MAA HL

Test on Sequences and Binomial theorem

by Christos Nikolaidis

Date: 16 January 2020

Marks: /40

Paper 1: without GDC

Name of student: SOLUTIONS

1. [Maximum mark: 5]

The **eleventh** term of an arithmetic sequence is 69 while the **sum** of the first 3 terms is 45. Find the third term.

$u_{11} = 69 \Rightarrow u_1 + 10d = 69$ (1)

$S_3 = 45 \Rightarrow \frac{3}{2}(2u_1 + 2d) = 45 \Rightarrow u_1 + d = 15$ (2)

(1) − (2) : $9d = 54 \Leftrightarrow \boxed{d = 6}$

(2): $\boxed{u_1 = 9}$

Hence,

$u_3 = u_1 + 2d = 9 + 2 \times 6 = \boxed{21}$

Turn over

2. [Maximum mark: 6]

The sum of the **first four terms** of an infinite geometric series is 15 while the **sum of all terms** is 16. Find the possible values of the fourth term.

$S_4 = 15 \Rightarrow \frac{u_1(1-r^4)}{1-r} = 15$ (1)

$S_\infty = 16 \Rightarrow \frac{u_1}{1-r} = 16$ (2)

$\frac{(1)}{(2)}: \quad 1-r^4 = \frac{15}{16} \Leftrightarrow r^4 = \frac{1}{16} \Leftrightarrow r = \pm\frac{1}{2}$

- If $r = \frac{1}{2}$ (2): $\frac{u_1}{\frac{1}{2}} = 16 \Rightarrow u_1 = 8$

Hence $u_4 = 8\left(\frac{1}{2}\right)^3 \Rightarrow \boxed{u_4 = 1}$

- If $r = -\frac{1}{2}$ (2): $\frac{u_1}{\frac{3}{2}} = 16 \Rightarrow u_1 = 24$

Hence, $u_4 = 24\left(-\frac{1}{2}\right)^3 \Rightarrow \boxed{u_4 = -3}$

3. [Maximum mark: 5]

(a) Find $10C4$ [2]

(b) Prove that

$$100C20 = 99C20 + 99C19$$ [3]

(a) $^{10}C_4 = \frac{10!}{4!6!} = \frac{7\cdot \cancel{8}^{3}\cdot 9\cdot 10}{1\cdot \cancel{2}\cdot 3\cdot \cancel{4}} = \boxed{210}$

(b) $RHS = \frac{99!}{20!\cdot 79!} + \frac{99!}{19!\cdot 80!}$

$= \frac{99!}{19!\cdot 79!}\left(\frac{1}{20} + \frac{1}{80}\right)$

$= \frac{99!}{19!\cdot 79!} \cdot \frac{100}{20\cdot 80}$

$= \frac{100!}{20!\cdot 80!} = {}^{100}C_{20}$ (LHS)

Turn over

4. [Maximum mark: 5]

Expand $(3-\sqrt{5})^4$ and express the result in the form $a+b\sqrt{5}$

$(3-\sqrt{5})^4 = 3^4 - 4\cdot 3^3\sqrt{5} + 6\cdot 3^2\sqrt{5}^2 - 4\cdot 3\sqrt{5}^3 + \sqrt{5}^4$

$= 81 - 108\sqrt{5} + 270 - 60\sqrt{5} + 25$

$= \boxed{376 - 168\sqrt{5}}$

5. [Maximum mark: 5]

The coefficient of x^4 is twice the coefficient of x^2 in the expansion of $(x^2+3)^n$. Find the value of n.

Term in x^4 $\quad \binom{n}{2}(x^2)^2 \cdot 3^{n-2}$

$$= \binom{n}{2} \cdot \frac{3^n}{9} x^4$$

Term in x^2 $\quad \binom{n}{1}(x^2)^1 3^{n-1}$

$$= n \frac{3^n}{3} \cdot x^2$$

Hence

$$\binom{n}{2} \frac{\cancel{3^n}}{\cancel{9}_3} = 2n \frac{\cancel{3^n}}{\cancel{3}} \Rightarrow \frac{\cancel{n}(n-1)}{2} \frac{1}{3} = 2\cancel{n}$$

$$\Rightarrow n-1 = 12$$

$$\Rightarrow \boxed{n = 13}$$

Turn over

6. [Maximum mark: 7]

The **third** term, the **fifth** term and the **eleventh** term of an arithmetic sequence are the first three terms of a geometric sequence.

