Topic 6 Part 4 *[219 marks]*

Consider the quadratic function $y = f(x)$, where $f(x) = 5 - x + ax^2$.

1a. It is given that $f(2) = -5$. Find the value of a . *[2 marks]*

1b. Find the equation of the axis of symmetry of the graph of $y = f(x)$. *[2 marks]*

1c. Write down the range of this quadratic function. *[2 marks]*

Consider the function
$g(x) = bx - 3 + \frac{1}{x^2}, \ x \neq 0.$

2a. Write down the equation of the vertical asymptote of the graph of $y = g(x)$. *[2 marks]*

2b. Write down $g'(x)$. *[3 marks]*

2c. The line T is the tangent to the graph of $y = g(x)$ at the point where $x = 1$. The gradient of T is 3. *[2 marks]*

Show that $b = 5$.

2d. The line T is the tangent to the graph of $y = g(x)$ at the point where $x = 1$. The gradient of T is 3. *[3 marks]*

Find the equation of T.

2e. Using your graphic display calculator find the coordinates of the point where the graph of $y = g(x)$ intersects the x-axis. *[2 marks]*

2f. (i) Sketch the graph of $y = g(x)$ for $-2 \le x \le 5$ and $-15 \le y \le 25$, indicating clearly your answer to part (e). *[6 marks]*

(ii) Draw the line T on your sketch.

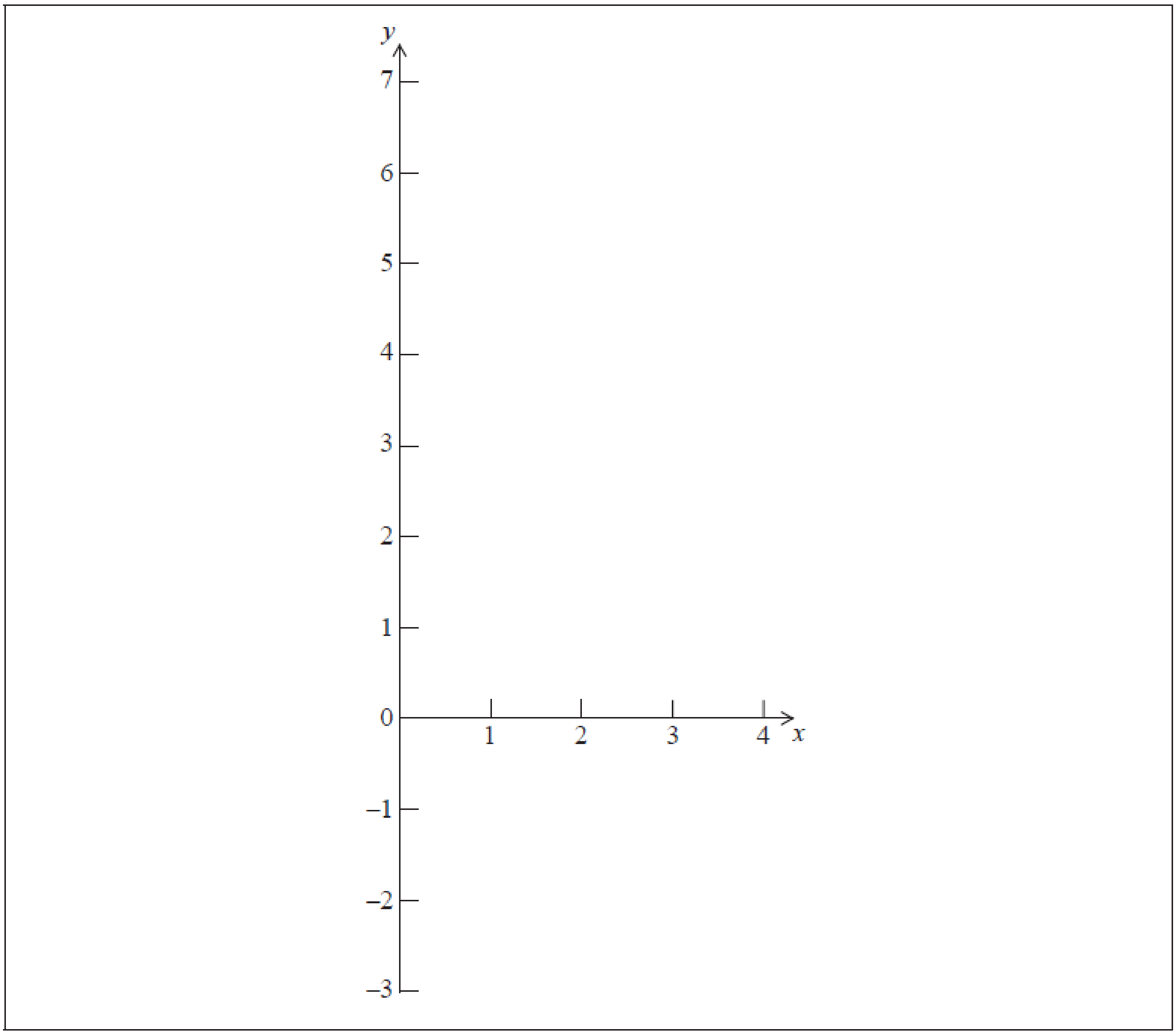
2g. Using your graphic display calculator find the coordinates of the local minimum point of $y = g(x)$. *[2 marks]*

