HL Paper 3

The integral I_n is defined by $I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x}|\sin x|\mathrm{d}x,\ \text{for } n \in \mathbb{N}$.

a. Show that $I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$. [6]

b. By letting $y = x - n\pi$, show that $I_n = \mathrm{e}^{-n\pi} I_0$. [4]

c. Hence determine the exact value of $\int_0^{\infty} \mathrm{e}^{-x}|\sin x|\mathrm{d}x$. [5]

Markscheme

a. $I_0 = \int_0^{\pi} \mathrm{e}^{-x} \sin x \mathrm{d}x$ ***M1***

Note: Award ***M1*** for $I_0 = \int_0^{\pi} \mathrm{e}^{-x}|\sin x|\mathrm{d}x$

Attempt at integration by parts, even if inappropriate modulus signs are present. ***M1***

$= -[\mathrm{e}^{-x}\cos x]_0^{\pi} - \int_0^{\pi} \mathrm{e}^{-x}\cos x \mathrm{d}x$ **or** $= -[\mathrm{e}^{-x}\sin x]_0^{\pi} - \int_0^{\pi} \mathrm{e}^{-x}\cos x \mathrm{d}x$ ***A1***

$= -[\mathrm{e}^{-x}\cos x]_0^{\pi} - [\mathrm{e}^{-x}\sin x]_0^{\pi} - \int_0^{\pi} \mathrm{e}^{-x}\sin x \mathrm{d}x$ **or** $= -[\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x]_0^{\pi} - \int_0^{\pi} \mathrm{e}^{-x}\sin x \mathrm{d}x$ ***A1***

$= -[\mathrm{e}^{-x}\cos x]_0^{\pi} - [\mathrm{e}^{-x}\sin x]_0^{\pi} - I_0$ **or** $-[\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x]_0^{\pi} - I_0$ ***M1***

Note: Do not penalise absence of limits at this stage

$I_0 = \mathrm{e}^{-\pi} + 1 - I_0$ ***A1***

$I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$ ***AG***

Note: If modulus signs are used around $\cos x$, award no accuracy marks but do not penalise modulus signs around $\sin x$.

[6 marks]

b. $I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x}|\sin x|\mathrm{d}x$

Attempt to use the substitution $y = x - n\pi$ ***M1***

(putting $y = x - n\pi$, $\mathrm{d}y = \mathrm{d}x$ and $[n\pi,\ (n+1)\pi] \to [0,\ \pi]$)

so $I_n = \int_0^{\pi} \mathrm{e}^{-(y+n\pi)}|\sin(y + n\pi)|\mathrm{d}y$ ***A1***

$= \mathrm{e}^{-n\pi} \int_0^{\pi} \mathrm{e}^{-y}|\sin(y + n\pi)|\mathrm{d}y$ ***A1***

$= \mathrm{e}^{-n\pi} \int_0^{\pi} \mathrm{e}^{-y} \sin y \mathrm{d}y$ ***A1***

$= \mathrm{e}^{-n\pi} I_0$ ***AG***

[4 marks]

c. $\int_0^{\infty} \mathrm{e}^{-x}|\sin x|\mathrm{d}x = \sum_{n=0}^{\infty} I_n$ ***M1***

$= \sum_{n=0}^{\infty} e^{-n\pi} I_0$ ***(A1)***

the $\sum$ term is an infinite geometric series with common ratio $e^{-\pi}$ ***(M1)***

therefore

$\int_0^\infty e^{-x}|\sin x|dx = \frac{I_0}{1-e^{-\pi}}$ ***(A1)***

$= \frac{1+e^{-\pi}}{2(1-e^{-\pi})} \left(= \frac{e^{\pi}+1}{2(e^{\pi}-1)}\right)$ ***A1***

[5 marks]

Examiners report

a. Part (a) is essentially core work requiring repeated integration by parts and many candidates realised that. However, some candidates left the modulus signs in I_0 which invalidated their work. In parts (b) and (c) it was clear that very few candidates had a complete understanding of the significance of the modulus sign and what conditions were necessary for it to be dropped. Overall, attempts at (b) and (c) were disappointing with few correct solutions seen.

