Chapter 7: Gravitation

7.1: Central Forces

A central force on a particle is directed toward or away from a fixed point in three dimensions and is spherically symmetric about that point. For example, gravitational attraction force between earth and sun is a central force, where F is equal to:

$$F = -G\frac{m_1 m_2}{r^2}\hat{r}$$

The corresponding gravitational potential energy is"

$$U = -\int F(r)dr = -G\frac{m_1 m_2}{r}$$

with the choice $U(\infty) = 0$. Similarly, for a spring, we have:

$$F = -kr\hat{r}$$

$$U(r) = \frac{1}{2}kr^2$$

7.2: The Two Body Problem

The section shows that the two body problem is equivalent to a one-body central force problem with the right choice of coordinate. For a two body system, there is a kinetic energy for each body and a potential energy between two bodies. Altogether, there are six coordinates: three coordinates $r_1 = (x_1, y_1, z_1)$ for the first body and three coordinates $r_2 = (x_2, y_2, z_2)$ for the second body.

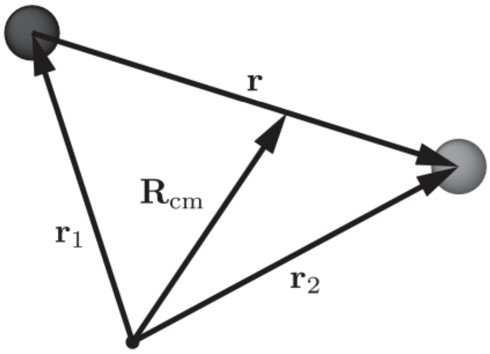
In order to reduce a two-body problem to a one-body central force problem, we use center of mass R_{cm} with three coordinates.

$$R_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} + \frac{m_1 r_1 + m_2 r_2}{M} \tag{1}$$

There are also three relative coordinates:

$$r \equiv r_2 - r_1 \tag{2}$$

where the relative coordinate vector points from the first body to the second, and the length is the distance between the



Vectors r_1 and r_2 can be expressed in terms of R_{cm} and r, giving us:

$$r_1 = R_{cm} - \frac{m_2}{M} r \quad \text{and} \quad r_2 = R_{cm} + \frac{m_1}{M} r \tag{3}$$

▼ `Derivation`

Let r_1' and r_1' be positions of m_1 and m_2 and the CM frame. By vector decompositions, we know that

$$r_1' = r_1 - R_{cm}, \quad r_2' = r_2 - R_{cm}$$

Substituting the expression for R_{cm}, equations are simplified into:

$$r_1' = \frac{m_2}{m_1 + m_2}(r_1 - r_2) = -\frac{m_2}{m_1 + m_2} r$$

$$r_2' = \frac{m_1}{m_1 + m_2}(r_1 - r_2) = \frac{m_1}{m_1 + m_2} r$$

Then, r_1 and r_2 are equal to:

$$r_1 = -R_{cm} + \frac{m_2}{m_1 + m_2} r$$

$$r_1 = R_{cm} + \frac{m_1}{m_1 + m_2} r$$

The total kinetic energy of the two bodies is equal to:

$$T = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 \tag{4}$$

This is equal to:

$$\boxed{T = \frac{1}{2}M\dot{R}_{cm}^2 + \frac{1}{2}\mu\dot{r}^2} \tag{5}$$

where μ is the reduced mass and is equal to:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

The Lagrangian L is equal to:

$$L = T - U = \frac{1}{2}M\dot{R}_{cm} + \frac{1}{2}\mu\dot{r}^2 - U(r)$$

The Lagrangian shows that R_{cm} is cyclic, so the corresponding momentum $p = MR_{cm}$ is conserved. This means that the center of mass of the two-body system drifts through space with a constant velocity and momentum, reducing the two-body system to a one-body central force problem (The statement is explained further in the following section).

7.21: Interpretation of the Lagrangian(CM Frame)

he Lagrangian is we derived has two portions: the Lagrangian of the center of mass and the Lagrangian for μ, the relative motion between m_1 and m_2.

$$L = L_{cm} + L_{rel} = \frac{1}{2}M\dot{R}_{cm} + \frac{1}{2}\mu\dot{r}^2 - U(r)$$

The portion of the Lagrangian $1/2\mu\dot{r}^2 - U(r)$ has the same form as that a single particle with mass μ orbiting around a force centered at the origin. Using spherical coordinates, position $\vec{r} = (r, r\theta, r\sin\theta\phi)$ and velocity is equal to $\vec{v} = \dot{r} = (\dot{r}, r\dot{\theta}, r\sin\theta\dot{\phi})$.

$$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - U(r)$$

Written in spherical coordinates and let $\theta = \pi/2$, the Lagrangian is equal to:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

The notion simplifies the two-body problem significantly if considering the inertial CM frame. The center of mass drifts with a constant velocity. In the CM frame, $\dot{R} = 0$, so $L_{cm} = 0$. The Lagrangian is simplified into:

$$L = \frac{1}{2}\mu\dot{r}^2 - U(r) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Interpretation of the Lagrangian and Motion in CM Frame