2h. Write down the interval for which $g(x)$ is increasing in the domain $0 < x < 5$. *[2 marks]*

$y = f(x)$ is a quadratic function. The graph of $f(x)$ intersects the y-axis at the point A(0, 6) and the x-axis at the point B(1, 0). The vertex of the graph is at the point C(2, –2).

3a. Write down the equation of the axis of symmetry. *[2 marks]*

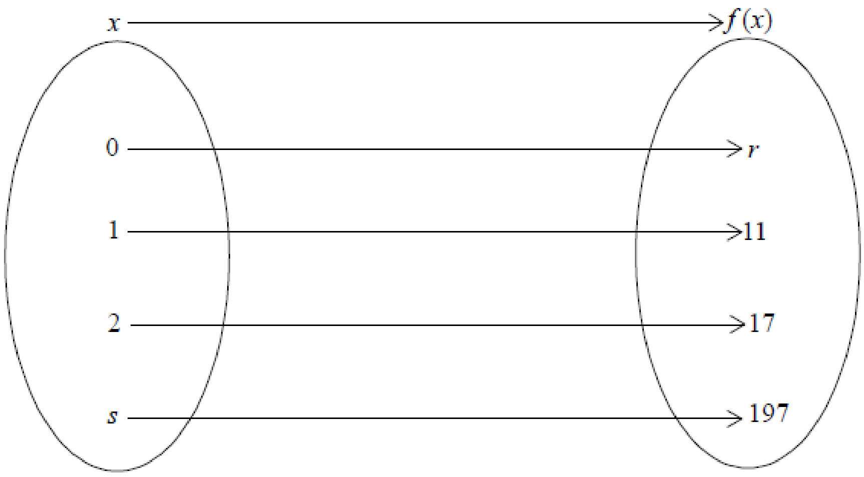
3b. Sketch the graph of $y = f(x)$ on the axes below for $0 \le x \le 4$. Mark clearly on the sketch the points A , B , and C. *[3 marks]*



3c. The graph of $y = f(x)$ intersects the x-axis for a second time at point D. *[1 mark]*

Write down the x-coordinate of point D.

A function $f(x) = p \times 2^x + q$ is defined by the mapping diagram below.



4a. Find the value of *[3 marks]*

(i) p ;

(ii) q .

4b. Write down the value of r . *[1 mark]*

4c. Find the value of s . *[2 marks]*

Water has a lower boiling point at higher altitudes. The relationship between the boiling point of water (T) and the height above sea level (h) can be described by the model
$T = -0.0034h + 100$ where T is measured in degrees Celsius (°C) and h is measured in metres from sea level.

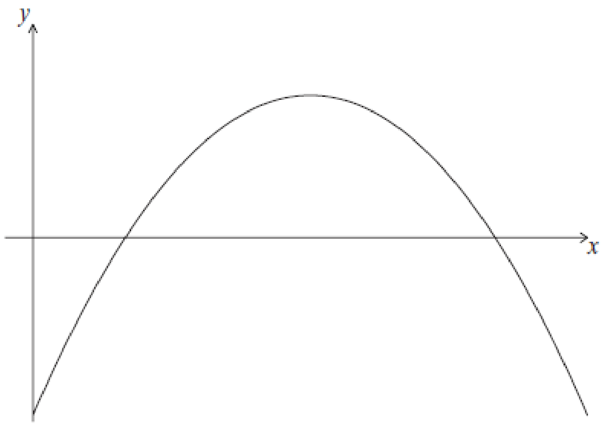
5a. Write down the boiling point of water at sea level. *[1 mark]*

5b. Use the model to calculate the boiling point of water at a height of 1.37 km above sea level. *[3 marks]*

5c. Water boils at the top of Mt. Everest at 70 °C. *[2 marks]*

Use the model to calculate the height above sea level of Mt. Everest.

