HL Paper 2

Boxes of mixed fruit are on sale at a local supermarket.

Box A contains 2 bananas, 3 kiwifruit and 4 melons, and costs \$6.58.

Box B contains 5 bananas, 2 kiwifruit and 8 melons and costs \$12.32.

Box C contains 5 bananas and 4 kiwifruit and costs \$3.00.

Find the cost of each type of fruit.

Given that $\log_{10}\left(\frac{1}{2\sqrt{2}}(p+2q)\right) = \frac{1}{2}(\log_{10}p + \log_{10}q)$, $p > 0,\ q > 0$, find p in terms of q.

Use the method of mathematical induction to prove that $5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$.

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a. Show that $\left|\mathrm{e}^{\mathrm{i}\theta}\right| = 1$. [1]

b. Consider the geometric series $1 + \frac{1}{3}\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{9}\mathrm{e}^{2\mathrm{i}\theta} + \ldots$. [2]

Write down the common ratio, z, of the series, and show that $|z| = \frac{1}{3}$.

c. Find an expression for the sum to infinity of this series. [2]

d. Hence, show that $\sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta + \ldots = \frac{9\sin\theta}{10-6\cos\theta}$. [8]

Consider the system of equations

$$0.1x - 1.7y + 0.9z = -4.4$$

$$-2.4x + 0.3y + 3.2z = 1.2$$

$$2.5x + 0.6y - 3.7z = 0.8.$$

a. Express the system of equations in matrix form. [2]

b. Find the solution to the system of equations. [3]

(a) Solve the equation $z^3 = -2 + 2\mathrm{i}$, giving your answers in modulus-argument form.

(b) **Hence** show that one of the solutions is $1 + \mathrm{i}$ when written in Cartesian form.

Six people are to sit at a circular table. Two of the people are not to sit immediately beside each other. Find the number of ways that the six people can be seated.

A metal rod 1 metre long is cut into 10 pieces, the lengths of which form a geometric sequence. The length of the longest piece is 8 times the length of the shortest piece. Find, to the nearest millimetre, the length of the shortest piece.

Find, in its simplest form, the argument of $(\sin\theta + \mathrm{i}(1 - \cos\theta))^2$ where θ is an acute angle.

The complex numbers z_1 and z_2 have arguments between 0 and π radians. Given that $z_1 z_2 = -\sqrt{3} + \mathrm{i}$ and $\frac{z_1}{z_2} = 2\mathrm{i}$, find the modulus and argument of z_1 and of z_2.

Ava and Barry play a game with a bag containing one green marble and two red marbles. Each player in turn randomly selects a marble from the bag, notes its colour and replaces it. Ava wins the game if she selects a green marble. Barry wins the game if he selects a red marble. Ava starts the game.

a. Find the probability that Ava wins on her first turn. [1]

b. Find the probability that Barry wins on his first turn. [2]

c. Find the probability that Ava wins in one of her first three turns. [4]

d. Find the probability that Ava eventually wins. [4]

Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^5 - 1 = 0$ which has the smallest positive argument.

a. (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$. [6]

(ii) Hence use De Moivre's theorem to prove

$$\sin 5\theta = 5\cos^4\theta \sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

(iii) State a similar expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

b. Find the value of r and the value of α. [4]

c. Using (a) (ii) and your answer from (b) show that $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$. [4]

d. Hence express $\sin 72^\circ$ in the form $\frac{\sqrt{a+b\sqrt{c}}}{d}$ where $a,\ b,\ c,\ d \in \mathbb{Z}$. [5]

Determine the first three terms in the expansion of $(1-2x)^5(1+x)^7$ in ascending powers of x.

The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

a. Show that $d = \frac{a}{2}$. [3]

b. The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200. [6]

Find the least value of n for which this occurs.