- In the CM frame, the center of mass is stationary, and we treat it as a fixed origin. Because momentum $p = M\dot{R}$ is conserved, masses m_1 and m_2 move with an equal and opposite momentum. If $m_2 >> m_1$ (the mass of a planet is is much smaller than mass of a star), the CM is close to m_2 and m_2 has a much smaller speed)

We make several observations from the Lagrangian:

i. L is not an explicit function of time, so Hamiltonian H is conserved, which in this case is the sum of kinetic and potential energies:

$$E = H = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + U(r)$$

ii. The angle ϕ is cyclic, so the corresponding angular momentum p_ϕ is conserved.

$$p_\phi = \mu r^2\dot{\phi} = r \times p = l$$

7.3: The Effective Potential Energy

Recall that in section 7.2, we have derived the equation

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + U(r)$$

where angular momentum $p_\phi = r(\mu r\dot{\phi}) = l$ is conserved. Expanding the expression, E is equal to:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r)$$

The first term is similar $1/2mv^2$ for linear kinetic energy, and the second term $1/2\mu r^2\dot{\phi}^2$ rotational kinetic energy. We rewrite the equation in the form

$$E = \frac{1}{2}\mu\dot{r}^2 + (\frac{1}{2}\mu r^2\dot{\phi}^2 + U(r)) = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) \tag{6}$$

and defined effective potential as:

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r) \tag{7}$$

Then, energy E is equal to:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) \tag{8}$$

The angular momentum l allows us to convert rotational kinetic energy $1/2\mu r^2\dot{\theta}^2$ into a term depending on position r only, which behaves like a potential energy. The sum of this term and the “real” potential energy is defined as effective potential energy. The “fake” potential energy is called centrifugal potential

$$U_{cent}(r) = \frac{l^2}{2\mu r^2} \tag{9}$$

Because its corresponding “force” $F_{cent} = -dU_{cent}/dt = l^3/\mu r^3$ tends to push the orbiting particle away from the force center at the origin (this can be seen by having a positive sign. An attractive force towards the origin is defined as the negative direction and vice versa). In general, $U_{eff}(r)$ has a minimum:

$$U'_{eff}(r) = -\frac{l^3}{\mu r^3} + U'(r) = 0$$

the system admits circular orbit at $r = R$. Such an orbit would be stable if $U''_{eff}(r) > 0$ and unstable if $U''_{eff}(r) < 0$.

$$U''_{eff}\Big|_{r=R} = 3\frac{l^3}{\mu r^4} + U''(r)\Big|_{r=R} \begin{cases} > 0 : \text{stable} \\ < 0 : \text{unstable} \\ = 0 : \text{critically stable} \end{cases} \tag{10}$$

$$\begin{cases} r_{min} < r < r_{max} : \text{bounded noncircular orbits} \\ -\infty < r < r_{min} : \text{unbounded} \\ r_{max} < r < \infty : \text{unbounded} \end{cases} \tag{11}$$

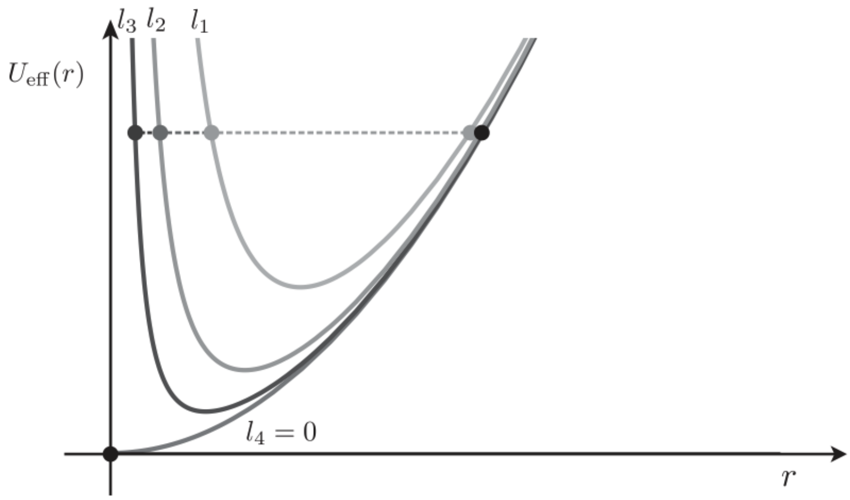
7.31: Radial Motion for the Central-Spring Problem

Recall that effective potential energy is defined as:

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r)$$

where $U(r) = \frac{1}{2}kr^2$ for a spring. Then, $U_{eff}(r)$ is equal to:

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + \frac{1}{2}kr^2$$



For $l = 0$, the angular momentum barrier vanishes, corresponding to a particle moving toward the origin. The motion is then entirely radial (truly on-dimensional) and the particle oscillates back and force through the origin.

