Option B Engineering physics

B1 Rotational dynamics

In the mechanics we have studied so far, we have assumed that we were dealing with point masses. A net force applied to a point mass will accelerate it. However, things change when we deal with extended bodies such as cylinders and spheres. Forces will not only accelerate a body's centre of mass but may also make the body rotate.

Learning objectives

- Understand torque.
- Apply the conditions of rotational and translational equilibrium.
- Work with the kinematic equations for rotational motion.
- Solve problems of rotational dynamics.
- Apply conservation of angular momentum.

B1.1 Kinematics

Figure **B.1** shows a body that rotates around an axis that is perpendicular to the plane of the paper. In time Δt the body sweeps out an angle $\Delta\theta$.

We define the average angular velocity of the body to be:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

and the instantaneous angular velocity to be:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

The unit of angular velocity is rad s^{-1}.

The angular velocity will increase or decrease if there is angular acceleration. We define the average angular acceleration as:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

and the instantaneous angular acceleration as:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

The unit of angular acceleration is rad s^{-2}.

We see that these definitions are completely analogous to our definitions of linear quantities: we have a 'translation' dictionary between linear and angular quantities:

Linear quantity	Angular quantity
Position, *s*	Angle, θ
Linear velocity, *v*	Angular velocity, ω
Acceleration, *a*	Angular acceleration, α

Precisely because of this correspondence between linear and angular quantities, where a formula applies to linear quantities a similar formula will apply to the angular quantities. Thus we have the relations

$$s = ut + \frac{1}{2}at^2 \qquad \theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$s = \frac{u+v}{2}t \qquad \theta = \frac{\omega_i + \omega_f}{2}t$$

$$v = u + at \qquad \omega_f = \omega_i + \alpha t$$

$$v^2 = u^2 + 2as \qquad \omega_f^2 = \omega_i^2 + 2\alpha\theta$$

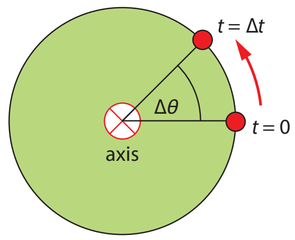


Figure B.1 A body rotating about an axis.

Exam tip
Angular velocity and angular acceleration are actually vector quantities, but we will not make use of their vector nature in this course.

Exam tip
Be careful to understand how angular acceleration α is related to linear acceleration a. This linear acceleration is the rate of change of speed, not the centripetal acceleration of circular motion.

where the subscripts i and f denote initial and final values, respectively. We saw in Topic **6** that angular velocity ω and linear velocity v are related by $v = \omega r$ (Figure **B.2**). A similar relation holds between angular and linear acceleration: $a = \alpha r$.

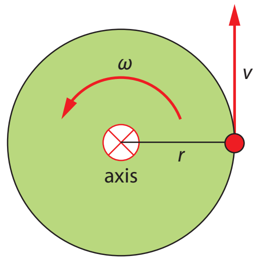


Figure B.2 Linear and angular velocities.

Similarly, whatever applies to graphs involving linear quantities also applies to graphs of angular quantities. Thus:

- in a graph of angle θ versus time t, the gradient (slope) is the angular velocity ω
- in a graph of angular velocity ω versus time t, the gradient (slope) is the **angular acceleration** and the area is the angle turned, $\Delta\theta$
- in a graph of angular acceleration α versus time t, the area is the change in angular velocity, $\Delta\omega$.

Worked examples

B.1 The initial angular speed of a rotating disc is $24\,\text{rad}\,\text{s}^{-1}$. The disc suffers an angular deceleration of $3.0\,\text{rad}\,\text{s}^{-2}$.

a Calculate how many full revolutions the disc will make before stopping.

b Determine when the disc will stop rotating.

a From $\omega^2 = \omega_i^2 + 2\alpha\theta$ we find

$$0 = 24^2 + 2 \times (-3.0) \times \theta \Rightarrow \theta = \frac{24^2}{2 \times 3.0} = 96\,\text{rad}$$

This corresponds to $\frac{96}{2\pi} = 15.3 \approx 15$ revolutions.

b From $\omega = \omega_i + \alpha t$ we find

$$0 = 24 + (-3.0)t$$

$$t = 8.0\,\text{s}$$

B.2 A wheel of radius $0.80\,\text{m}$ is rotating about its axis with angular speed $2.5\,\text{rad}\,\text{s}^{-1}$. Find the linear speed of a point on the circumference of the wheel.

The answer is just $v = \omega R = 2.5 \times 0.80 = 2.0\,\text{m}\,\text{s}^{-1}$.

B.3 Figure **B.3** shows a disc of radius R rotating about its axis with constant angular velocity ω.

A point X is at the circumference of the disc and another point Y is at a distance $\frac{R}{2}$ from the axis.

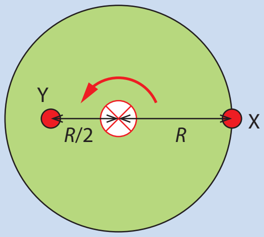


Figure B.3

Calculate the ratios

a $\frac{\omega_X}{\omega_Y}$ **b** $\frac{v_X}{v_Y}$

a All points on the disc have the same angular velocity, so $\frac{\omega_X}{\omega_Y} = 1$.

b From $v = \omega r$ we have that $\frac{v_X}{v_Y} = \frac{\omega R}{\omega R/2} = 2$.

B.4 The graph in Figure **B.4** shows the variation of the angular velocity of a rotating body. Calculate **a** the angular acceleration and **b** the total angle the body turns in 4.0 s.

Figure B.4

a The angular acceleration is the gradient of the curve, so $\alpha = -3.0\,\text{rad}\,\text{s}^{-2}$.

b The angle turned is the area under the curve, 24 rad.

B1.2 Torque

Torque refers to the ability of a force to produce rotation.

Figure **B.5** shows a rigid body that is free to rotate about an axis through point P. A force F acts at point Q. The force is applied a distance r from the axis. The force makes an angle θ with the vector from P to Q. We define the torque Γ produced by the force F about the axis as the product of the force, the magnitude of $\boldsymbol{r}$ and the sine of θ:

$$\Gamma = Fr\sin\theta$$

The unit of torque is N m. Although this combination of units is equivalent to J (joules), by convention torque is never expressed in J.

Exam tip
The angle θ (is the angle between the vector $\boldsymbol{r}$ and the force $\boldsymbol{F}$.