(a) Find the common ratio of the geometric sequence. [5]

(b) Find the second term of the arithmetic sequence. [2]

(a) u_n : A.S

$u_3 = u_1 + 2d$

$u_5 = u_1 + 4d$

$u_{11} = u_1 + 10d$

$$\frac{u_5}{u_3} = \frac{u_{11}}{u_5} \Rightarrow \frac{u_1 + 4d}{u_1 + 2d} = \frac{u_1 + 10d}{u_1 + 4d}$$

$$\Rightarrow \cancel{u_1^2} + 8u_1d + 16d^2 = \cancel{u_1^2} + 10u_1d + 2u_1d + 20d^2$$

$$\Rightarrow -4u_1d = 4d^2$$

$$\Rightarrow u_1 = -d \qquad (*)$$

Hence $r = \frac{u_1 + 4d}{u_1 + 2d} = \frac{-d + 4d}{-d + 2d} = \frac{3d}{d} = \boxed{3}$

(b) $u_2 = u_1 + d = \boxed{0}$ by $(*)$

7. [Maximum mark: 7]

The integers $x, y, 15$ are consecutive terms of an arithmetic sequence.

The integers $1, x, x+2y$ are consecutive terms of a geometric sequence.

(a) Find the possible values of x and y. [5]

(b) Confirm the results by stating the common difference and the common ratio in each case. [2]

(a) $y - x = 15 - y \Rightarrow 2y - x = 15$ ①

$\frac{x}{1} = \frac{x+2y}{x} \Rightarrow x^2 = x + 2y$ ②

① and ② : $x^2 = x + x + 15$

$\Leftrightarrow x^2 - 2x - 15 = 0$

$\Delta = 4 + 4 \times 15 = 64 \quad x = \frac{2 \pm 8}{2} \Rightarrow x = 5$ or $x = -3$

• If $\boxed{x=5}$ ①: $2y = 20 \Rightarrow \boxed{y = 10}$

• If $\boxed{x=-3}$ ①: $2y = 12 \Rightarrow \boxed{y = 6}$

(b) if $x=5$, $y=10$

A.S : $5, 10, 15$ $d=5$

G.S : $1, 5, 25$ $r=5$

if $x=-3$ $y=6$

A.S : $-3, 6, 15$ $d=9$

G.S : $1, -3, 9$ $r=-3$

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Paper 2: with GDC

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1. [Maximum mark: 4]

Find the **constant** term in the expansion of $(2x - \frac{9}{x^3})^8$

Constant term

$$\binom{8}{2}(2x)^6\left(-\frac{9}{x^3}\right)^2 = 28 \cdot 2^6 \cdot (-9)^2$$

$$= \boxed{145152}$$

Turn over

2. [Maximum mark: 6]

The sum of the first 100 terms of an arithmetic sequence is 15250 while the sum of the next 100 terms is 45250. Find the sum of the next 100 terms.

$S_{100} = 15250$

$S_{200} - S_{100} = 45250 \Rightarrow S_{200} = 60500$

Hence

$\frac{100}{2}(2u_1 + 99d) = 15250 \Rightarrow 100u_1 + 4950d = 15250$

$\frac{200}{2}(2u_1 + 199d) = 60500 \Rightarrow 200u_1 + 19900d = 60500$

By GDC (simultaneous equations)

$u_1 = 4 \qquad d = 3$

$S_{300} = \frac{300}{2}(2u_1 + 299d) = 135750$

Hence $S_{300} - S_{200} = 135750 - 60500$

$= \boxed{75250}$

3. [Maximum mark: 6]

(a) Express $10x^2 - 19x + 6$ in the form $(ax - b)(cx - d)$, where a, b, c, d are positive integers. [1]

(b) **Hence** or otherwise find the coefficient of x^2 in the expansion of $(10x^2 - 19x + 6)^6$. [5]