From equation (8), we can see that energy E is equal to:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) \tag{12}$$

where $\dot{r} = dr/dr$. Solving for dr, we get:

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E - \frac{1}{2}kr^2 - \frac{l^2}{2\mu r^2})}$$

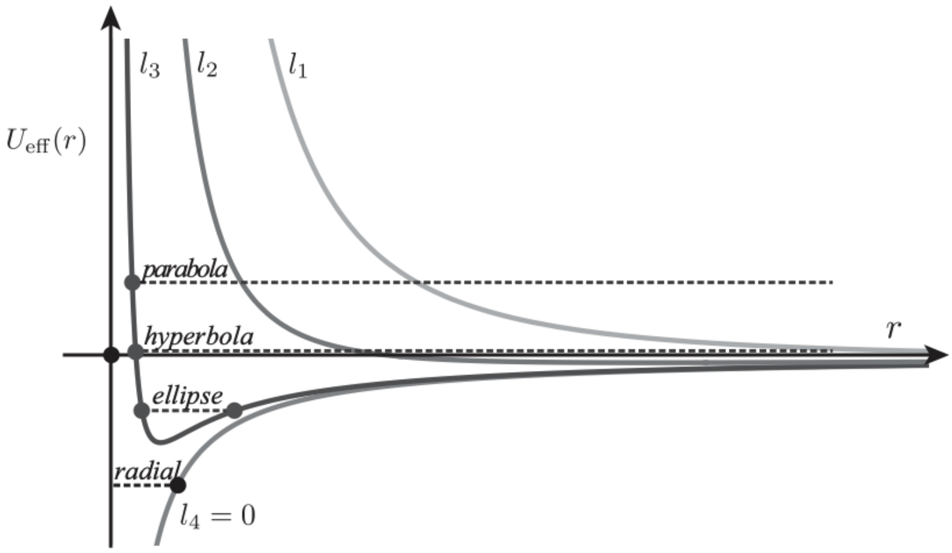
Using separable integration, the time $t(r)$ to move from radius r_0 to r is equal to:

$$t(r) = \pm\sqrt{\frac{\mu}{2}}\int_{r_o}^{r} \frac{rdr}{\sqrt{Er^2 - kr^4/2 - l^2/2\mu}} \quad (13)$$

7.32: Radial Motion in Central Gravity

The effective potential energy of a particle in a central gravitational field is:

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} - \frac{GMm}{r} \quad (14)$$



There are two types of orbits: bound orbits with energy $E < 0$ and unbound orbits with energy $E \geq 0$.

Bound Orbits:

- Bound orbits do not escaped to infinity. They includes circular orbits with an energy E_{min} corresponding to the energy at the bottom of the potential well, where only one radius is possible (circular motion).

- The minimum radius is called the periapse for orbits around an arbitrary object.
- The maximum radius r_{max} is called the apoapse in general

Unbound Orbits

- Unbound orbits are those with no outer turning point: these orbits extend out infinitely far. There are orbits with $E = 0$ that are barely unbounded. In this case, kinetic energy goes to zero in the limiting case as the orbiting particle infinitely far from the origin. Trajectories are parabolas
- For orbits with $E > 0$, the orbiting particle has a positive kinetic energy when infinitely far from the origin. Trajectories are hyperbolas

Energy E is equal to:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} - \frac{GMm}{r} \tag{15}$$

where $\dot{r} = dr/dt$. Solving for dr, we get:

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E + \frac{GM\mu}{r} - \frac{l^2}{2\mu r})}$$

with the fact $m_1 m_2 = M\mu$. Using separable integration, $t(r)$ is equal to:

$$t(r) = \pm\sqrt{\frac{2}{\mu}}\int_{r_0}^{r} \frac{r dr}{\sqrt{Er^2 + GM\mu r - l^2/2\mu}} \tag{16}$$

$$\begin{cases} E < 0 : \text{bound orbits, ellipse} \\ E_{min} : \text{bound orbit, circle} \\ E = 0 : \text{barely unbound, parabolas} \\ E > 0 : \text{ubound orbits, hyperbolas} \end{cases} \tag{17}$$

7.4: The Shape of Central-Force Orbits

In this section, we want to determine the shape of the orbits for the time evolution determined above. To do this, we will eliminate t from the equations and find equations involving r and ϕ only. Recall that energy E is equal to:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r} + U(r), \quad l = mr^2\dot{\phi}$$

and $\dot{r} = dr/dt$. Because we want to find equations involving r and ϕ only (this is done by finding differential equations first), we use separable integration. We first find $dr/d\phi$. Using chain rule, we get

$$\frac{dr}{d\phi} = \frac{dr/dt}{d\phi/dt}$$

Solving for dr/dt using E and $d\phi/dt$ using angular momentum l, $dr/d\phi$ is equal to:

$$\frac{dr}{d\phi} = \sqrt{\frac{2m}{l^2}}r^2\sqrt{E - l^2/2mr^2 - U(r)}$$

Using separable integration, ϕ is equal to:

$$\phi = \int d\phi = \pm\int^r \frac{dr/r^2}{\sqrt{E - l^2/2mr^2 - U(r)}} \tag{18}$$

7.41: Central Spring-Force Orbits

$$\phi = \int d\phi = \pm\int^r \frac{dr/r^2}{\sqrt{E - l^2/2mr^2 - (1/2)kr^2}} \tag{19}$$