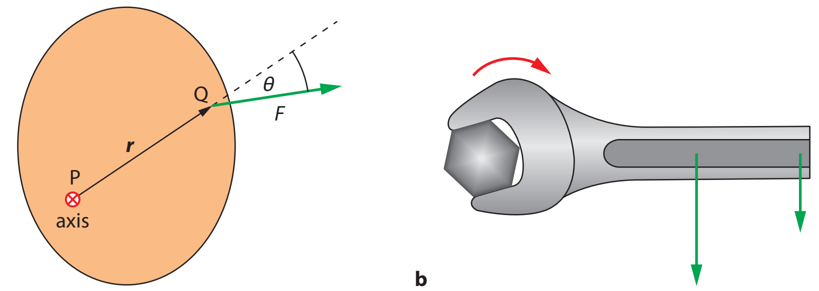


Figure B.5 **a** The force tends to rotate the body in a clockwise direction about an axis out of the page. This is because the force produces a torque about that axis.
b A spanner turning a screw: a real-life application of torque. A smaller force further from the axis would have the same turning effect as a larger force closer to the screw.

Worked example

B.5 Find the torque produced by a 20 N force on a 4 m-long rod free to rotate about an axis at its left end, for the three different force directions shown in Figure **B.6**.

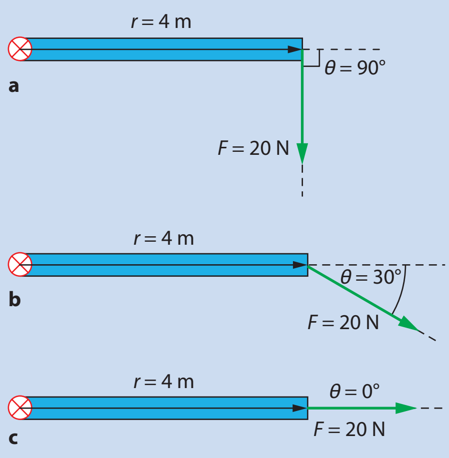


Figure B.6

a $\Gamma = Fr\sin\theta = 20 \times 4 \times \sin 90° = 80\,\text{N m}$

b $\Gamma = Fr\sin\theta = 20 \times 4 \times \sin 30° = 40\,\text{N m}$

c $\Gamma = Fr\sin\theta = 20 \times 4 \times \sin 0° = 0\,\text{N m}$. This force is directed through the axis; it cannot turn the body and has zero torque.

There are other, equivalent ways to find torque, and you may choose whichever way you find convenient. In Figure **B.7**, for example, $r\sin\theta$ is the perpendicular distance d between the axis and the line of action of the force, so $\Gamma = Fd$.

Yet another way (Figure **B.8**) is to use components of the force. Noticing that $F\sin\theta$ is the component of the force along a direction at right angles to the vector $\boldsymbol{r}$ (PQ), we write $F_\perp = F\sin\theta$, so $\Gamma = F_\perp r$.

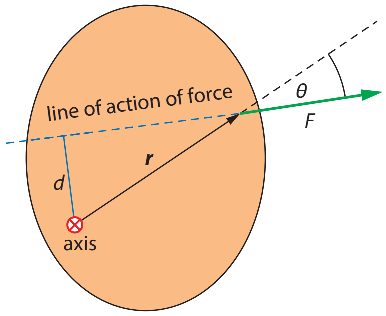


Figure B.7 Torque may be expressed as the product of the force and the perpendicular distance between the axis and the line of action of the force.

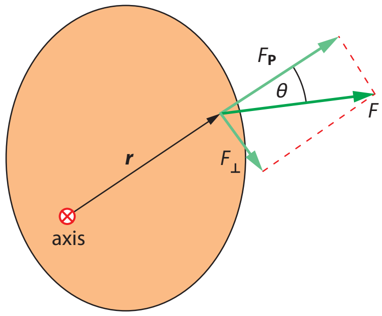


Figure B.8 Torque may be expressed as the product of the component of the force at right angles to the vector ***r*** and the distance between the axis and the point of application of the force.

Notice that if the force is directed through the axis, the torque is zero (Figure **B.9**). The force cannot produce a rotation in this case.

Torque is essential in discussing situations of equilibrium. For a point mass, equilibrium means that the net force on it is zero. For a rigid body, we have to distinguish between **translational** and **rotational equilibrium**. Translational equilibrium is similar to the equilibrium of a point mass: the net force on the body must be zero. The centre of mass of the rigid body remains at rest or moves in a straight line at constant velocity. Rotational equilibrium means that the net **torque** on the rigid body is zero. We examine these issues in Section **B1.3**.

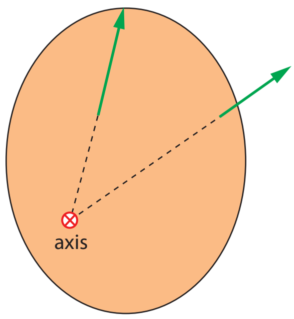


Figure B.9 Each of these forces passes through the axis and therefore produces zero torque about that axis.

B1.3 Equilibrium

Consider the simple example of a see-saw (Figure **B.10**). The plank can rotate about its pivot, the point of support. How can we find the force F_2 required for equilibrium? We do not want the plank to move or to rotate. We must apply the conditions for translational and rotational equilibrium.

Translational equilibrium requires that the net force be zero. In addition to the two forces F_1 and F_2 we have the upward reaction force N on the plank at the point of support. Translational equilibrium demands that,

$$F_1 + F_2 = N$$

For rotational equilibrium we demand that the net torque be zero. The reaction force N passes through the axis, so it produces no torque about the axis. Force F_1 tends to rotate the plank in a counter-clockwise direction, and force F_2 in a clockwise direction. The two forces have opposite torques, and we set them equal to one another to have zero net torque. This gives:

$$F_1 \times 1.2 = F_2 \times 0.8$$

$$240 \times 1.2 = F_2 \times 0.8$$

$$F_2 = 360\,\text{N}$$

From the translational-equilibrium condition, we then find that $N = 600\,\text{N}$.

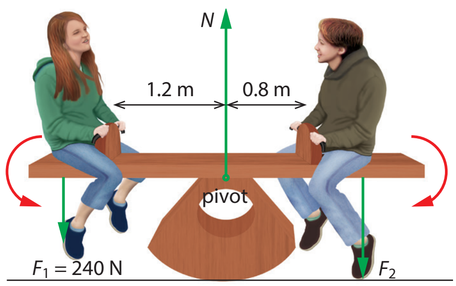


Figure B.10 A see-saw in equilibrium.