Part of the graph of the quadratic function f is given in the diagram below.



On this graph one of the x-intercepts is the point (5, 0). The x-coordinate of the maximum point is 3.

The function f is given by
$f(x) = -x^2 + bx + c$, where
$b, c \in \mathbb{Z}$

6a. Find the value of *[3 marks]*

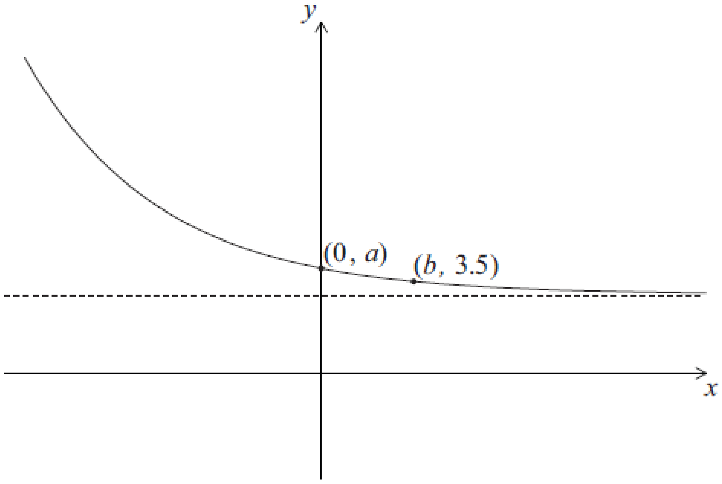
(i) b ;

(ii) c .

6b. The domain of f is $0 \leq x \leq 6$. *[3 marks]*

Find the range of f .

The diagram shows part of the graph of
$y = 2^{-x} + 3$, and its horizontal asymptote. The graph passes through the points (0, a) and (b, 3.5).



7a. Find the value of *[4 marks]*

(i) a ;

(ii) b .

7b. Write down the equation of the horizontal asymptote to this graph. *[2 marks]*

George leaves a cup of hot coffee to cool and measures its temperature every minute. His results are shown in the table below.

Time, t (minutes)	0	1	2	3	4	5	6
Temperature, y (°C)	94	54	34	24	k	16.5	15.25

8a. Write down the decrease in the temperature of the coffee *[3 marks]*

(i) during the first minute (between $t = 0$ and $t = 1$) ;

(ii) during the second minute;

(iii) during the third minute.

8b. Assuming the pattern in the answers to part (a) continues, show that $k = 19$. *[2 marks]*

8c. Use the **seven** results in the table to draw a graph that shows how the temperature of the coffee changes during the first six minutes. *[4 marks]*

Use a scale of 2 cm to represent 1 minute on the horizontal axis and 1 cm to represent 10 °C on the vertical axis.

8d. The function that models the change in temperature of the coffee is $y = p\,(2^{-t}) + q$. *[2 marks]*

(i) Use the values $t = 0$ and $y = 94$ to form an equation in p and q.

(ii) Use the values $t = 1$ and $y = 54$ to form a second equation in p and q.

8e. Solve the equations found in part (d) to find the value of p and the value of q. *[2 marks]*

8f. The graph of this function has a horizontal asymptote. *[2 marks]*

Write down the equation of this asymptote.