The arithmetic sequence $\{u_n : n \in \mathbb{Z}^+\}$ has first term $u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence $\{v_n : n \in \mathbb{Z}^+\}$ has first term $v_1 = 3$ and common ratio $r = 1.2$.

a. Find an expression for $u_n - v_n$ in terms of n. [2]

b. Determine the set of values of n for which $u_n > v_n$. [3]

c. Determine the greatest value of $u_n - v_n$. Give your answer correct to four significant figures. [1]

Each time a ball bounces, it reaches 95 % of the height reached on the previous bounce.
Initially, it is dropped from a height of 4 metres.

a. What height does the ball reach after its fourth bounce? [2]

b. How many times does the ball bounce before it no longer reaches a height of 1 metre? [3]

c. What is the total distance travelled by the ball? [3]

The three planes having Cartesian equations $2x + 3y - z = 11,\ x + 2y + z = 3$ and $5x - y - z = 10$ meet at a point P. Find the coordinates of P

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(a) Show that the complex number i is a root of the equation

$$x^4 - 5x^3 + 7x^2 - 5x + 6 = 0\ .$$

(b) Find the other roots of this equation.

From a group of five males and six females, four people are chosen.

a. Determine how many possible groups can be chosen. [2]

b. Determine how many groups can be formed consisting of two males and two females. [2]

c. Determine how many groups can be formed consisting of at least one female. [2]

Consider the following system of equations

$$2x + y + 6z = 0$$

$$4x + 3y + 14z = 4$$

$$2x - 2y + (\alpha - 2)z = \beta - 12.$$

a. Find conditions on α and β for which [6]

(i) the system has no solutions;

(ii) the system has only one solution;

(iii) the system has an infinite number of solutions.

b. In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form. [3]

One root of the equation $x^2 + ax + b = 0$ is $2 + 3\mathrm{i}$ where $a,\ b \in \mathbb{R}$. Find the value of a and the value of b.

a. Find the set of values of x for which the series $\sum_{n=1}^{\infty}\left(\frac{2x}{x+1}\right)^n$ has a finite sum. [4]

b. Hence find the sum in terms of x. [2]

The system of equations

$$2x - y + 3z = 2$$

$$3x + y + 2z = -2$$

$$-x + 2y + az = b$$

is known to have more than one solution. Find the value of a and the value of b.

Consider $w = \frac{z}{z^2+1}$ where $z = x + \mathrm{i}y$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that $\operatorname{Im} w = 0$, show that $|z| = 1$.

The equations of three planes, are given by

$$ax + 2y + z = 3$$

$$-x + (a+1)\,y + 3z = 1$$

$$-2x + y + (a+2)\,z = k$$

where $a \in \mathbb{R}$.

a. Given that $a = 0$, show that the three planes intersect at a point. [3]

b. Find the value of a such that the three planes do not meet at a point. [5]

c. Given a such that the three planes do not meet at a point, find the value of k such that the planes meet in one line and find an equation of this line in the form [6]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

a. (i) Express the sum of the first n positive odd integers using sigma notation. [4]

(ii) Show that the sum stated above is n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers.

b. A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points. [7]

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is $\frac{n(n-3)}{2}$ if there are n points, where $n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible.

c. The random variable $X \sim B(n,\ p)$ has mean 4 and variance 3. [8]

(i) Determine n and p.

(ii) Find the probability that in a single experiment the outcome is 1 or 3.

Given that $z = \frac{2-\mathrm{i}}{1+\mathrm{i}} - \frac{6+8\mathrm{i}}{u+\mathrm{i}}$, find the values of u, $u \in \mathbb{R}$, such that $\mathrm{Re}\, z = \mathrm{Im}\, z$.

The fourth term in an arithmetic sequence is 34 and the tenth term is 76.

(a) Find the first term and the common difference.

(b) The sum of the first n terms exceeds 5000. Find the least possible value of n.

It is known that the number of fish in a given lake will decrease by 7% each year unless some new fish are added. At the end of each year, 250 new fish are added to the lake.

At the start of 2018, there are 2500 fish in the lake.

a. Show that there will be approximately 2645 fish in the lake at the start of 2020. [3]

b. Find the approximate number of fish in the lake at the start of 2042. [5]

It has been suggested that in rowing competitions the time, T seconds taken to complete a 2000 m race can be modelled by an equation of the form $T = aN^b$, where N is the number of rowers in the boat and a and b are constants for rowers of a similar standard.

To test this model the times for the finalists in all the 2000 m men's races at a recent Olympic games were recorded and the mean calculated. The results are shown in the following table for $N = 1$ and $N = 2$.