Worked example

B.6 Figure **B.11** shows a uniform ladder of length 5.0 m resting against a vertical wall, which is assumed to be frictionless. The other end rests on the floor, where a frictional force f prevents the ladder from slipping. The weight of the ladder is 350 N. Calculate the minimum coefficient of static friction between the ladder and the floor so that the ladder does not slip. The ladder is to make an angle of 60° with the floor.

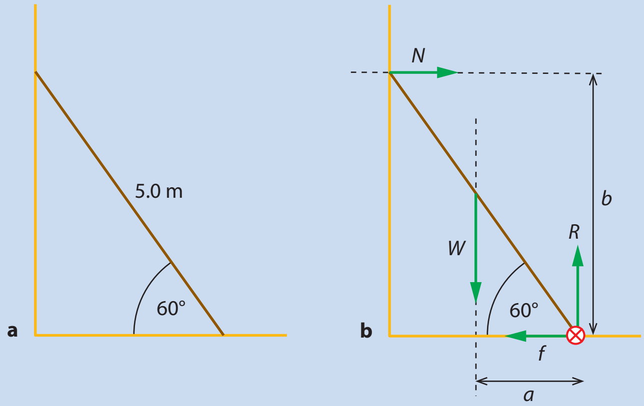


Figure B.11

Exam tip

- For translational equilibrium, require that the net force be zero
- For rotational equilibrium, require that the net torque be zero.
- You may take torques about any axis, not just the actual axis of rotation.

Figure **B.11a** shows the situation and Figure **B.11b** shows all the forces on the ladder.

Translational equilibrium demands that

$$N = f$$

$$R = W$$

where W is the weight of the ladder, R is the reaction force of the floor on the ladder and N is the normal force of the wall on the ladder. Since $W = 350$ N, we find $R = 350$ N right away.

We must now apply the condition for rotational equilibrium. But what is the axis of rotation? As stated in the Exam tip, we can take any point we wish as the rotation axis. It is convenient to choose a point through which as many forces as possible pass. In this way, their torques will be zero and we will not have to deal with them. Choosing the axis to pass through the point where the ladder touches the floor, we must then find the torques produced by W and N. The torque from N is $N \times b = N \times 5.0 \times \sin 60°$. The torque from W is $W \times a = W \times 2.5 \times \cos 60°$. The two torques must balance:

$$N \times 5.0 \times \sin 60° = W \times 2.5 \times \cos 60°$$

$$N = \frac{W \times 2.5 \times \cos 60°}{5.0 \times \sin 60°}$$

$$= 101 \text{ N}$$

This implies that $f = 101$ N. Now $f_{max} = \mu_s R$, so $\mu_s = \frac{f_{max}}{R} = \frac{101}{350} \approx 0.29$.

B1.4 Kinetic energy of a rotating body

Consider a rigid body that rotates with angular velocity ω about the axis shown in Figure **B.12**.

We break up the body into small bits, each of mass m_i. The kinetic energy of the rotating body is the sum of the kinetic energies of all the bits:

$$E_K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \ldots$$

$$= \sum \frac{1}{2} m_i v_i^2$$

Now, each bit is at a different distance r_i from the axis of rotation but all bits have the same angular velocity ω. Since $v_i = \omega r_i$ we deduce that:

$$E_K = \frac{1}{2}m_1 \omega^2 r_1^2 + \frac{1}{2}m_2 \omega^2 r_2^2 + \frac{1}{2}m_3 \omega^2 r_3^2 + \ldots$$

$$= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

This is similar to the formula for the kinetic energy of a point mass, but here velocity is replaced by angular velocity and mass is replaced by the quantity $I = \sum m_i r_i^2$. We call this quantity the **moment of inertia** of the body about the rotation axis. Before continuing with kinetic energy, we must therefore discuss the idea of moment of inertia.

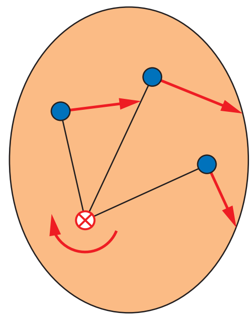


Figure B.12 The sum of the kinetic energies of each bit of the body equals the total kinetic energy of the entire body.

Exam tip
Moment of inertia is to rotational motion what mass is to linear motion.

B1.5 Moment of inertia

Consider a point particle of mass m that is free to rotate about an axis a distance R from the particle (Figure **B.13**).

The moment of inertia of the particle about the given axis is:

$$I = \sum m_i r_i^2$$

But here there is just one 'bit' making up the body (the body itself), so the sum just has one term: $I = mR^2$. For two bodies of equal mass (Figure **B.14**), the moment of inertia is:

$$I = \sum m_i r_i^2 = mR^2 + mR^2 = 2mR^2$$

Notice that if we had chosen a different axis we would get a different moment of inertia.

Look now at a rigid body that is free to rotate about some axis. Consider a ring of radius R as in Figure **B.15**. We imagine that the ring consists of small bits of mass m_i. In this case, each bit has the same distance from the axis, so:

$$I = \sum m_i r_i^2 = \sum m_i R^2 = R^2 \sum m_i = MR^2$$

where $M = \sum m_i$ is the total mass of the ring.

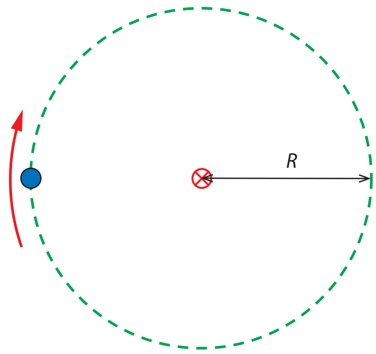


Figure B.13 A body rotating about a fixed axis.

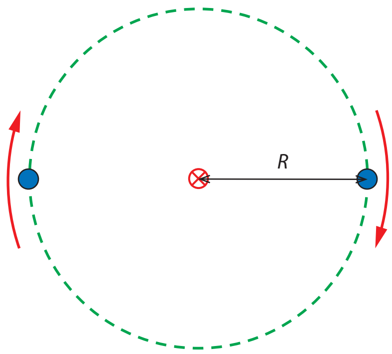


Figure B.14 Two bodies rotating about a common fixed axis.