8g. George decides to model the change in temperature of the coffee with a linear function using correlation and linear regression. *[4 marks]*

Use the **seven** results in the table to write down

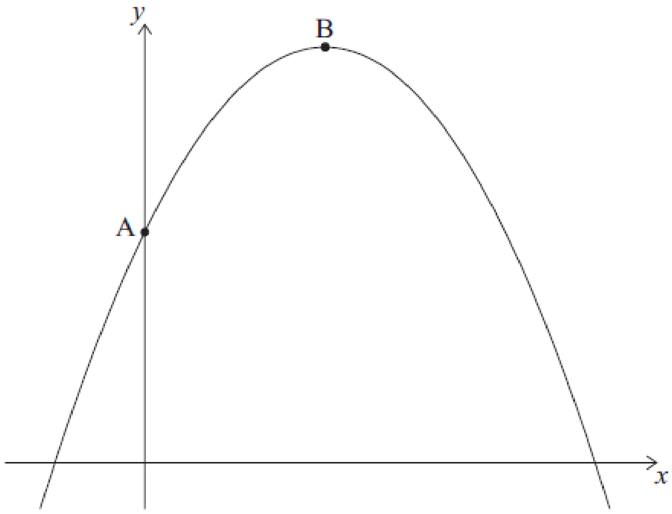
(i) the correlation coefficient;

(ii) the equation of the regression line y on t.

8h. Use the equation of the regression line to estimate the temperature of the coffee at $t = 3$. *[2 marks]*

8i. Find the percentage error in this estimate of the temperature of the coffee at $t = 3$. *[2 marks]*

The graph of the quadratic function
$f(x) = c + bx - x^2$ intersects the y-axis at point A(0, 5) and has its vertex at point B(2, 9).



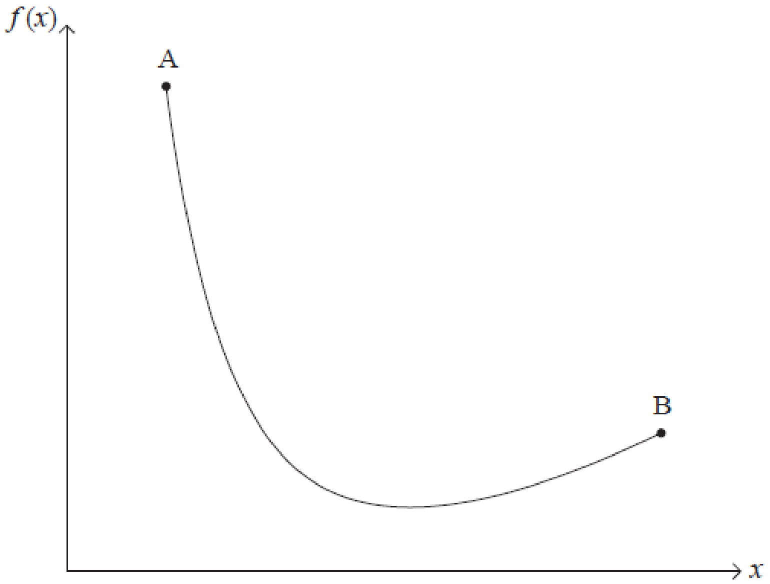
9a. Write down the value of c. *[1 mark]*

9b. Find the value of b. *[2 marks]*

9c. Find the x-intercepts of the graph of f . *[2 marks]*

9d. Write down *[1 mark]*
$f(x)$ in the form
$f(x) = -(x - p)(x + q)$.

The graph of the function
$f(x) = \frac{14}{x} + x - 6$, for $1 \le x \le 7$ is given below.



10a. Calculate
$f(1)$. *[2 marks]*

10b. Find
$f'(x)$. *[3 marks]*

10c. **Use your answer to part (b)** to show that the x-coordinate of the local minimum point of the graph of f is 3.7 correct to 2 significant figures. *[3 marks]*

10d. Find the range of
f. *[3 marks]*

10e. Points A and B lie on the graph of
f. The x-coordinates of A and B are 1 and 7 respectively. *[1 mark]*

Write down the y-coordinate of B.

10f. Points A and B lie on the graph of f . The x-coordinates of A and B are 1 and 7 respectively. *[2 marks]*

Find the gradient of the straight line passing through A and B.

10g. M is the midpoint of the line segment AB. *[2 marks]*

Write down the coordinates of M.

10h. L is the tangent to the graph of the function
$y = f(x)$, at the point on the graph with the same x-coordinate as M. *[2 marks]*

Find the gradient of L.

10i. Find the equation of L. Give your answer in the form
$y = mx + c$. *[3 marks]*

The number of bacteria in a colony is modelled by the function

$$N(t) = 800 \times 3^{0.5t}, \ t \geqslant 0,$$

where
N is the number of bacteria and
t is the time in hours.