Substitute $z = r^2$, the intergral becomes:

$$\phi = \int d\phi = \pm\int^r \frac{dz/z^2}{\sqrt{E - l^2/2mr^2 - (1/2)kz}} \tag{20}$$

Multiplying top and bottom of the integrand by r and substituting $z = r^2$ gives

$$\phi(z) = \pm\frac{\ell}{2\sqrt{2m}}\int^z \frac{dz/z}{\sqrt{-\ell^2/2m + Ez - (k/2)z^2}}. \tag{7.34}$$

From integral tables online or in a book, we find that

$$\int^z \frac{dz/z}{\sqrt{a + bz + cz^2}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bz + 2a}{z\sqrt{b^2 - 4ac}}\right), \tag{7.35}$$

where a, b, and c are constants, with $a < 0$. In our case $a = -\ell^2/2m, b = E$, and $c = -k/2$, so

$$\begin{aligned}\phi - \phi_0 &= \pm\frac{\ell}{2\sqrt{2m}}\frac{1}{\sqrt{\ell^2/2m}}\sin^{-1}\left(\frac{bz + 2a}{z\sqrt{b^2 - 4ac}}\right) \\ &= \pm\frac{1}{2}\sin^{-1}\left(\frac{Er^2 - \ell^2/m}{r^2\sqrt{E^2 - k\ell^2/m}}\right), \end{aligned} \tag{7.36}$$

where ϕ_0 is a constant of integration. Multiplying by ± 2, taking the sine of each side, and solving for r^2 gives the orbital shape equation

$$r^2(\phi) = \frac{\ell^2/m}{E \mp (\sqrt{E^2 - k\ell^2/m})\sin 2(\phi - \phi_0)}. \tag{7.37}$$

7.42: The Shape of Gravitational Orbits

7.421: By Integration

Gravitational potential energy is equal to $U(r) = -GMm/r$, so $\phi(r)$ is equal to:

$$\phi = \pm\int^r \frac{dr/r^2}{\sqrt{E - +GMm/r - l^2/2mr^2}} = \pm\int^r \frac{dr/r}{\sqrt{Er^2 + GMmr - l^2/2m}}$$

In order to simplify the integral, we use the technique that:

$$\int \frac{dr/r}{a + br + cr^2} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{br + 2a}{r\sqrt{b^2 - 4ac}}\right)$$

where $a = -l^2/2m, b = GMm$ and $c = E$. Then, ϕ is equal to:

$$\begin{aligned}\phi &= \frac{1}{\sqrt{l^2/2m}}\sin^{-1}\left(\frac{GMmr - l^2/m}{r\sqrt{G^2M^2m^2 + 2l^2E/m}}\right) \\ &= \frac{1}{\sqrt{l^2/2m}}\sin^{-1}\left(\frac{GMmr - l^2/m}{r\sqrt{1 + 2El^2/G^2M^2m^3}}\frac{1}{GMm}\right) \\ &= \frac{1}{\sqrt{l^2/2m}}\sin^{-1}\left(\frac{GMm^2r - l^2}{r\epsilon}\frac{1}{GMm^2}\right) \\ &= \frac{1}{\sqrt{l^2/2m}}\sin^{-1}\left(\frac{GMm^2r - l^2}{\epsilon GMm^2r}\right)\end{aligned}$$

where ϵ is defined as the eccentricity:

$$\epsilon = \sqrt{1 + \frac{2El^2}{G^2M^2m^3}} \tag{21}$$

Solving for r, we get:

$$r = \frac{l^2/GMm^2}{1 \pm \epsilon\sin(\phi - \phi_0)} \tag{22}$$

By convention, we choose the plus sign in the denominator with $\phi_0 = \pi/2$, which in effect locates $\phi = 0$ at the point of closest approach to the center, the periapse of the ellipse. The choice changes sine to cosine, giving us:

$$r = \frac{l^2/GMm^2}{1 + \epsilon\cos(\phi)} \tag{23}$$

7.422: By Differentiation

Return to the step for $dr/d\phi$

$$\frac{dr}{d\phi} = \sqrt{\frac{2m}{l^2}}r^2\sqrt{E - l^2/2mr^2 - U(r)}$$

Let $u = 1/r$. Using chain rule, the equation is equal to:

$$\frac{d(1/u)}{dt} = -\frac{1}{u^2}\frac{du}{d\phi}$$

Square the equation, we get:

$$(\frac{du}{d\phi})^2 = (u')^2 = -u^4(\frac{dr}{d\phi})^2$$

which is equal to:

$$(u')^2 = \frac{2m}{l^2}(E - \frac{l^2}{2mr^2} + \frac{GMm}{r}) = \frac{2m}{l^2}(E - \frac{l^2u^2}{2m} + GMmu)$$