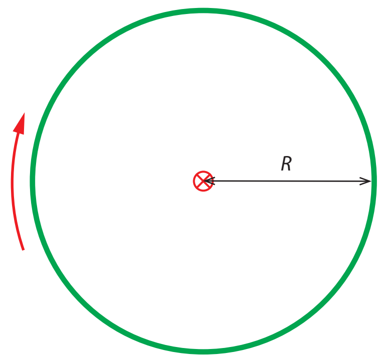


Figure B.15 A ring rotating about a fixed axis.

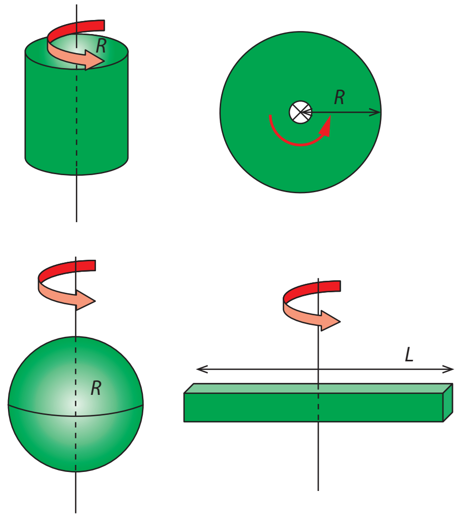


Figure B.16 Moments of inertia of a few common shapes, about the axes shown.

It is not possible to find the moments of inertia of other arrangements as easily as in the preceding cases. Here we just quote some results. The axis of rotation is indicated for each shape in Figure **B.16**.

Disc or cylinder: $I = \frac{1}{2}MR^2$

Sphere: $I = \frac{2}{5}MR^2$

Rod: $I = \frac{1}{12}ML^2$

We see that the moment of inertia is a quantity that depends on the mass of the body and how this mass is distributed around the rotation axis. The closer the mass is to the axis, the smaller the moment of inertia. The unit of moment of inertia is kg m^2.

For a body rotating about some axis, its kinetic energy is then

$$E_K = \frac{1}{2}I\omega^2$$

where I is the body's moment of inertia about the rotation axis and ω is its angular velocity about the (same) rotation axis.

Worked example

B.7 A ring of mass 0.45 kg and radius 0.20 m rotates with angular velocity $5.2\,\text{rad s}^{-1}$. Calculate the kinetic energy of the ring.

The moment of inertia is $I = MR^2 = 0.45 \times 0.20^2 = 1.8 \times 10^{-2}\,\text{kg m}^2$. The kinetic energy is therefore

$$E_K = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 1.8 \times 10^{-2} \times (5.2)^2 = 0.24\,\text{J}$$

B1.6 Rolling without slipping

We will be dealing with bodies that do not just rotate about some axis but roll as well. For example, when you ride a bicycle the wheels rotate but at the same time they roll forward, making the bicycle move.

Consider a wheel of radius R that rotates about its axis (Figure **B.17a**). Every point in the body has the same angular velocity ω and the same

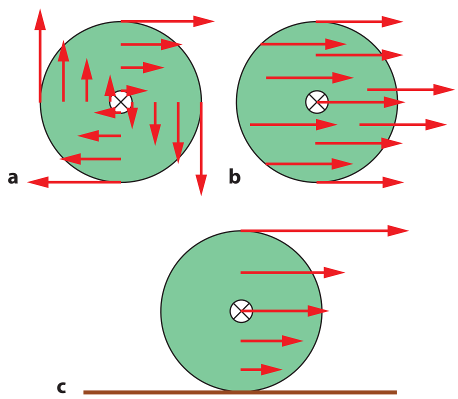


Figure B.17 **a** A rotating disc: each point of the disc has a different linear velocity. **b** A sliding disc: each point on the disc has the same velocity. **c** A disc that rolls without slipping: the point of contact has zero velocity.

linear speed $v = \omega R$. Now imagine the wheel sliding with velocity u along a horizontal floor but without rotating. In this case every point on the circumference has a velocity u to the right. Now let the wheel rotate as it slides to the right. Every point in the body now has two motions: due to rotation and due to sliding. Consequently, the top point has a velocity $v = \omega R + u$ to the right and the bottom point a velocity $v = \omega R - u$ to the left.

Rolling without slipping means that the point of contact between the wheel and the floor is instantaneously at rest. This means that $v = \omega R - u = 0$, so the velocity due to sliding, u, must be $u = \omega R$. So the bottom point is instantaneously at rest and the top point has velocity $v = \omega R + u = 2\omega R$ to the right.

This means that if we have a sphere of radius R and apply a force to it, the centre of mass of the sphere will start to slide in the direction of the force, and may also rotate about an axis. It all depends on where the force is applied. If the force is applied horizontally through the centre of mass, the sphere will slide but will not rotate. (There is no torque to cause rotation.) If the force is applied anywhere else it will cause sliding as well as rotation. But there is only one point where the sphere may be hit so that it slides without slipping. This is examined in a later section.

So sliding without slipping means that if the body rotates with angular velocity ω, the centre of mass of the body must have a velocity ωR, where R is the distance of the centre of mass from the point of contact.

Because angular acceleration is the rate of change of angular velocity, the condition of rolling without slipping may be expressed in terms of angular acceleration as $a = \alpha r$.

B1.7 Kinetic energy of a body that rolls without slipping

We saw that a rotating body has kinetic energy $E_K = \frac{1}{2}I\omega^2$. If the body rolls in addition, then we have to include the kinetic energy due to the translational motion of the centre of mass. The total kinetic energy is then $E_K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$, where v is the speed of the centre of mass.

If the centre of mass of the body is at a height h from some horizontal level then the gravitational potential energy relative to that level is Mgh, so the total mechanical energy is:

$$E_T = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 + Mgh$$

In the absence of resistance forces, the total mechanical energy is conserved.

Worked example

B.8 A cylinder of mass M and radius R (moment of inertia $I = \frac{1}{2}MR^2$) begins to roll from rest down an inclined plane (Figure **B.18**). Calculate the linear speed of the cylinder when it reaches level ground a height h lower.