11a. Write down the number of bacteria in the colony at time $t = 0$. *[1 mark]*

11b. Calculate the number of bacteria present at 2 hours and 30 minutes. Give your answer correct to the nearest hundred bacteria. *[3 marks]*

11c. Calculate the time, in hours, for the number of bacteria to reach 5500. *[2 marks]*

On Monday Paco goes to a running track to train. He runs the first lap of the track in 120 seconds. Each lap Paco runs takes him 10 seconds longer than his previous lap.

12a. Find the time, in seconds, Paco takes to run his fifth lap. *[3 marks]*

12b. Paco runs his last lap in 260 seconds. *[3 marks]*

Find how many laps he has run on Monday.

12c. Find the **total** time, in **minutes,** run by Paco on Monday. *[4 marks]*

12d. On Wednesday Paco takes Lola to train. They both run the first lap of the track in 120 seconds. Each lap Lola runs takes 1.06 times as long as her previous lap. *[3 marks]*

Find the time, in seconds, Lola takes to run her third lap.

12e. Find the **total** time, in seconds, Lola takes to run her first four laps. *[3 marks]*

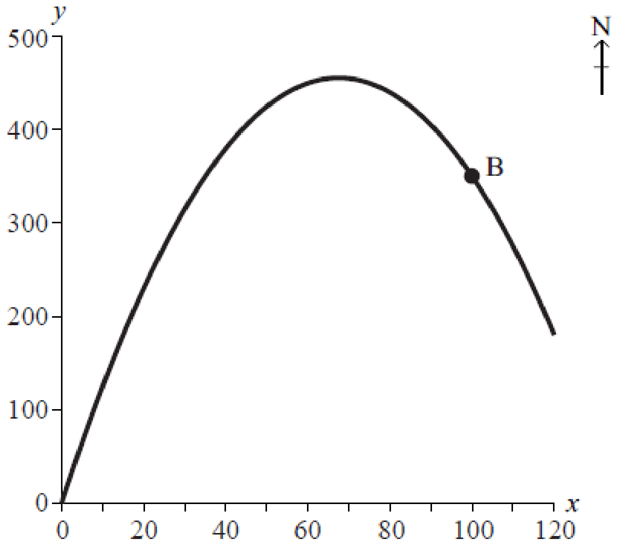
12f. Each lap Paco runs again takes him 10 seconds longer than his previous lap. After a certain number of laps Paco takes less time per lap than Lola. *[3 marks]*

Find the number of the lap when this happens.

The diagram shows an **aerial** view of a bicycle track. The track can be modelled by the quadratic function

$y = \frac{-x^2}{10} + \frac{27}{2}x$, where
$x \geqslant 0,\ y \geqslant 0$

(x , y) are the coordinates of a point x metres east and y metres north of O , where O is the origin (0, 0) . B is a point on the bicycle track with coordinates (100, 350) .



13a. The coordinates of point A are (75, 450). Determine whether point A is on the bicycle track. Give a reason for your answer. *[3 marks]*

13b. Find the derivative of *[2 marks]*
$y = \frac{-x^2}{10} + \frac{27}{2}x.$

13c. Use the answer in part (b) to determine if A (75, 450) is the point furthest north on the track between O and B. Give a reason for your answer. *[4 marks]*

13d. (i) Write down the midpoint of the line segment OB. *[3 marks]*

(ii) Find the gradient of the line segment OB.

13e. Scott starts from a point C(0,150) . He hikes along a straight road towards the bicycle track, parallel to the line segment OB. *[3 marks]*

Find the equation of Scott's road. Express your answer in the form
$ax + by = c$, where
a, b and $c \in \mathbb{R}$.

13f. Use your graphic display calculator to find the coordinates of the point where Scott first crosses the bicycle track. *[2 marks]*

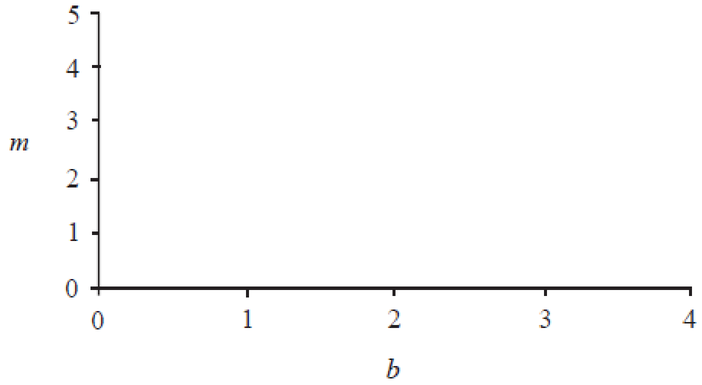
A store sells bread and milk. On Tuesday, 8 loaves of bread and 5 litres of milk were sold for \$21.40. On Thursday, 6 loaves of bread and 9 litres of milk were sold for \$23.40.