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In this question you may assume that $\arctan x$ is continuous and differentiable for $x \in \mathbb{R}$.

a. Consider the infinite geometric series [1]

$$1 - x^2 + x^4 - x^6 + \dots \quad |x| < 1.$$

Show that the sum of the series is $\frac{1}{1+x^2}$.

b. Hence show that an expansion of $\arctan x$ is $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ [4]

c. f is a continuous function defined on $[a,\ b]$ and differentiable on $]a,\ b[$ with $f'(x) > 0$ on $]a,\ b[$. [4]

Use the mean value theorem to prove that for any $x,\ y \in [a,\ b]$, if $y > x$ then $f(y) > f(x)$.

d. (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$. [4]

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$.

e. Use the result from part (c) to prove that $\arctan x > x - \frac{x^3}{3}$, for $x > 0$. [5]

f. Hence show that $\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$. [4]

Markscheme

a. $r = -x^2, \quad S = \frac{1}{1+x^2}$ ***A1AG***

[1 mark]

b. $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

EITHER

$\int \frac{1}{1+x^2}\mathrm{d}x = \int 1 - x^2 + x^4 - x^6 + \ldots \mathrm{d}x$ ***M1***

$\arctan x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***A1***

Note: Do not penalize the absence of c at this stage.

when $x = 0$ we have $\arctan 0 = c$ hence $c = 0$ ***M1A1***

OR

$\int_0^x \frac{1}{1+t^2}\mathrm{d}t = \int_0^x 1 - t^2 + t^4 - t^6 + \ldots \mathrm{d}t$ ***M1A1A1***

Note: Allow x as the variable as well as the limit.

M1 for knowing to integrate, ***A1*** for each of the limits.

$[\arctan t]_0^x = \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right]_0^x$ ***A1***

hence $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***AG***

[4 marks]

c. applying the MVT to the function f on the interval $[x,\ y]$ ***M1***

$\frac{f(y)-f(x)}{y-x} = f'(c) \quad (\text{for some } c \in]x,\ y[)$ ***A1***

$\frac{f(y)-f(x)}{y-x} > 0 \quad (\text{as } f'(c) > 0)$ ***R1***

$f(y) - f(x) > 0 \text{ as } y > x$ ***R1***

$\Rightarrow f(y) > f(x)$ ***AG***

Note: If they use x rather than c they should be awarded ***M1A0R0***, but could get the next ***R1***.

[4 marks]

d. (i) $g(x) = x - \arctan x \Rightarrow g'(x) = 1 - \frac{1}{1+x^2}$ ***A1***

this is greater than zero because $\frac{1}{1+x^2} < 1$ ***R1***

so $g'(x) > 0$ ***AG***

(ii) (g is a continuous function defined on $[0,\ b]$ and differentiable on $]0,\ b[$ with $g'(x) > 0$ on $]0,\ b[$ for all $b \in \mathbb{R}$)

(If $x \in [0,\ b]$ then) from part (c) $g(x) > g(0)$ ***M1***

$x - \arctan x > 0 \Rightarrow \arctan x < x$ ***M1***

(as b can take any positive value it is true for all $x > 0$) ***AG***

[4 marks]

e. let $h(x) = \arctan x - \left(x - \frac{x^3}{3}\right)$ ***M1***

(h is a continuous function defined on $[0,\ b]$ and differentiable on $]0,\ b[$ with $h'(x) > 0$ on $]0,\ b[$)

$h'(x) = \frac{1}{1+x^2} - (1 - x^2)$ ***A1***

$= \frac{1-(1-x^2)(1+x^2)}{1+x^2} = \frac{x^4}{1+x^2}$ ***M1A1***

$h'(x) > 0$ hence $(\text{for } x \in [0,\ b])\ h(x) > h(0)(= 0)$ ***R1***

$\Rightarrow \arctan x > x - \frac{x^3}{3}$ ***AG***

Note: Allow correct working with $h(x) = x - \frac{x^3}{3} - \arctan x$.

[5 marks]

f. use of $x - \frac{x^3}{3} < \arctan x < x$ ***M1***

choice of $x = \frac{1}{\sqrt{3}}$ ***A1***

$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***M1***

$\frac{8}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***A1***

Note: Award final ***A1*** for a correct inequality with a single fraction on each side that leads to the final answer.