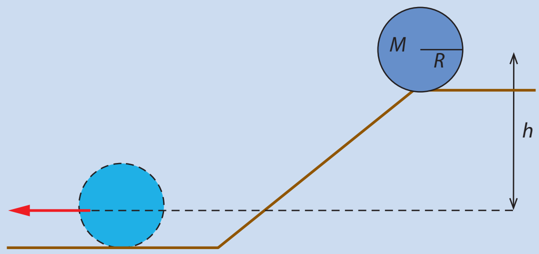


Figure B.18

The initial energy, at the upper level, is Mgh. The energy at the lower level is $E_K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$. Equating the two energies gives:

$$\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh$$

The moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Because the cylinder rolls without slipping, $v = \omega R$, or $\omega = \frac{v}{R}$. Therefore,

$$\frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{v^2}{R^2} = Mgh$$

$$\frac{1}{2}v^2 + \frac{1}{4}v^2 = gh$$

$$\frac{3v^2}{4} = gh$$

The linear speed is $\sqrt{\frac{4gh}{3}}$

Exam tip
Notice that if the body were a point mass the answer for the speed would be $v = \sqrt{2gh}$, greater than the answer here.

Exam tip
Moment of inertia is to rotational motion what mass is to linear motion.
Torque is to rotational motion what force is to linear motion.

B1.8 Newton's second law applied to rotational motion

When we solve problems in mechanics with rigid bodies we must apply Newton's second law to the centre of mass of the body:

$F_{net} = Ma$ (Newton's second law for translational motion)

that is, in exactly the way we would treat a point mass. But we must also apply Newton's second law to the rotational motion of the body; it can be shown that this law takes the form

$\Gamma_{net} = I\alpha$ (Newton's second law for rotational motion)

(This is to be expected since torque plays the role of force and moment of inertia plays the role of mass.)

We begin with an example of a sphere rolling without slipping down an inclined plane that makes an angle θ with the horizontal (Figure **B.19**). The sphere has mass M and radius R. We want to find the linear acceleration of the sphere as it comes down the plane. The diagram shows the forces on the sphere. Notice right away that a frictional force f must be present. Without one, the sphere would just slide down the plane and would not roll. (The problem would then be identical to that for a point mass.)

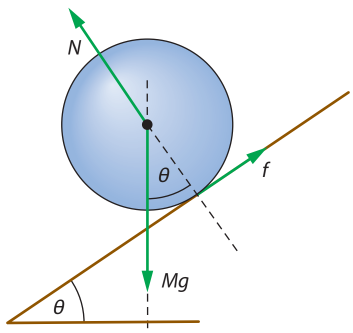


Figure B.19 A sphere that rolls without slipping down an inclined plane: there must be a frictional force to provide the torque for it to turn.

The net force on the sphere along the incline is $Mg\sin\theta - f$, so:

$$F_{net} = Ma$$
$$Mg\sin\theta - f = Ma \quad \text{(Newton's second law for translational motion)}$$

Because the forces perpendicular to the incline are in balance, we also have that $N = Mg\cos\theta$.

The net torque about a horizontal axis through the centre of mass is fR (N and Mg have zero torques about this point) and so, because the moment of inertia of a sphere about its axis is $I = \frac{2}{5}MR^2$:

$$\Gamma = I\alpha$$
$$fR = \frac{2}{5}MR^2\alpha \quad \text{(Newton's second law for rotational motion)}$$

We assume the sphere is rolling without slipping, so

$$\alpha = \frac{a}{R}$$

So our two equations become:

$$Mg\sin\theta - f = Ma$$
$$fR = \frac{2}{5}MR^2\frac{a}{R}$$

These simplify to:

$$Mg\sin\theta - f = Ma$$
$$f = \frac{2}{5}Ma$$

and so:

$$Mg\sin\theta - \frac{2}{5}Ma = Ma$$

giving finally

$$a = \frac{5g\sin\theta}{7}$$

Let us now calculate the speed of the sphere when its centre of mass is lowered by a vertical distance h. The distance travelled down the plane is s and, from Figure **B.20**, $h = s\sin\theta$. So:

$$v^2 = 2as = 2 \times \frac{5gs\sin\theta}{7} = \frac{10gh}{7}$$

We get the same result if we apply conservation of energy:

$$\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh$$
$$\frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\frac{v^2}{R^2} = Mgh$$
$$\frac{1}{2}v^2 + \frac{1}{5}v^2 = gh$$
$$\frac{7v^2}{10} = gh$$
$$v^2 = \frac{10gh}{7}$$

The power of formal analogies

Once the correspondence between linear and angular quantities is established, the formulas relating the angular quantities can be guessed from the corresponding formulas for linear quantities. The formalism is the same and so the formulas are the same. We have seen similar connections between electricity and gravitation, to name one example, before.

Exam tip
Notice that if the body were a point mass, the acceleration would be just $a = g\sin\theta$.

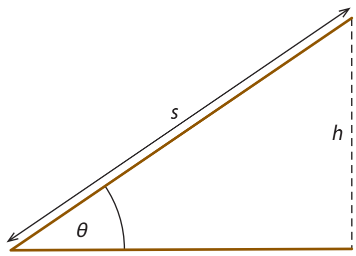


Figure B.20 Relation between the vertical distance and the distance along the plane.

At this point you may be wondering how we can claim energy conservation when there is a frictional force present! The point is that the point of contact where the frictional force acts is not sliding; it is a point that is instantaneously at rest and so the frictional force does not transfer any energy into thermal energy. It would do so if we had rolling with slipping, but we will not consider such cases in this course.

Worked examples

B.9 A block of mass m is attached to a string that goes over a cylindrical pulley of mass M and radius R (Figure **B.21**). When the block is released, the pulley begins to turn as the block falls. Calculate the acceleration of the block.

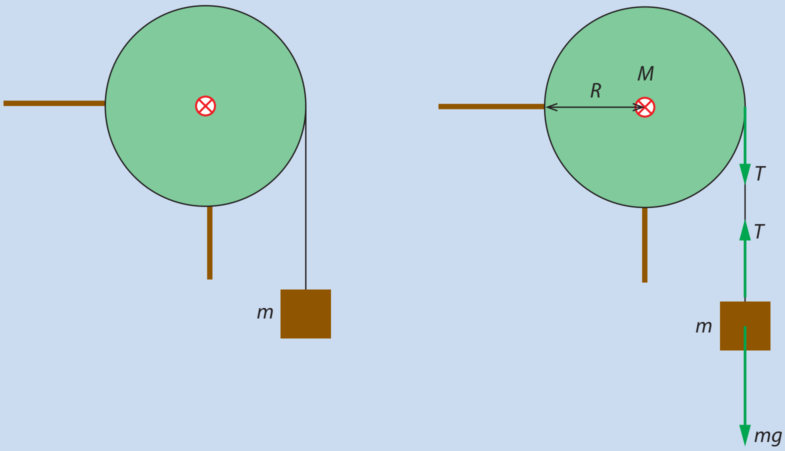


Figure B.21

Applying the second law to the motion of the hanging mass,

$$mg - T = ma$$

Applying the second law to the rotation of the pulley,

$$TR = I\alpha$$

The angular and linear accelerations are related by $a = \alpha R$. So the two equations become

$$TR = I\frac{a}{R}$$

$$mg - T = ma$$

Solving the first equation for T, $T = \frac{Ia}{R^2} = \frac{1}{2}MR^2\frac{a}{R^2} = \frac{1}{2}Ma$. Substitute this in the second equation to get

$$mg - \frac{1}{2}Ma = ma$$

$$mg = a\left(m + \frac{M}{2}\right)$$

$$a = \frac{mg}{m + \frac{M}{2}}$$

B.10 A snooker ball of mass M and radius R is hit horizontally at a point a distance d above its centre of mass (Figure **B.22**). Determine d such that the ball rolls without slipping.

