gy/c^2)} = \frac{t_{em}}{1 + gy/c^2}$$

Becase $t_{low} = t_{ob} < t_{em} = t_{high}$, it is natural for observers at the top to conclude that bottom clocks run slower than top clocks (as their clocks have a smaller reading). The must be true for all clocks at the top and bottom. Rearranging the equation, we get

$$\boxed{\Delta t_{high} = \Delta t_{low}(1 + gy/c^2)}$$

The equation shows the time difference for two clocks at rest, but at different altitudes in a uniform gravitational field.



Clock C travels with a nonzero speed by an arbitrary path between stationary ground ground clocks A and B. All clocks are synchronized initially. Clock B reads t_f when C arrives, and the reading of clock C depends on the path it takes in moving from A to B.

Recall that the proper time for time dilation is:

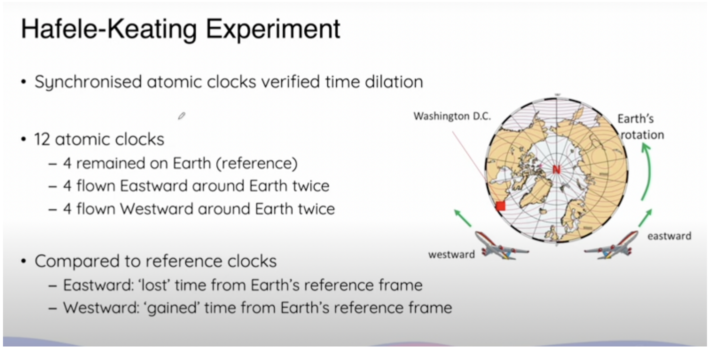
$$\tau = \Delta t\sqrt{1 - \frac{v^2}{c^2}}$$

where v is the velocity of the moving object relative to the outside observer. Because clock C moves with a non-zero speed, by time dilation, we know that the reading of C is different from the reading of B. In an infinitesimal time dt according to the top clock, clock C advances by the proper time:

$$d\tau = dt(1 + gy/c^2)\sqrt{1 - v^2/c^2} \approx dt(1 + gy/c^2 - v^2/2c^2)$$

$$\boxed{\tau_f = \int_0^{t_f} dt(1 + gy/c^2 - v^2/2c^2)}$$

3.7.3: The Hafele–Keating Experiments



144 ± 14	altitude effect
-184 ± 18	velocity effect
-40 ± 23	net effect predicted.

Time for eastward clock

179 ± 18	altitude effect
96 ± 10	velocity effect
275 ± 21	net effect predicted.

Westward clock

Explanation

The opposite sign of the two results suggests that *the rate at which time flows depends on the motion of the observer*. The eastward clock is moving in the same direction as earth's rotation, so its velocity relative to earth is greater than the westward clock. Thus, a greater time dilation effect. The westward clock is moving in the opposite direction as the earth's rotation, so it has a smaller relative velocity and time dilation effect.

Previously we have derived that clock C advances by the proper time

$$\tau_f = \int_0^{t_f} dt(1 + gy/c^2 - v^2/2c^2)$$

We can rewrite the integral in the form of energy, giving us:

$$\begin{aligned} \tau_f &= \int_0^{t_f} dt + \int (gy/c^2 - v^2/2c^2)dt \\ &= t_f + \int (-v^2/2c^2 + gy/c^2)dt \\ &= t_f - \frac{1}{mc^2} \int (\frac{1}{2}mv^2 - mgy)dt \end{aligned}$$

We know that $1/2mv^2 = K$ is kinetic energy and $mgy = U$ is potential energy. Then, the integral becomes:

$$\tau_f = t_f - \int_0^{t_f} (K - U)dt \tag{8}$$

In order to find a path that extremeizes (We use the term "extremeize" because it is uncertain if the proper time τ is maximized or minimized. Euler's equation only provides a path (a function) that maximizes or minimizes a function) the proper time τ, we apply Euler's equation to the integral I, giving us:

$$I = \int_0^{t_f} (K - U)dt$$

Let $F = F(\dot{x}, x, \dot{y}, y, \dot{z}, z)$ be the integrand, we get:

$$F = F(\dot{x}, x, \dot{y}, y, \dot{z}, z) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgy$$

Since F is independent of x and z, we get:

$$0 = \frac{d}{dt}\frac{\partial F}{\partial x'}, \quad 0 = mg - \frac{d}{dt}\frac{\partial F}{\partial y'}, \quad 0 = \frac{d}{dt}\frac{\partial F}{\partial z'}$$

giving us:

$$0 = \dot{x}\ddot{x}, \quad 0 = g - \dot{y}\ddot{y}, \quad 0 = \dot{z}\ddot{z}$$

Then,

$$0 = \ddot{x}, \quad \dot{y} = g, \quad 0 = \ddot{z}$$

When $\ddot{x}, \ddot{z} = 0$, we recognize these differential equations as results of Newton's equation of motion $F = ma$ in a uniform gravitational field.

$$x = v_x t, \quad y = \frac{1}{2}gt^2, \quad z = 0$$

Our goal in identifying a variational principle for the motion of a particle in a uniform gravitational field is successful. Without every using Newton's law, we found the correct equations of motion. In the next section, we will derive $F = -\nabla U$ using variational calculus.

3.8: Arbitrary Potential Energy

$$I = \int_0^{t_f} (K - U)dt$$

Let F be the integrand, we get:

$$F(\dot{x}, x, \dot{y}, y, \dot{z}, z) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^+ \dot{z}^2) - U(x, y, z)$$

Using Euler's equation, we get:

$$0 = \frac{d}{dt}\frac{\partial F}{\partial x'}, \quad 0 = mg - \frac{d}{dt}\frac{\partial F}{\partial y'}, \quad 0 = \frac{d}{dt}\frac{\partial F}{\partial z'}$$

Summary:

$$\text{Spherical:} \quad ds^2 = r^2 + r^2 d\theta^2 + r^2 \sin^2\theta\phi^2$$

$$\text{Cylinderical:} \quad ds = \sqrt{r^2 d\theta^2 + z^2}$$

$$t = \int dt = t \int \frac{d}{C/n(r)} = \frac{1}{C}\int \frac{ds}{n(r)}$$

$$0 = \frac{\partial F}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right)$$

$$\frac{\partial F}{\partial y_i(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'(x)}\right) = 0$$

$$\Delta t_{high} = \Delta t_{low}(1 + gy/c^2)$$

$$d\tau = dt(1 + \frac{gy}{c^2} - \frac{v^2}{2c^2})$$