N	T (seconds)
1	420.65
2	390.94

It is now given that the mean time in the final for boats with 8 rowers was 342.08 seconds.

a. Use these results to find estimates for the value of a and the value of b. Give your answers to five significant figures. [4]

b. Use this model to estimate the mean time for the finalists in an Olympic race for boats with 8 rowers. Give your answer correct to two decimal places. [1]

c. Calculate the error in your estimate as a percentage of the actual value. [1]

d. Comment on the likely validity of the model as N increases beyond 8. [2]

The three planes

$2x - 2y - z = 3$

$4x + 5y - 2z = -3$

$3x + 4y - 3z = -7$

intersect at the point with coordinates (a, b, c).

a. Find the value of each of a, b and c. [2]

b. The equations of three planes are [4]

$2x - 4y - 3z = 4$

$-x + 3y + 5z = -2$

$3x - 5y - z = 6.$

Find a vector equation of the line of intersection of these three planes.

Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0,\ n \in \mathbb{Z}^+;$

(b) $\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0,\ z \neq -1.$

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a. Express the binomial coefficient $\binom{3n+1}{3n-2}$ as a polynomial in n. [3]

b. Hence find the least value of n for which $\binom{3n+1}{3n-2} > 10^6$. [3]

Three Mathematics books, five English books, four Science books and a dictionary are to be placed on a student's shelf so that the books of each subject remain together.

(a) In how many different ways can the books be arranged?

(b) In how many of these will the dictionary be next to the Mathematics books?

Twelve students are to take an exam in advanced combinatorics.

The exam room is set out in three rows of four desks, with the invigilator at the front of the room, as shown in the following diagram.

INVIGILATOR

Desk 1	Desk 2	Desk 3	Desk 4
Desk 5	Desk 6	Desk 7	Desk 8
Desk 9	Desk 10	Desk 11	Desk 12

Two of the students, Helen and Nicky, are suspected of cheating in a previous exam.

a. Find the number of ways the twelve students may be arranged in the exam hall. [1]

b. Find the number of ways the students may be arranged if Helen and Nicky must sit so that one is directly behind the other (with no desk in between). For example Desk 5 and Desk 9. [2]

c. Find the number of ways the students may be arranged if Helen and Nicky must not sit next to each other in the same row. [3]

Three boys and three girls are to sit on a bench for a photograph.

a. Find the number of ways this can be done if the three girls must sit together. [3]

b. Find the number of ways this can be done if the three girls must all sit apart. [4]

Prove, by mathematical induction, that $7^{8n+3} + 2,\ n \in \mathbb{N}$, is divisible by 5.

Consider the arithmetic sequence 8, 26, 44, ...

(a) Find an expression for the n^{th} term.

(b) Write down the sum of the first n terms using sigma notation.

(c) Calculate the sum of the first 15 terms.

Find the sum of all three-digit natural numbers that are not exactly divisible by 3.

On the day of her birth, 1st January 1998, Mary's grandparents invested $\$x$ in a savings account. They continued to deposit $\$x$ on the first day of each month thereafter.

The account paid a fixed rate of 0.4% interest per month. The interest was calculated on the last day of each month and added to the account. Let $\$A_n$ be the amount in Mary's account on the last day of the $n\text{th}$ month, immediately after the interest had been added.

a. Find an expression for A_1 and show that $A_2 = 1.004^2x + 1.004x$. [2]

b. (i) Write down a similar expression for A_3 and A_4. [6]

(ii) Hence show that the amount in Mary's account the day before she turned 10 years old is given by $251(1.004^{120} - 1)x$.

c. Write down an expression for A_n in terms of x on the day before Mary turned 18 years old showing clearly the value of n. [1]

d. Mary's grandparents wished for the amount in her account to be at least $\$20\,000$ the day before she was 18. Determine the minimum value of the monthly deposit $\$x$ required to achieve this. Give your answer correct to the nearest dollar. [4]

e. As soon as Mary was 18 she decided to invest $\$15\,000$ of this money in an account of the same type earning 0.4% interest per month. She withdraws $\$1000$ every year on her birthday to buy herself a present. Determine how long it will take until there is no money in the account. [5]

The complex numbers u and v are represented by point A and point B respectively on an Argand diagram.