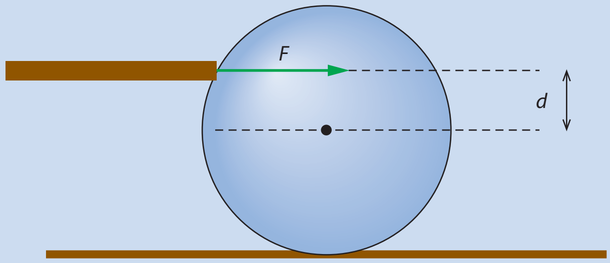


Figure B.22

The torque on the ball about an axis through the centre of mass is Fd. Therefore

$$Fd = \frac{2}{5}MR^2\alpha$$

The net force on the ball is F, so

$$F = Ma$$

The ball does not slip, and so the angular and linear accelerations are related by $a = \alpha R$, so

$$Fd = \frac{2}{5}MR^2\frac{a}{R}$$

$$F = Ma$$

Dividing one equation by the other to get rid of F, we get $d = \frac{2}{5}R$.

B1.9 Work and power

Consider a constant force F applied to a rigid body for a time Δt. Let this force move its point of application by a distance Δs, rotating the body by an angle $\Delta\theta$ in the process. The force does work $W = F\Delta s$. But $\Delta s = R\Delta\theta$, so the work is also given by:

$$W = FR\Delta\theta$$

The torque of the force is $\Gamma = FR$, so

$$W = \Gamma\Delta\theta.$$

The familiar result from linear mechanics that the work done by a net force equals the change in kinetic energy holds here as well, for the work done by a net torque.

The power developed by the force is:

$$P = \frac{W}{\Delta t} = \frac{\Gamma\Delta\theta}{\Delta t}$$

$$P = \Gamma\omega$$

Both formulas are direct analogues of the corresponding quantities in linear motion (Table **B.1**).

Linear motion	Rotational motion
$F = ma$	$\Gamma = I\alpha$
$W = F\Delta s$	$W = \Gamma\Delta\theta$
$P = Fv$	$P = \Gamma\omega$

Table B.1 Force, work and power for linear and angular quantities.

Worked example

B.11 A disc of moment of inertia $25\,\text{kg}\,\text{m}^2$ and radius 0.80 m which is rotating at 320 revolutions per minute must be stopped in 12 s. Calculate **a** the work needed to stop it, and **b** the average power developed in stopping it. **c** Determine the torque that stopped the disc (assuming it is constant).

a The initial kinetic energy of the rotating disc is $E_K = \frac{1}{2}I\omega^2$. We must find ω: $\omega = \frac{320 \times 2\pi}{60} = 33.5\,\text{rad}\,\text{s}^{-1}$.

Hence $E_K = \frac{1}{2} \times 25 \times 33.5^2 = 1.4 \times 10^4\,\text{J}$. This is the work required.

b The average power is therefore $P = \frac{1.4 \times 10^4}{12} = 1.2 \times 10^3\,\text{W}$.

c We will use $P = \Gamma\omega$. For the average power we use the average angular speed, so $\overline{P} = \Gamma\overline{\omega}$.

Since $\overline{\omega} = \frac{33.5}{2} = 16.75\,\text{rad}\,\text{s}^{-1}$, the torque stopping the disc is $\Gamma = \frac{1.2 \times 10^3}{16.75} \approx 72\,\text{N}\,\text{m}$.

B1.10 Angular momentum

The angular momentum of a rigid body with moment of inertia I rotating about a fixed axis with angular speed ω is defined as:

$$L = I\omega$$

(See Figure **B.23**.) If the body is a point mass, the angular momentum is $L = I\omega = mr^2\omega = m(\omega r)r = mvr$, an expression we used in Topic **12** in relation to the Bohr model. The unit of angular momentum is $\text{kg}\,\text{m}\,\text{s}^{-2}$ (equivalent to J s).

Just as there is a relation between linear momentum and net force in particle mechanics, here there is a relation between angular momentum and net torque:

$$\Gamma_{\text{net}} = \frac{\Delta L}{\Delta t}$$

The net torque on a body is the rate of change of the angular momentum of the body.

Similarly, we have the law of **conservation of angular momentum**:

When the net torque on a system is zero, the angular momentum is conserved, that is, it stays constant.

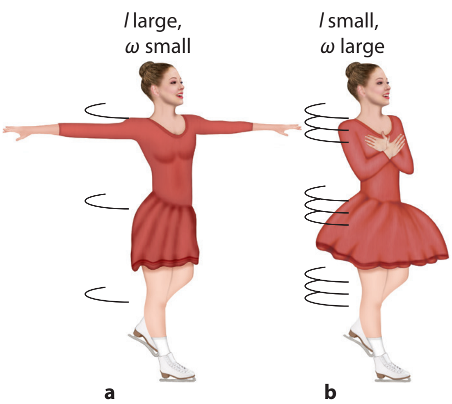


Figure B.23 **a** A skater rotates about a vertical axis. **b** When she brings her arms in, her moment of inertia is reduced and her angular velocity increases in order to conserve angular momentum.

Worked examples

B.12 A spherical star of mass M and radius R rotates about its axis with angular speed ω. The star explodes, ejecting mass into space. The star that is left behind has mass $\frac{M}{10}$ and radius $\frac{R}{20}$. Calculate the new angular speed of the star.