If
$b =$ the price of a loaf of bread and
$m =$ the price of one litre of milk, Tuesday's sales can be written as
$8b + 5m = 21.40.$

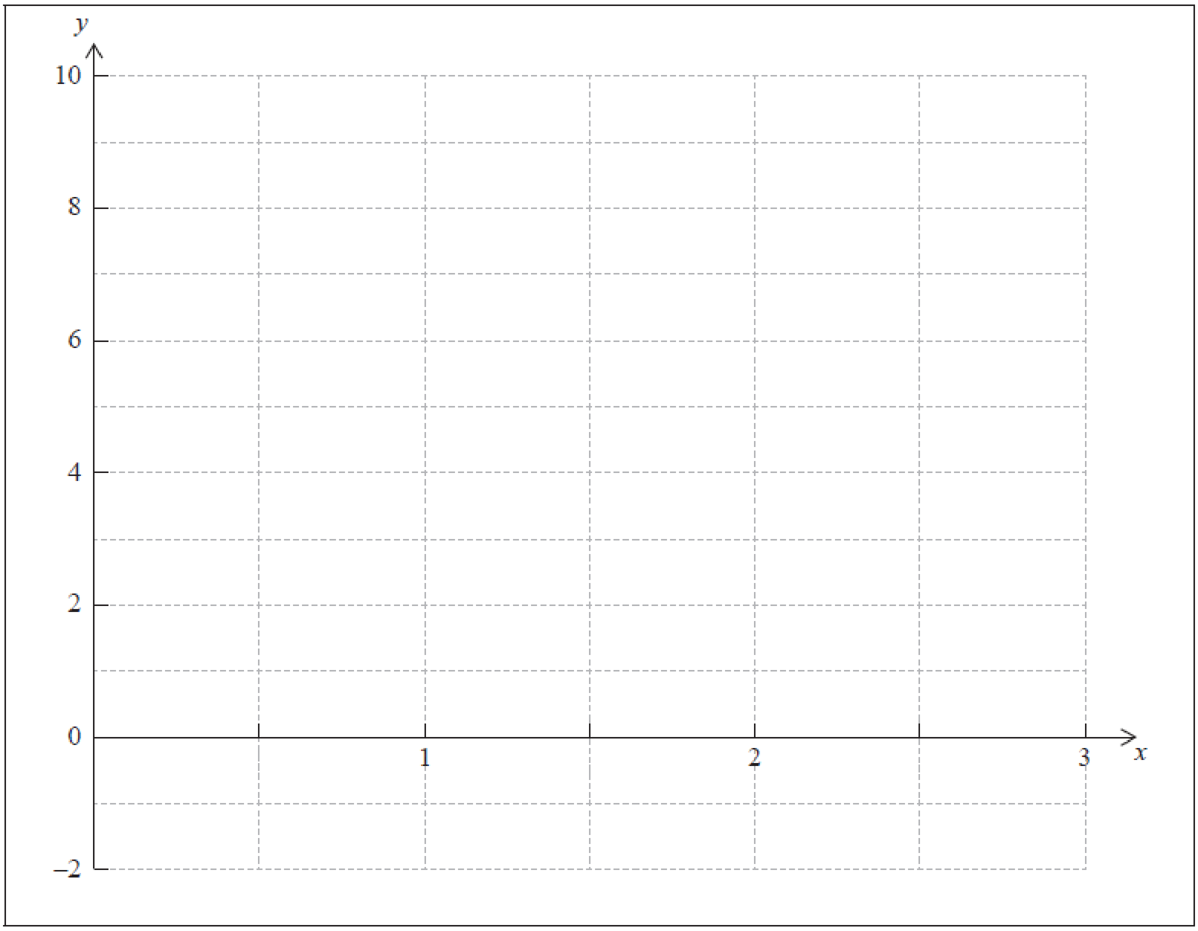
14a. Using simplest terms, write an equation in b and m for Thursday's sales. *[2 marks]*