Differentiating both sides again, we get:

$$2u'u'' = \frac{2m}{l^2}(-\frac{l^2u}{m}u' + GMmu')$$

$$2u'u'' = -2uu' + \frac{2GMm^2}{l}u'$$

This gives us a second order differential equation:

$$u'' + u = \frac{GMm^2}{l^2}u'$$

$$r_p = \frac{l^2/GMm^2}{1+\epsilon} \tag{24}$$

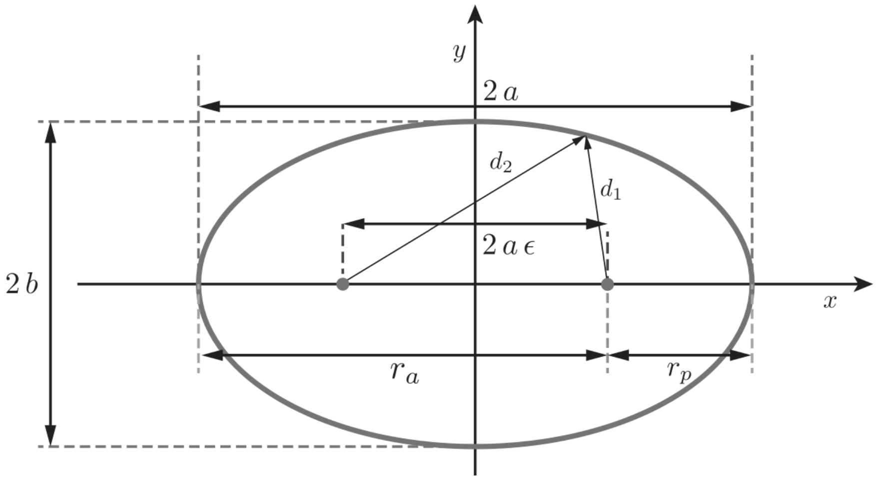
Then, eccentricity is equal to:

$$\epsilon = \sqrt{1 + \frac{2El^2}{G^2M^2m^3}} \tag{25}$$

Conic sections:

- For circles, eccentricity is zero, so radius $r = r_p$, a constant independent of ϕ
- ellipses: $0 < \epsilon < 1$

The long axis of the ellipse is called the major axis, and half of this distance is the semi-major axis, denoted by a. The shorter axis is the minor axis, and half of this distance is the semi-minor axis, denoted by b.



One can derive several properties of the ellipses using the equation for eccentricity.

- The periapse and apoapse of the ellipses (the shortest and farthest points of the orbit from the right-hand focus) are given by:

$$r_p = a(1-\epsilon), \quad r_a = a(1+\epsilon) \tag{26}$$

- The sum of distances d_1 and d_2 from the two foci to a point on the ellipses is the same for all points on the ellipses.
- The distance between the two foci is $d = 2a\epsilon$, so the eccentricity of an ellipses is the ratio of this interfocal distance to the length of the major axis.

$$\epsilon = \frac{d}{2a}$$

- The semi-major and semi-minor axes is related by:

$$b = a\sqrt{1-\epsilon^2}$$

- The are is the ellipse is $A = \pi ab$, where a and b are the semi-major and semi-minor axes.

$$A = \pi ab \tag{27}$$

- The shape of an orbit is entirely is through two parameters we choose: energy E and angular momentum l, or eccentricity ϵ and the periapse radius r_p (shortest point of the orbit from the right-hand focus), or semi-major and semi-minor axes a and b.

Example 7.1: Orbital Geometry and Orbital Physics

Let us relate the geometrical parameters of a gravitational orbit to the physical parameters, the energy E and angular momentum l. The relationships follow from equations we derived. We first consider circles and ellipses, and then parabolas and hyperbolas. Find energy E.

▼ `Solution`

Recall that eccentricity ϵ is defined as:

$$\epsilon = \sqrt{1 + \frac{2El^2}{G^2M^2m^3}}$$

As mentioned before, the shape of an orbit can be determined using two parameters, such as energy and angular momentum, eccentricity and peripase radius, and semi-major and semi-minor axes. Because we want to relate eccentricity to semi-major axis a, we need to think about an equation that includes both a and ϵ, reminding us about the peripase radius.

$$r_p = a(1 - \epsilon)$$

However, because we wan to get rid of the square root in ϵ, we will use $a(1 - \epsilon^2) = l^2/GMm^2$ as we derived in section 7.422. Solving for a, we get:

$$a = \frac{l^2}{GMm^2(1 - \epsilon^2)}$$

Substituting the equation for ϵ into r_p, the expression becomes:

$$a = -\frac{GMm}{2E}$$

From the equation, we can see that the semi-major axis depends on energy E only and not angular momentum l. In summary, for ellipses and circles, the geometric parameters are related to physical parameters by:

$$a = -\frac{GMm}{2E}$$

These can be inverted to give physical parameters in terms of the geometric parameters, giving us:

$$E = -\frac{GMm}{2a}, \quad l = \sqrt{GMm^2a(1-\epsilon^2)}$$

For a circular orbit, eccentricity $\epsilon = 0$, giving us $l = \sqrt{GMm^2a}$, so

$$l^2 = GMm^2a$$

We want to express energy in term of angular momentum, so we will replace a, giving us:

$$E = -\frac{G^2M^2m^3}{2l^2}$$

7.5: Bertrand's Theorem

Betrand's theorem: The only central force potential $U(r)$ for which all bounded orbits are closed are the following:

1. The gravitational potential $U(r)$ is directly proportional to $-1/r$
2. The central spring potential $U(r)$ is directly proportional to r^2

7.6: Orbital Dynamics

Recall from chapter 5 From Classical to Quantum Back, Kepler identified three laws governing planetary motion.