(a) roots $\frac{3}{2}, \frac{2}{5}$

Factorisation: $10\left(x - \frac{3}{2}\right)\left(x - \frac{2}{5}\right)$

$= (2x - 3)(5x - 2)$

(b) $(10x^2 - 19x + 6)^6 = (3 - 2x)^6 (2 - 5x)^6$

$= \left[3^6 - 6 \cdot 3^5 (2x) + 15 \cdot 3^4 (2x)^2 + \cdots\right]$

$\cdot \left[2^6 - 6 \cdot 2^5 (5x) + 15 \cdot 2^4 (5x)^2 + \cdots\right]$

$= (729 - 2916x + 4860x^2 + \cdots)$

$\cdot (64 - 960x + 6000x^2 + \cdots)$

Term in x^2: $729 \times 6000x^2 + 2916 \times 960x^2 + 64 \times 4860x^2$

$= 7484400x^2$

Coeff of x^2: $\boxed{7484400}$

Turn over

4. [Maximum mark: 5]

Find

$$\ln x + \ln 2x + \ln 4x + \ln 8x + \cdots + \ln 1024x\,.$$

Express the result in the form $a\ln(bx)$ where a, b are integers to be determined.

$u_1 = \ln x$

$u_2 = \ln x + \ln 2$

$u_3 = \ln x + \ln 4 = \ln x + 2\ln 2 \ \ldots$

A.S. with $d = \ln 2$

There are $\underline{\underline{11}}$ terms

$S_{11} = \frac{11}{2}(2u_1 + 10d) = 11(u_1 + 5d)$

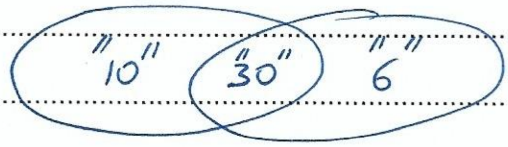
$= 11(\ln x + 5\ln 2)$

$= 11\ln(2^5 x)$

$= 11\ln(32x)$

5. [Maximum mark: 6]

Find the sum of all integers between 100 and 999 which are **either** multiples of 10 or multiples of 6.



$\sum 10x + \sum 6x - \sum 30x$

$100/10 = 10 \qquad 999/10 = 99.9 \rightarrow \sum_{x=11}^{99} 10x = 48950$

$100/6 \cong 16.6 \qquad 999/6 = 166.5 \rightarrow \sum_{x=17}^{166} 6x = 82350$

$100/30 \cong 3.33 \qquad 999/30 = 33.3 \rightarrow \sum_{x=4}^{33} 30x = 16650$

Hence

$48950 + 82350 - 16650 = \boxed{114650}$

Turn over

6. [Maximum mark: 6]

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = 3n^2 + 5n$$

(a) Find the **tenth** term. [2]

(b) Find the sum of the terms which are less than 1000. [4]

(a) $S_9 = 288$

$S_{10} = 350$

$u_{10} = S_{10} - S_9 = 62$

(b) $S_1 = 8 \Rightarrow u_1 = 8$

$S_2 = 22 \Rightarrow u_2 = 14$ so $d = 6$

Hence, $u_n = 8 + (n-1)6 = 6n + 2$

$u_n < 1000 \Rightarrow 6n + 2 < 1000 \Rightarrow 6n < 998$

$\Rightarrow n < 166.33$, so $\boxed{n = 166}$

$S_{166} = \frac{166}{2}(2 \cdot 8 + 165 \cdot 6) = \boxed{83498}$

7. [Maximum mark: 7]

Stefanos invests 800€ at 8% per year (compounded yearly).

Eleonora invests 780€ at 8% per year compounded monthly.

Margarita invests 500€ at 10% per year (compounded yearly).

(a) Find whether Stefanos or Eleonora receives more money after 10 years. [3]

(b) Stefanos estimates that he will receive more than 3000€ after n **complete** years. Find the minimum value of n. [2]

(c) Margarita receives more money that Stefanos after m **complete** years. Find the minimum value of m. [2]

(a) For Stefanos:

$FV = 800\left(1+\frac{8}{100}\right)^{10} \cong \boxed{1727.14}$

For Eleonora:

$FV = 780\left(1+\frac{8}{100\times 12}\right)^{10\times 12} \cong \boxed{1731.32}$

Hence Eleonora receives more money.

(b) $800\left(1+\frac{8}{100}\right)^{n} > 3000$

Solving the equation by GDC

$n \cong 17.17$

Hence $\boxed{n = 18}$

(c) $500\left(1+\frac{10}{100}\right)^{m} > 800\left(1+\frac{8}{100}\right)^{m}$

Now $m \cong 25.6$ Hence $\boxed{m = 26}$

Turn over