Point A is rotated through $\frac{\pi}{2}$ in the anticlockwise direction about the origin O to become point A'. Point B is rotated through $\frac{\pi}{2}$ in the clockwise direction about O to become point B'.

a. Consider $z = r(\cos\theta + \mathrm{i}\sin\theta),\ z \in \mathbb{C}$. [7]

Use mathematical induction to prove that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta),\ n \in \mathbb{Z}^+$.

b. Given $u = 1 + \sqrt{3}\mathrm{i}$ and $v = 1 - \mathrm{i}$, [4]

(i) express u and v in modulus-argument form;

(ii) hence find u^3v^4.

c. Plot point A and point B on the Argand diagram. [1]

d. Find the area of triangle $\mathrm{OA'B'}$. [3]

e. Given that u and v are roots of the equation $z^4 + bz^3 + cz^2 + dz + e = 0$, where $b,\ c,\ d,\ e \in \mathbb{R}$, find the values of $b,\ c,\ d$ and e. [5]

Phil takes out a bank loan of \$150 000 to buy a house, at an annual interest rate of 3.5%. The interest is calculated at the end of each year and added to the amount outstanding.

To pay off the loan, Phil makes annual deposits of \$*P* at the end of every year in a savings account, paying an annual interest rate of 2% . He makes his first deposit at the end of the first year after taking out the loan.

David visits a different bank and makes a single deposit of \$*Q* , the annual interest rate being 2.8%.

a. Find the amount Phil would owe the bank after 20 years. Give your answer to the nearest dollar. [3]

b. Show that the total value of Phil's savings after 20 years is $\frac{(1.02^{20}-1)P}{(1.02-1)}$. [3]

c. Given that Phil's aim is to own the house after 20 years, find the value for P to the nearest dollar. [3]

d.i. David wishes to withdraw \$5000 at the end of each year for a period of n years. Show that an expression for the minimum value of Q is [3]

$\frac{5000}{1.028} + \frac{5000}{1.028^2} + \ldots + \frac{5000}{1.028^n}$.

d.ii. Hence or otherwise, find the minimum value of Q that would permit David to withdraw annual amounts of \$5000 indefinitely. Give your answer to the nearest dollar. [3]

A system of equations is given below.

$x + 2y - z = 2$

$2x + y + z = 1$

$-x + 4y + az = 4$

(a) Find the value of a so that the system does not have a unique solution.

(b) Show that the system has a solution for any value of a.

Let $\omega = \cos\theta + \mathrm{i}\sin\theta$. Find, in terms of θ , the modulus and argument of $(1-\omega^2)^*$.

Fifteen boys and ten girls sit in a single line.

a. In how many ways can they be seated in a single line so that the boys and girls are in two separate groups? [3]

b. Two boys and three girls are selected to go the theatre. In how many ways can this selection be made? [3]

$z_1 = (1 + \mathrm{i}\sqrt{3})^m$ and $z_2 = (1 - \mathrm{i})^n$.

(a) Find the modulus and argument of z_1 and z_2 in terms of m and n, respectively.

(b) **Hence**, find the smallest positive integers m and n such that $z_1 = z_2$.

Use mathematical induction to prove that $(1 - a)^n > 1 - na$ for $\{n : n \in \mathbb{Z}^+,\ n \geqslant 2\}$ where $0 < a < 1$.

Find the constant term in the expansion of $\left(x - \frac{2}{x}\right)^4 \left(x^2 + \frac{2}{x}\right)^3$.

Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

Given that $1 + \mathrm{i}$ and $1 - 2\mathrm{i}$ are zeros of $p(x)$, find the values of a, b, c and d.

The 3rd term of an arithmetic sequence is 1407 and the 10th term is 1183.

a. Find the first term and the common difference of the sequence. [4]

b. Calculate the number of positive terms in the sequence. [3]

The sum of the second and third terms of a geometric sequence is 96.

The sum to infinity of this sequence is 500.

Find the possible values for the common ratio, r.

(a) Find the set of values of k for which the following system of equations has no solution.

$$x + 2y - 3z = k$$
$$3x + y + 2z = 4$$
$$5x + 7z = 5$$

(b) Describe the geometrical relationship of the three planes represented by this system of equations.