$\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$ ***AG***

[4 marks]

Total [22 marks]

Examiners report

a. Most candidates picked up this mark for realizing the common ratio was $-x^2$.

b. Quite a few candidates did not recognize the importance of ‘hence’ in this question, losing a lot of time by trying to work out the terms from first principles.

Of those who integrated the formula from part (a) only a handful remembered to include the ‘$+c$’ term, and to verify that this must be equal to zero.

c. Most candidates were able to achieve some marks on this question. The most commonly lost mark was through not stating that the inequality was unchanged when multiplying by $y - x \text{ as } y > x$.

d. The first part of this question proved to be very straightforward for the majority of candidates.

In (ii) very few realized that they had to replace the lower variable in the formula from part (c) by zero.

e. Candidates found this part difficult, failing to spot which function was required.

f. Many candidates, even those who did not successfully complete (d) (ii) or (e), realized that these parts gave them the necessary inequality.

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a. Prove by induction that $n! > 3^n$, for $n \geq 7,\ n \in \mathbb{Z}$. [5]

b. Hence use the comparison test to prove that the series $\sum\limits_{r=1}^{\infty} \frac{2^r}{r!}$ converges. [6]

Markscheme

a. if $n = 7$ then $7! > 3^7$ ***A1***

so true for $n = 7$

assume true for $n = k$ ***M1***

so $k! > 3^k$

consider $n = k + 1$

$(k+1)! = (k+1)k!$ ***M1***

$> (k+1)3^k$

$> 3.3k \quad (\text{as } k > 6)$ ***A1***

$= 3^{k+1}$

hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 7$, so true for all $n \geq 7$. ***R1***

Note: Do not award the ***R1*** if the two ***M*** marks have not been awarded.

[5 marks]

b. consider the series $\sum\limits_{r=7}^{\infty} a_r$, where $a_r = \frac{2^r}{r!}$ ***R1***

Note: Award the ***R1*** for starting at $r = 7$

compare to the series $\sum\limits_{r=7}^{\infty} b_r$ where $b_r = \frac{2^r}{3^r}$ ***M1***

$\sum\limits_{r=7}^{\infty} b_r$ is an infinite Geometric Series with $r = \frac{2}{3}$ and hence converges ***A1***

Note: Award the ***A1*** even if series starts at $r = 1$.

as $r! > 3^r$ so $(0 <)a_r < b_r$ for all $r \geq 7$ ***M1R1***

as $\sum\limits_{r=7}^{\infty} b_r$ converges and $a_r < b_r$ so $\sum\limits_{r=7}^{\infty} a_r$ must converge

Note: Award the ***A1*** even if series starts at $r = 1$.

as $\sum_{r=1}^{6} a_r$ is finite, so $\sum_{r=1}^{\infty} a_r$ must converge ***R1***

Note: If the limit comparison test is used award marks to a maximum of ***R1M1A1M0A0R1***.

[6 marks]

Total [11 marks]

Examiners report

a. [N/A]
b. [N/A]

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a. Show that $n! \geqslant 2^{n-1}$, for $n \geqslant 1$. [2]

b. Hence use the comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges or diverges. [3]

Markscheme

a. for $n \geqslant 1,\ n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1 \geqslant 2 \times 2 \times 2 \ldots 2 \times 2 \times 1 = 2^{n-1}$ ***M1A1***

$\Rightarrow n! \geqslant 2^{n-1}$ for $n \geqslant 1$ ***AG***

[2 marks]

b. $n! \geqslant 2^{n-1} \Rightarrow \frac{1}{n!} \leqslant \frac{1}{2^{n-1}}$ for $n \geqslant 1$ ***A1***

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a positive converging geometric series ***R1***

hence $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test ***R1***

[3 marks]

Examiners report

a. Part (a) of this question was found challenging by the majority of candidates, a fairly common 'solution' being that the result is true for $n = 1$, 2, 3 and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between $\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

b. Part (a) of this question was found challenging by the majority of candidates, a fairly common 'solution' being that the result is true for $n = 1$, 2, 3 and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between $\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.