14b. Find b and m. *[2 marks]*

14c. Draw a sketch, in the space provided, to show how the prices can be found graphically. *[2 marks]*



15a. On the grid below sketch the graph of the function $f(x) = 2(1.6)^x$ for the domain $0 \leqslant x \leqslant 3$. *[2 marks]*

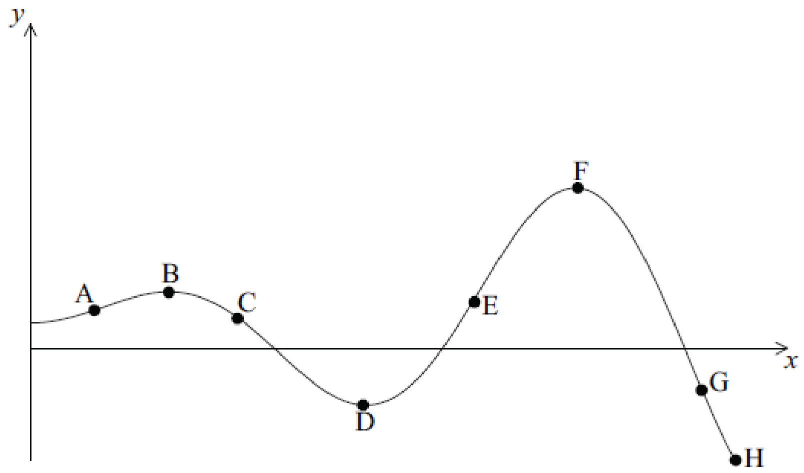


15b. Write down the coordinates of the y-intercept of the graph of $y = f(x)$. *[1 mark]*

15c. On the grid draw the graph of the function $g(x) = 5 - 2x$ for the domain $0 \leqslant x \leqslant 3$. *[2 marks]*

15d. Use your graphic display calculator to solve $f(x) = g(x)$. *[1 mark]*

Consider the graph of the function $y = f(x)$ defined below.



Write down **all** the labelled points on the curve

16a. that are local maximum points; *[1 mark]*

16b. where the function attains its least value; *[1 mark]*

16c. where the function attains its greatest value; *[1 mark]*

16d. where the gradient of the tangent to the curve is positive; *[1 mark]*

16e. where $f(x) > 0$ and $f'(x) < 0$. *[2 marks]*

A liquid is heated so that after 20 seconds of heating its temperature, T , is $25^{\circ}\mathrm{C}$ and after 50 seconds of heating its temperature is $37^{\circ}\mathrm{C}$.

The temperature of the liquid at time t can be modelled by $T = at + b$, where t is the time in seconds after the start of heating.

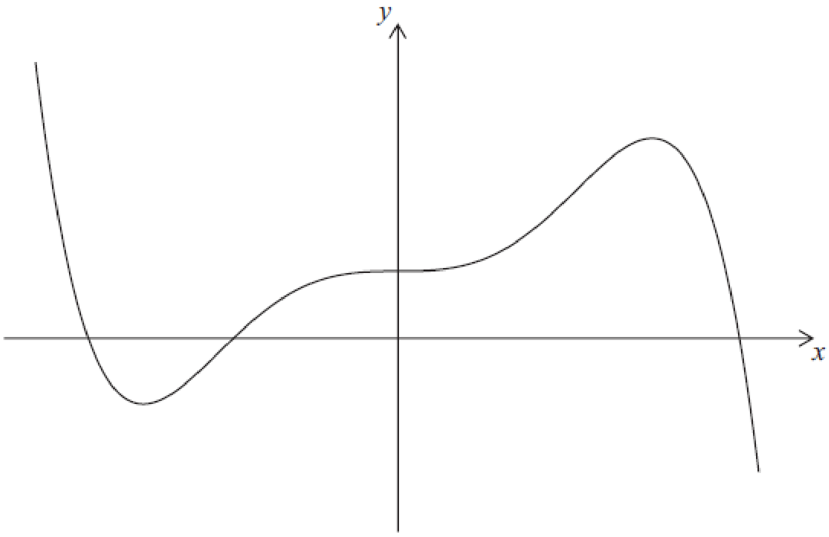
Using this model one equation that can be formed is $20a + b = 25$.

