Proof *[182 marks]*

1a. Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n\in\mathbb{Z}$. *[2 marks]*

1b. Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. *[3 marks]*

2a. Explain why any integer can be written in the form $4k$ or $4k+1$ or $4k+2$ or $4k+3$, where $k\in\mathbb{Z}$. *[2 marks]*

2b. Hence prove that the square of any integer can be written in the form $4t$ or $4t+1$, where $t\in\mathbb{Z}^+$. *[6 marks]*

The function f is defined by $f(x)=\frac{ax+b}{cx+d}$, for $x\in\mathbb{R},\ x\neq-\frac{d}{c}$.

The function g is defined by $g(x)=\frac{2x-3}{x-2},\ x\in\mathbb{R},\ x\neq2$

3. Express $g(x)$ in the form $A+\frac{B}{x-2}$ where A, B are constants. *[2 marks]*

4a. Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}}=\sqrt{n+1}-\sqrt{n}$ where $n\geq0,\ n\in\mathbb{Z}$. *[2 marks]*

4b. Hence show that $\sqrt{2}-1<\frac{1}{\sqrt{2}}$. *[2 marks]*

4c. Prove, by mathematical induction, that $\sum_{r=1}^{r=n}\frac{1}{\sqrt{r}}>\sqrt{n}$ for $n\geq2,\ n\in\mathbb{Z}$. *[9 marks]*

5. Use mathematical induction to prove that $\frac{d^n}{dx^n}(xe^{px})=p^{n-1}(px+n)e^{px}$ for $n\in\mathbb{Z}^+,\ p\in\mathbb{Q}$. *[7 marks]*

6. Consider the function $f(x) = x\mathrm{e}^{2x}$, where $x \in \mathbb{R}$. The n^{th} derivative of $f(x)$ is denoted by $f^{(n)}(x)$. *[7 marks]*

Prove, by mathematical induction, that $f^{(n)}(x) = \left(2^n x + n2^{n-1}\right)\mathrm{e}^{2x}$, $n \in \mathbb{Z}^+$.

7a. Solve the inequality $x^2 > 2x + 1$. *[2 marks]*

7b. Use mathematical induction to prove that $2^{n+1} > n^2$ for $n \in \mathbb{Z}$, $n \geqslant 3$. *[7 marks]*

8. *[6 marks]*

Use mathematical induction to prove that $\sum_{r=1}^{n} r\,(r!) = (n+1)! - 1$, for $n \in \mathbb{Z}^+$.

9. Use the principle of mathematical induction to prove that *[7 marks]*

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$, where $n \in \mathbb{Z}^+$.

10. Use mathematical induction to prove that $(1-a)^n > 1 - na$ for $\{n : n \in \mathbb{Z}^+, n \geqslant 2\}$ where $0 < a < 1$. *[7 marks]*

Consider the function $f_n(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^n x)$, $n \in \mathbb{Z}^+$.

11a. Determine whether f_n is an odd or even function, justifying your answer. *[2 marks]*

11b. By using mathematical induction, prove that *[8 marks]*

$f_n(x) = \frac{\sin 2^{n+1}x}{2^n \sin 2x}$, $x \neq \frac{m\pi}{2}$ where $m \in \mathbb{Z}$.

11c. Hence or otherwise, find an expression for the derivative of $f_n(x)$ with respect to x. *[3 marks]*

11d. Show that, for $n > 1$, the equation of the tangent to the curve $y = f_n(x)$ at $x = \frac{\pi}{4}$ is $4x - 2y - \pi = 0$. *[8 marks]*

12. Use the method of mathematical induction to prove that $4^n + 15n - 1$ is divisible by 9 for $n \in \mathbb{Z}^+$. *[6 marks]*

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13. Prove by mathematical induction that *[9 marks]*

$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n-1}{2} = \binom{n}{3}$, where $n \in \mathbb{Z}, n \geqslant 3$

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14a. Find the value of $\sin \frac{\pi}{4} + \sin \frac{3\pi}{4} + \sin \frac{5\pi}{4} + \sin \frac{7\pi}{4} + \sin \frac{9\pi}{4}$. *[2 marks]*

14b. Show that $\frac{1-\cos 2x}{2\sin x} \equiv \sin x,\ x \neq k\pi$ where $k \in \mathbb{Z}$. *[2 marks]*

14c. Use the principle of mathematical induction to prove that *[9 marks]*

$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x},\ n \in \mathbb{Z}^+,\ x \neq k\pi$ where $k \in \mathbb{Z}$.

14d. Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$. *[6 marks]*

15a. Use de Moivre’s theorem to find the value of $\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3$. *[2 marks]*

15b. Use mathematical induction to prove that *[6 marks]*

$$(\cos\theta - \mathrm{i}\sin\theta)^n = \cos n\theta - \mathrm{i}\sin n\theta \text{ for } n \in \mathbb{Z}^+.$$

Let $z = \cos\theta + i\sin\theta$.

15c. Find an expression in terms of θ for $(z)^n + (z^*)^n,\ n \in \mathbb{Z}^+$ where z^* is the complex conjugate of z. *[2 marks]*

15d. (i) Show that $zz^* = 1$. *[5 marks]*

(ii) Write down the binomial expansion of $(z + z^*)^3$ in terms of z and z^*.

(iii) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

15e. Hence solve $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$ for $0 \leqslant \theta < \pi$. *[6 marks]*

16. Use mathematical induction to prove that $n(n^2+5)$ is divisible by 6 for $n \in \mathbb{Z}^+$. *[8 marks]*

17a. Show that $\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$. *[1 mark]*

17b. Consider $f(x)=\sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x)=a^n\sin\left(ax+\frac{n\pi}{2}\right)$ where $n\in\mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. *[7 marks]*

Let $y(x)=xe^{3x}, x\in\mathbb{R}$.

18a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[2 marks]*

18b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}=n3^{n-1}\mathrm{e}^{3x}+x3^n\mathrm{e}^{3x}$ for $n\in\mathbb{Z}^+$. *[7 marks]*

18c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. *[5 marks]*

Justify whether any such point is a maximum or a minimum.

18d. Find the coordinates of any points of inflexion on the graph of $y(x)$. *[5 marks]*
Justify whether any such point is a point of inflexion.