Angular momentum is conserved, so

$$\frac{2}{5}MR^2\omega = \frac{2}{5}\frac{M}{10}\left(\frac{R}{20}\right)^2\omega' \Rightarrow \omega' = 10 \times 20^2\omega = 4000\omega$$

B.13 A horizontal disc of mass $M = 0.95$ kg and radius $R = 0.35$ m rotates about a vertical axis with angular speed $3.6\,\text{rad s}^{-1}$. A piece of modelling clay of mass $m = 0.30$ kg lands vertically on the disc, attaching itself at a point $r = 0.25$ m from the centre. Find the new angular speed of the disc.

The original angular momentum of the disc is

$$L_{in} = I\omega = \tfrac{1}{2}MR^2\omega = 0.209\,\text{J s}$$

The final angular momentum is

$$L_{fin} = I\omega' + mr^2\omega' = \left(\frac{1}{2}2.2 \times 0.35^2 + 0.12 \times 0.15^2\right)\omega'$$

Angular momentum will be conserved because there are no external torques on the disc–clay system, so

$$\left(\frac{1}{2}0.95 \times 0.35^2 + 0.30 \times 0.25^2\right)\omega' = 0.209$$

$$\omega' = 2.7\,\text{rad s}^{-1}$$

Nature of science

Adapting models to the real world

The point particle model simplifies interactions between objects, but does not predict how real objects behave. By adapting models to take rotation into account, engineers can apply the laws of physics to design structures and machines that aim to make our lives safe, comfortable and enjoyable. The great technological advances in energy production, food production, transportation, telecommunications, space exploration, weather prediction, medicine, the entertainment industry and a myriad of other fields are all the result of the application of the basic laws and principles of the sciences to practical problems.

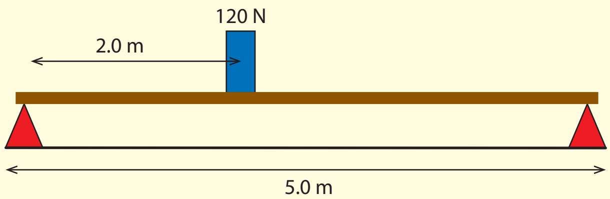
Test yourself

1 A disc has an initial angular velocity 3.5 rad s^{-1} and after 5.0 s the angular velocity increases to 15 rad s^{-1}. Determine the angle through which the disc has turned during that 5.0 s.

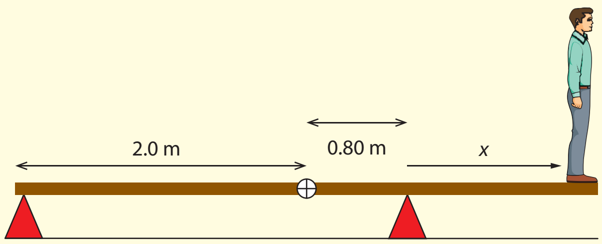
2 A body rotates about an axis with an angular velocity of 5.0 rad s^{-1}. The angular acceleration is 2.5 rad s^{-2}. Calculate the body's angular velocity after it has turned through an angle of 54 rad.

3 A body rotates with an initial angular velocity of 3.2 rad s^{-1}. The angular velocity increases to 12.4 rad s^{-1} in the course of 20 full revolutions. Calculate the angular acceleration.

4 A uniform plank of weight 450 N and length 5.0 m is supported at both ends. A block of weight 120 N is placed at a distance of 2.0 m from the left end. Calculate the force at each support.

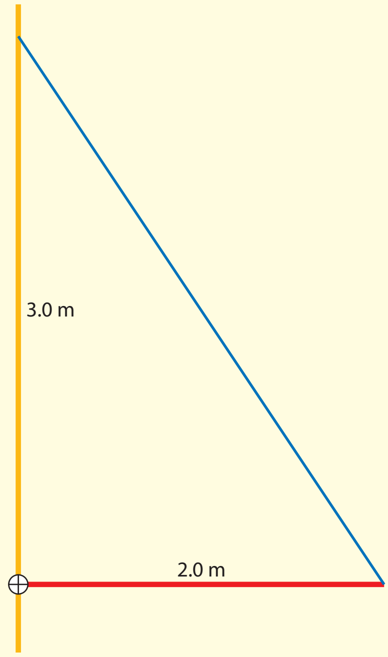


5 A uniform plank of mass 30 kg and length 4.0 m is supported at its left end and at a point 0.80 m from the middle.



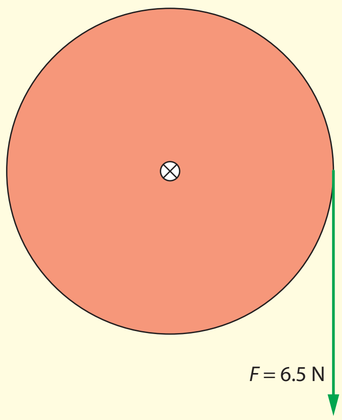
Calculate the largest distance x to which a boy of mass 40 kg can walk without tipping the rod over.

6 A uniform plank of length 2.0 m and weight 58 N is supported horizontally by a cable attached to a vertical wall.



Calculate **a** the tension in the cable, and **b** the magnitude and direction of the force exerted by the wall on the rod.

7 A cylinder of mass 5.0 kg and radius 0.20 m is attached to an axle parallel to its axis and through its centre of mass. A constant force of 6.5 N acts on the cylinder as shown in below. Find the angular speed of the cylinder after 5.0 s.



8 A point mass, a sphere, a cylinder and a ring each have mass M. The solid bodies have the same radius R. The four bodies are released from the same position on an inclined plane (with the ring upright, so it rolls). Determine the order, from least to greatest, of their speeds when they reach level ground. Assume rolling without slipping.

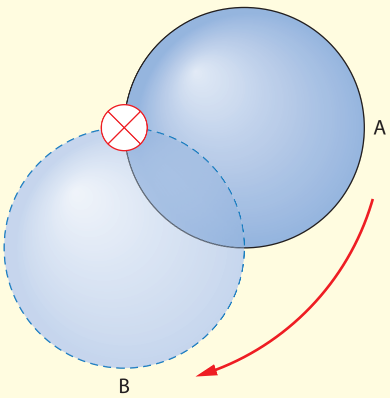
9 A disc of mass 12 kg and radius 0.35 m is spinning with angular velocity 45 rad s^{-1}.

a Determine how many revolutions per minute (rpm) the disc is making.

b A force is applied to the rim of the disc, tangential to the disc's circumference. Determine the magnitude of this force if the disc is to stop spinning after 4.0 s.

c Calculate the number of revolutions it makes before stopping.