17a. Using the model, write down a second equation in a and b . *[2 marks]*

17b. Using your graphic display calculator or otherwise, find the value of a and of b . *[2 marks]*

17c. Use the model to predict the temperature of the liquid 60 seconds after the start of heating. *[2 marks]*

A sketch of the function
$f(x) = 5x^3 - 3x^5 + 1$ is shown for
$-1.5 \leqslant x \leqslant 1.5$ and
$-6 \leqslant y \leqslant 6$.



18a. Write down $f'(x)$. *[2 marks]*

18b. Find the equation of the tangent to the graph of $y = f(x)$ at $(1, 3)$. *[2 marks]*

18c. Write down the coordinates of the second point where this tangent intersects the graph of $y = f(x)$. *[2 marks]*

A small manufacturing company makes and sells
x machines each month. The monthly cost
C , in dollars, of making
x machines is given by

$$C(x) = 2600 + 0.4x^2.$$

The monthly income
I , in dollars, obtained by selling
x machines is given by

$$I(x) = 150x - 0.6x^2.$$

$P(x)$ is the monthly profit obtained by selling
x machines.

19a. Find $P(x)$. *[2 marks]*

19b. Find the number of machines that should be made and sold each month to maximize $P(x)$. *[2 marks]*

19c. Use your answer to part (b) to find the selling price of **each machine** in order to maximize $P(x)$. *[2 marks]*

A deep sea diver notices that the intensity of light, I, below the surface of the ocean decreases with depth, d, according to the formula

$$I = k(1.05)^{-d},$$

where
I is expressed as a percentage,
d is the depth in metres below the surface and
k is a constant.

The intensity of light at the surface is 100%.

20a. Calculate the value of k. *[2 marks]*

20b. Find the intensity of light at a depth 25 m below the surface. *[2 marks]*

20c. To be able to see clearly, a diver needs the intensity of light to be at least 65%. *[2 marks]*

Using your graphic display calculator, find the greatest depth below the surface at which she can see clearly.

20d. The table below gives the intensity of light (correct to the nearest integer) at different depths. *[4 marks]*

Depth (d)	0	10	20	50	80
Intensity (I)	100	61	38	9	2

Using this information draw the graph of I against d for $0 \leqslant d \leqslant 100$. Use a scale of 1 cm to represent 10 metres on the horizontal axis and 1 cm to represent 10% on the vertical axis.

20e. Some sea creatures have adapted so they can see in low intensity light and cannot tolerate too much light. *[2 marks]*

Indicate clearly on your graph the range of depths sea creatures could inhabit if they can tolerate between 5% and 35% of the light intensity at the surface.

Given
$f(x) = x^2 - 3x^{-1}, x \in \mathbb{R}, -5 \leqslant x \leqslant 5, x \neq 0,$

21a. Write down the equation of the vertical asymptote. *[1 mark]*

21b. Find $f'(x)$. *[2 marks]*

21c. Using your graphic display calculator or otherwise, write down the coordinates of any point where the graph of $y = f(x)$ has zero gradient. *[2 marks]*

21d. Write down all intervals in the given domain for which $f(x)$ is increasing. *[3 marks]*

A football is kicked from a point A (a, 0), 0 < a < 10 on the ground towards a goal to the right of A.

The ball follows a path that can be modelled by **part** of the graph

$$y = -0.021x^2 + 1.245x - 6.01, x \in \mathbb{R}, y \geqslant 0.$$

x is the horizontal distance of the ball from the origin

y is the height above the ground

Both *x* and *y* are measured in metres.

21e. Using your graphic display calculator or otherwise, find the value of *a*. *[1 mark]*

21f. Find $\frac{dy}{dx}$. *[2 marks]*

21g. (i) Use your answer to part (b) to calculate the horizontal distance the ball has travelled from A when its height is a maximum. *[4 marks]*

(ii) Find the maximum vertical height reached by the football.

21h. Draw a graph showing the path of the football from the point where it is kicked to the point where it hits the ground again. Use 1 cm to represent 5 m on the horizontal axis and 1 cm to represent 2 m on the vertical scale. *[4 marks]*

21i. The goal posts are 35 m from **the point where the ball is kicked**. *[2 marks]*

At what height does the ball pass over the goal posts?

© International Baccalaureate Organization 2017
International Baccalaureate® - Baccalauréat International® - Bachillerato Internacional®

Printed for Sierra High School