A complex number z is given by $z = \frac{a+\mathrm{i}}{a-\mathrm{i}}$, $a \in \mathbb{R}$.

(a) Determine the set of values of a such that

(i) z is real;

(ii) z is purely imaginary.

(b) Show that $|z|$ is constant for all values of a.

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a. Find the term in x^5 in the expansion of $(3x + A)(2x + B)^6$. [4]

b. Mina and Norbert each have a fair cubical die with faces labelled 1, 2, 3, 4, 5 and 6; they throw it to decide if they are going to eat a cookie. [4]

Mina throws her die just once and she eats a cookie if she throws a four, a five or a six.

Norbert throws his die six times and each time eats a cookie if he throws a five or a six.

Calculate the probability that five cookies are eaten.

In a trial examination session a candidate at a school has to take 18 examination papers including the physics paper, the chemistry paper and the biology paper. No two of these three papers may be taken consecutively. There is no restriction on the order in which the other examination papers may be taken.

Find the number of different orders in which these 18 examination papers may be taken.

The interior of a circle of radius 2 cm is divided into an infinite number of sectors. The areas of these sectors form a geometric sequence with common ratio k. The angle of the first sector is θ radians.

(a) Show that $\theta = 2\pi(1 - k)$.

(b) The perimeter of the third sector is half the perimeter of the first sector.

Find the value of k and of θ.

Solve the following system of equations.

$$\log_{x+1} y = 2$$

$$\log_{y+1} x = \frac{1}{4}$$

Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

Two distinct roots for the equation $z^4 - 10z^3 + az^2 + bz + 50 = 0$ are $c + \mathrm{i}$ and $2 + \mathrm{i}d$ where $a,\ b,\ c,\ d \in \mathbb{R},\ d > 0$.

a. Write down the other two roots in terms of c and d. [1]

b. Find the value of c and the value of d. [6]

When $\left(1 + \frac{x}{2}\right)^2$, $n \in \mathbb{N}$, is expanded in ascending powers of x, the coefficient of x^3 is 70.

(a) Find the value of n.

(b) Hence, find the coefficient of x^2.

Find the constant term in the expansion of $\left(4x^2 - \frac{3}{2x}\right)^{12}$.

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a. Find the values of k for which the following system of equations has no solutions and the value of k for the system to have an infinite number of solutions. [5]

$$x - 3y + z = 3$$

$$x + 5y - 2z = 1$$

$$16y - 6z = k$$

b. Given that the system of equations can be solved, find the solutions in the form of a vector equation of a line, $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where the components of $\boldsymbol{b}$ are integers. [7]

c. The plane ÷ is parallel to both the line in part (b) and the line $\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-2}{0}$. [5]

Given that ÷ contains the point $(1, 2, 0)$, show that the Cartesian equation of ÷ is $16x + 24y - 11z = 64$.

d. The z-axis meets the plane ÷ at the point P. Find the coordinates of P. [2]

e. Find the angle between the line $\frac{x-2}{3} = \frac{y+5}{4} = \frac{z}{2}$ and the plane ÷ . [5]

Find the sum of all the multiples of 3 between 100 and 500.

The complex number $z = -\sqrt{3} + \mathrm{i}$.

a. Find the modulus and argument of z , giving the argument in degrees. [2]

b. Find the cube root of z which lies in the first quadrant of the Argand diagram, giving your answer in Cartesian form. [2]

c. Find the smallest positive integer n for which z^n is a positive real number. [2]

(a) (i) Find the sum of all integers, between 10 and 200, which are divisible by 7.

(ii) Express the above sum using sigma notation.

An arithmetic sequence has first term 1000 and common difference of -6 . The sum of the first n terms of this sequence is negative.

(b) Find the least value of n.

A geometric sequence has a first term of 2 and a common ratio of 1.05. Find the value of the smallest term which is greater than 500.