1. Planets move in elliptical orbits, which the sun at one focus.
2. Planetary orbits sweep out equal areas in equal time.
3. The periods squared of planetary orbits are proportional to their semi-major axes cubed.

$$T^2...a^3$$

7.61: Kepler's Second Law

Kepler's law states that planetary orbits sweep out equal areas in equal time. The theorem is purely a result of angular momentum conservation with some approximations.

For a thin slice of triangle, where angle ϕ approaches zero, area$A$ is equal to $\mathrm{Area{=}(1/2)(Base)(height)}$. Because ϕ is small, base is approximated using the orbit's arc length and height is approximately equal to radius r. Then, area A is equal to:

$$A = \frac{1}{2} r(r\phi)$$

Differentiating A with respect to time t (notice we have assumed radius r as a constant), we get:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{\mu r^2 \dot{\phi}}{2\mu} = \frac{l^2}{2\mu}$$

Using separable integration, the area swiped between t_1 and t_2 is equal to:

$$A = \int_{t_1}^{t_2} \frac{l^2}{2\mu} dt = \frac{l^2}{2\mu}(t_2 - t_1)$$

which is equal to the area sweeped between t_3 and t_4 if time intervals are equal. Thus, we have proved Kepler's second law.

7.62: Kepler's Third Law

Period T is equal to:

$$T = \frac{2m}{l} A$$

where A is the area of the ellipses, given by $A = \pi ab$. Then, T is equal to:

$$T = \frac{2\pi}{\sqrt{GM}} a^{3/2} \tag{28}$$

Example 7.2:

From the comet's (Halley's Comet) current period of $T = 75.3 years$ and observed perihelion distance $r_p = 0.586 AU$ (which lies between the orbits of Mercury and Venus), we can calculate the orbit's (a) semi-major axis a, (b) aphelion distance r_a , and (c) eccentricity ε. (Note that 1 AU is the length of the semi-major axis of earth's orbit, $1 AU = 1.5 \times 10^{11} m$.)

▼ `Solution`

(a) Recall from Kepler's third law, we know that the period of an orbit directly proportional to the $3/2th$ power of its semi-major axis. This gives us:

$$\frac{T_H}{T_E} = (\frac{a_H}{a_e})^{3/2}$$

where T_e is the period is earth and is equal to one year. Solving for T_H, we get:

$$a_H = a_e(\frac{T_H}{T_e})^{2/3}$$

Substituting numbers, we get

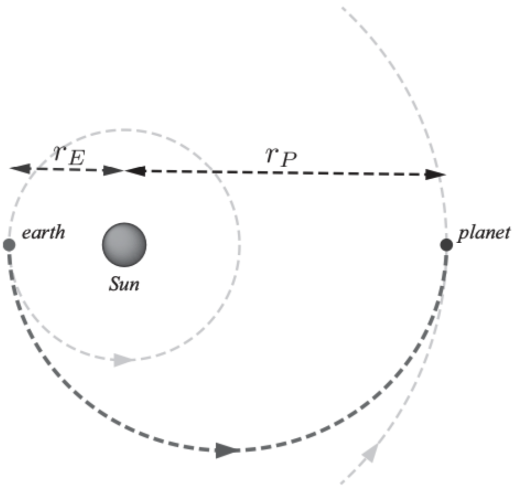
$$a_H = 17.8\text{AU}$$

(b)

7.6.3: Minimum Energy Transfer Orbits

When sending a spacecraft, we want to find a trajectory requiring least fuel (ignoring gravitational assist from other planets). The trajectory is called minimum-energy transfer orbit or or Hohmann transfer orbit, taking advantages to earth's motion.

Typically, the spacecraft is first lifted into low-earth orbit (LEO), where it circles the earth a few hundred kilometers above the surface. Then, at the right time, the spacecraft is given a velocity boost Δv the sends it away from the earth and into an orbit around the sun that reaches all the way to its destination. Once the spacecraft coasts far enough from earth and the sun's gravity dominates, it obeys central motion we have discussed so far, including Kepler's law. **It coasts towards its destination in an elliptical orbit with the sun at one focus.**



7.6.3.1: Time Taking to Reach the Destination

We will find the time taking to reach the destination using Kepler's third law. The major axis of the craft's orbit is $2a_c = r_e + r_p$, where r_e is the distance between the earth and the sun and r_p is the distance between the sun and the planet. The semi-major axis is equal to:

$$a_c = \frac{r_e + r_p}{2} \tag{29}$$

From Kepler's third law, we know that $(T_c/T_e)^2 = (a_c/a_e)^3$, where T_c is the craft's elliptical orbit period and T_e is the period when orbiting around earth. However, because the craft only use one half of the period when traveling from earth to planet, indicated by dotted lines in the figure, period T is equal to:

$$\boxed{T = \frac{1}{2}T_c = \frac{1}{2}(\frac{r_e + r_p}{2r_e})^{3/2}T_e} \tag{30}$$

7.6.3.2: Velocity When a Away

Now, we can outline the steps required for the spacecraft to reach Mars or an outer planet.