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a. Write down the quadratic expression $2x^2 + x - 3$ as the product of two linear factors. [1]

b. Hence, or otherwise, find the coefficient of x in the expansion of $\left(2x^2 + x - 3\right)^8$. [4]

Consider the complex number $z = \frac{2+7\mathrm{i}}{6+2\mathrm{i}}$.

a. Express z in the form $a + \mathrm{i}b$, where $a,\ b \in \mathbb{Q}$. [2]

b. Find the exact value of the modulus of z. [2]

c. Find the argument of z, giving your answer to 4 decimal places. [2]

The coefficient of x^2 in the expansion of $\left(\frac{1}{x}+5x\right)^8$ is equal to the coefficient of x^4 in the expansion of $(a+5x)^7,\ a \in \mathbb{R}$. Find the value of a.

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a. In an arithmetic sequence the first term is 8 and the common difference is $\frac{1}{4}$. If the sum of the first $2n$ terms is equal to the sum of the next n terms, find n. [9]

b. If $a_1,\ a_2,\ a_3,\ \ldots$ are terms of a geometric sequence with common ratio $r \neq 1$, show that [7]

$(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_4)^2+\ldots+(a_n-a_{n+1})^2=\frac{a_1^2(1-r)(1-r^{2n})}{1+r}$.

The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8.

a. Find the first term and the common difference. [4]

b. Find the smallest value of n such that the sum of the first n terms is greater than 600. [3]

The first term and the common ratio of a geometric series are denoted, respectively, by a and r where $a\ ,r \in \mathbb{Q}$. Given that the third term is 9 and the sum to infinity is 64, find the value of a and the value of r .

Find the coefficient of x^{-2} in the expansion of $(x-1)^3\left(\frac{1}{x}+2x\right)^6$.

A bank offers loans of $\$P$ at the beginning of a particular month at a monthly interest rate of I . The interest is calculated at the end of each month and added to the amount outstanding. A repayment of $\$R$ is required at the end of each month. Let $\$S_n$ denote the amount outstanding immediately after the n^{th} monthly repayment.

a. (i) Find an expression for S_1 and show that [7]

$$S_2 = P\left(1+\frac{I}{100}\right)^2 - R\left(1+\left(1+\frac{I}{100}\right)\right).$$

(ii) Determine a similar expression for S_n . Hence show that

$$S_n = P\left(1+\frac{I}{100}\right)^n - \frac{100R}{I}\left(\left(1+\frac{I}{100}\right)^n - 1\right)$$

b. Sue borrows \$5000 at a monthly interest rate of 1 % and plans to repay the loan in 5 years (*i.e.* 60 months). [6]

(i) Calculate the required monthly repayment, giving your answer correct to two decimal places.

(ii) After 20 months, she inherits some money and she decides to repay the loan completely at that time. How much will she have to repay, giving your answer correct to the nearest \$?

In the arithmetic series with n^{th} term u_n , it is given that $u_4 = 7$ and $u_9 = 22$.

Find the minimum value of *n* so that $u_1 + u_2 + u_3 + \ldots + u_n > 10\,000$.

Solve the simultaneous equations

$$\ln\frac{y}{x} = 2$$

$$\ln x^2 + \ln y^3 = 7.$$

Let $f(x) = \ln x$. The graph of *f* is transformed into the graph of the function *g* by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, followed by a reflection in the *x*-axis.

Find an expression for $g(x)$, giving your answer as a single logarithm.

Find the value of *k* such that the following system of equations does not have a unique solution.

$$kx + y + 2z = 4$$

$$-y + 4z = 5$$

$$3x + 4y + 2z = 1$$

A. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$, [8]

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

B. (a) Using integration by parts, show that $\int \mathrm{e}^{2x} \sin x \mathrm{d}x = \frac{1}{5}\mathrm{e}^{2x}(2\sin x - \cos x) + C$. [17]

(b) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-y^2}\mathrm{e}^{2x} \sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form $y = f(x)$.

(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the x-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the x-axis, between the origin and P, is rotated 360° about the x-axis to form a solid of revolution.

Calculate the volume of this solid.

Consider the equation $z^3 + az^2 + bz + c = 0$, where a , b, $c \in \mathbb{R}$. The points in the Argand diagram representing the three roots of the equation form the vertices of a triangle whose area is 9. Given that one root is $-1 + 3\mathrm{i}$, find

(a) the other two roots;

(b) a , b and c .