1. **Orbital velocity v_0 when r_0 away from earth:** As mentioned before, the craft is lifted to lower orbits first with a orbital speed v_0, so it is in circular motion at a distance r_0 around earth. By equalizing centripetal and gravitational attraction force, v_0 is equal to:

$$\frac{GM_e}{r_0^2} = \frac{v_0^2}{r_0}$$

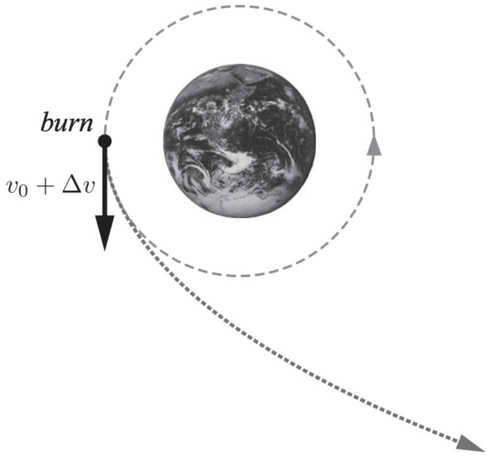
$$v_0 = \sqrt{\frac{GM_e}{r_0}} \tag{31}$$

2. **Velocity when escapes earth's gravity:** Then, at the right moment, a rocket provides a boost velocity Δv in the same direction of v_0, so the craft has a velocity $\Delta v + v_0$. The applied velocity at the right moment allows the craft to escape from earth in the most efficient way. Using energy conservation, we get

$$\frac{1}{2}mv_\infty^2 = \frac{1}{2}m(v_0 + \Delta v)^2 - \frac{GM_e m}{r_0} \tag{32}$$

where v_∞ is the speed at the destination. Solving for v_∞, we get:

$$v_\infty = \sqrt{(v_0 + \Delta v)^2 - 2GM_e/r_0} = \sqrt{(v_0 + \Delta v)^2 - 2v_0^2} \tag{33}$$



3. If the boost velocity Δv is provided at the time when the spacecraft is moving in the same direction as earth's velocity v_e around the same, the craft's velocity in the sun's frame of reference is:

$$v = v_\infty + v_e = \sqrt{(v_0 + \Delta v)^2 - 2v_0^2} + v_e \tag{34}$$

4. The velocity v we have just calculated will be the speed of the spacecraft at the perihelion point of some elliptical Hohmann transfer orbit. The required velocity v to reach the desired orbit for the semi-major axis is calculated using energy conservation, giving us

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} \tag{35}$$

Thus, the required velocity for craft to be injected from Earth is equal to:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{36}$$

Example 7.3: A Voyage to Marts

We will use this scenario to plan a trip to Mars by Hohmann transfer orbit. First, we can use Kepler's third law to find how long it will take for the spacecraft to arrive and the boost velocity Δv. Earth and Mar have radius of $1.5 \times 10^8 m$ and $2.28 \times 10^8 m$.

▼ `Solution`

The length of the semi-major axis of the elliptical orbit is equal to:

$$a_c = \frac{1.5 + 2.28}{2} \times 10^8 m = 1.89 \times 10^8 m$$

The craft travels through half of the period, so the travel is T is equal to:

$$T = \frac{1}{2}T_c = \frac{1}{2}(\frac{r_e + r_p}{2r_e})^{3/2}T_e$$

$$T = \frac{1}{2}T_c = \frac{1}{2}(\frac{1.89 \times 10^8/2}{2 \times 1.5 \times 10^8})^{3/2}(1year) = 258days$$

As derived in the pervious section, we know that the craft has a velocity v when it is distance a away from earth, where v is calculated using:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

In this problem, r is the earth's radius and a is the length of the orbit's semi-major axis (the distance between earth and the craft). When reaching Mars, the craft is $1.89 \times 10^8 m$ away from Earth. Substituting $r = r_e = 1.5 \times 10^8 m$ and $a = 1.89 \times 10^8 m$ in to the equation, velocity v is equal to:

$$v = \sqrt{GM\left(\frac{2}{1.5 \times 10^8} - \frac{1}{1.89 \times 10^8}\right)} = 32.7km/s$$

In order to find boost velocity Δv, we need to think about an equation relating v and Δv, reminding us about equation (29), where v_0 is the craft's orbital velocity and v is equal to:

$$v_\infty = \sqrt{(v_0 + \Delta v)^2 - 2v_0^2}$$

Given earth's radius, we can calculated v_0 and thus find v_0, which is equal to:

$$v_0 = \frac{GM}{r_e} = 7.5km/s$$

Solving for Δv, we get:

$$\Delta v = \sqrt{v_\infty^2 + 2v_0^2} - v_0$$

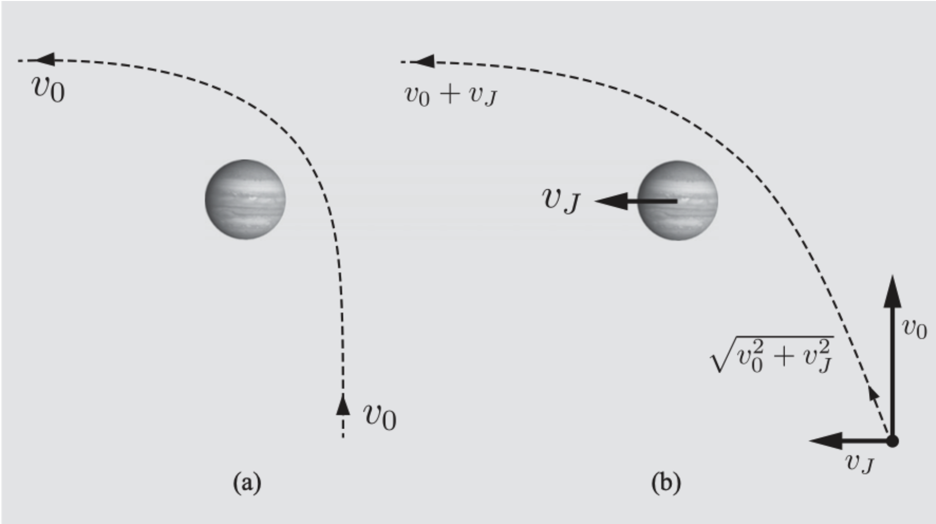
Ignoring earth's speed, we approximated $v \approx v_\infty$. Then, Δv is equal to:

$$\Delta v = \sqrt{(32.7)^2 + 2(7.5)} - 7.5 = 3.5km/s$$

Example 7.4: Gravitational Assists

Suppose we want to send a heavy spacecraft to Saturn, but it has only enough room for fuel to make it to Jupiter. If the timing is just right and the planets are also aligned just right, it is possible to aim for Jupiter, causing the spacecraft to fly just *behind* Jupiter as it swings by that planet. Jupiter can pull on the spacecraft, turning its orbit to give it an increased velocity in the sun's frame of reference, sufficient to propel it out to Saturn. ***Explain how***.

▼ `Solution`



7.7: The Virial Theorem in Astrophysics

Consider a collection of N point like non-relativistic partiles, where the ith particle is at position r_i, has momentum p_i, and is subject to a net force F_i. We define a quantity $G \equiv \sum_i p_i \cdot r_i$, whose time derivative is:

$$\frac{dG}{dt} = \sum_i \dot{p}_i \cdot r_i + \sum_i p_i \cdot \dot{r}_i \tag{37}$$

We know that $\dot{p}_i = F_i$ and $p_i \cdot r_i = m_i v_i^2 = 2T_i^2$. Then, the time derivative is equal to:

$$\frac{dG}{dt} = \sum_i F_i \cdot r_i + \sum_i 2T_i = \sum_i F_i \cdot r_i + 2T \tag{38}$$

We then take the time average of each term in the equation, where the total time is τ. This gives us:

$$\left\langle \frac{dG}{dt} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{dG}{dt} dt = \frac{1}{\tau}(G(t) - G(0)) \tag{39}$$

If the motion is periodic such that $G(t) = G(0)$, the average of time derivatives is equal to zero. This is true for central force motion, such as central gravitational or spring force, where τ is the orbital period. More generally, suppose all motion are at least bounded, with an upper limit to G. Then, over a long period of time, the left-hand side of the equation is zero, such that

$$\left\langle \frac{dG}{dt} \right\rangle = -\frac{1}{2}\left\langle \sum F_i \cdot r_i \right\rangle \tag{40}$$

💡 More generally, suppose all motion are at least bounded, with an upper limit to G. Then, over a long period of time, the average of time derivatives $dG/dt = 0$.

Because $F = -\nabla U$, the expression becomes:

$$\left\langle \frac{dG}{dt} \right\rangle = -\frac{1}{2}\left\langle -\nabla U \cdot r \right\rangle = -\frac{1}{2}\left\langle (-dU/dr) \cdot r \right\rangle \tag{41}$$

Example 7.5:

Now consider N particles that pull on one another with central gravitational forces, so all forces on a particle are due to other particles in the system. This might be a good approximation for the gravitational attractions of stars on one another in a globular cluster, for example, or for entire galaxies attracting one another in a cluster of galaxies like the Coma Cluster or the Virgo Cluster. Let us start simply, by considering the case $N = 3$. The force of particle 2 on particle 1 is F_{12} , and the force of particle 1 on particle 3 is F_{31}, and so on. Therefore counting all six interactions:

▼ `Solution`

Summary

Key Equations

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$

$$t(r) = \pm\sqrt{\frac{2}{\mu}}\int_{r_0}^{r}\frac{rdr}{\sqrt{Er^2 + GM\mu r - l^2/2\mu}}$$

$$r_p = \frac{l^2/GMm^2}{1+\epsilon}, \quad r_p = a(1-\epsilon)$$

$$\text{Eccentricity:} \quad \epsilon = \sqrt{1 + \frac{2El^2}{G^2M^2m^3}}$$

$$\text{Semi-major axis:} \quad a = -\frac{GMm}{2E}$$

$$\text{Required injection velocity from earth:} \quad v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Practice Problems

Chapter 7 Practice Problems