10 A uniform sphere rotates about a fixed horizontal axis through the edge of the sphere, as shown below. The axis is at the height of the initial position of the sphere's centre of mass. The moment of inertia of the sphere about this axis is $I = \frac{7}{5}MR^2$. The sphere is released from rest at position A.

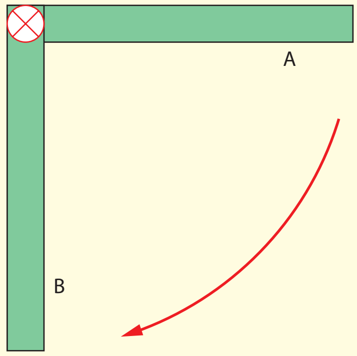


a Find an expression for the initial angular acceleration of the sphere.

b State and explain whether the angular acceleration is constant in magnitude as the sphere rotates.

c Find an expression for the angular velocity of the sphere as it moves past position B.

11 A rod of length $L = 1.20$ m and mass $M = 3.00$ kg is free to move about a fixed axis at its left end. Its moment of inertia about this axis is given by $\frac{1}{3}ML^2$.



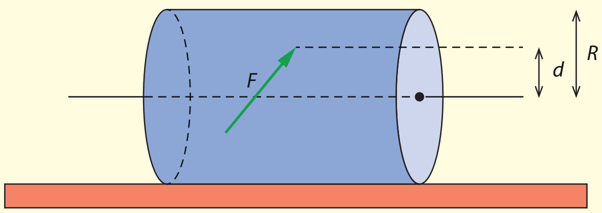
The rod is is released from rest in the horizontal position A.

a Calculate the initial angular acceleration of the rod.

b State and explain whether the angular acceleration is constant in magnitude as the rod rotates.

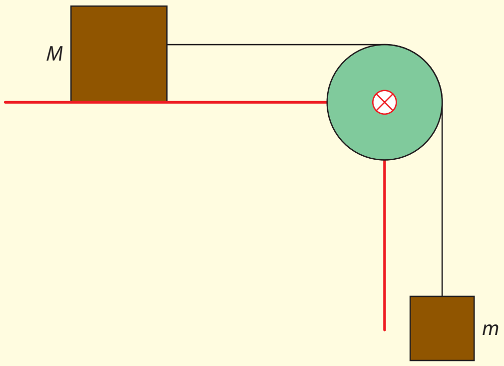
c Calculate the angular velocity of the rod as it moves past the vertical position B.

12 A horizontal force F is applied to the surface of a cylinder of mass M and radius R. The force is applied a vertical distance d above its centre of mass, as shown below.



Determine d as a fraction of R such that the cylinder rolls without slipping. The moment of inertia of a cylinder about its central axis is $\frac{1}{2}MR^2$.

13 A block of mass m hangs from the end of a string that goes over a pulley of radius R, and is connected to another block of mass M that rests on a horizontal table.

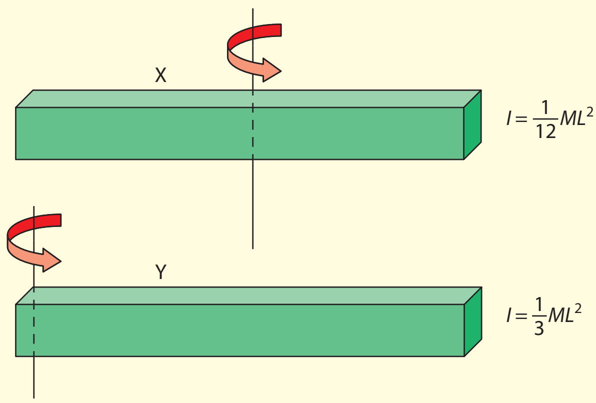


a Assuming that the pulley is massless and that there are no frictional forces, calculate the acceleration of each block when the smaller block is released.

b The pulley is now assumed to have mass M. The small block is again released and the pulley turns as the blocks move. Again calculate the acceleration of each block. (*Hint*: the tensions in the two strings are different.)

14 A horizontal disc, with radius 0.80 m and moment of inertia 280 kg m^2 about its vertical axis, rotates about this axis at 320 revolutions per minute. The disc is brought to rest in 12 s. Calculate **a** the work done, and **b** the power developed in stopping the disc.

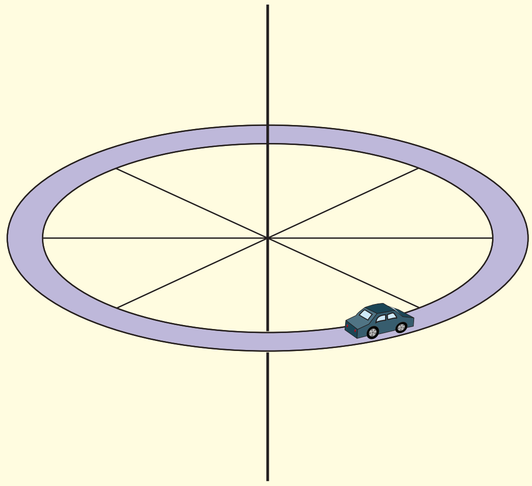
15 Two identical rods, X and Y, each of length 1.20 m and mass 2.40 kg, are made to rotate about different vertical axes, as shown below each with angular velocity 4.50 rad s^{-1}.



Calculate **a** the kinetic energy and **b** the angular momentum of X and of Y.

16 A star of mass M and radius R explodes radially and symmetrically. The star is left with a mass of $\frac{1}{10}M$ and a radius of $\frac{1}{50}R$. Calculate the ratio of the star's final angular velocity to its initial angular velocity.

17 A battery-driven toy car of mass 0.18 kg is placed on a circular track that is part of a horizontal ring with radius 0.50 m and moment of inertia 0.20 kg m^2 relative to its vertical axis. The ring can rotate about this axis without friction.



The car is started and its speed is measured to be 0.80 m s^{-1} relative to the ground. Calculate the angular speed of the ring.

B2 Thermodynamics

Thermodynamics deals with the conditions under which heat can be transformed into mechanical work. The **first law of thermodynamics** states that the **thermal energy** given to a system is used to increase the system's internal energy and to do mechanical work. The **second law of thermodynamics** invokes limitations on how much thermal energy can actually be transformed into work.

Learning objectives

- Understand the first and second laws of thermodynamics.
- Appreciate the concept of entropy.
- Work with cyclic processes.
- Identify and understand isovolumetric, isobaric, isothermal and adiabatic processes.
- Understand the Carnot cycle and thermal efficiency.

B2.1 Internal energy

In Topic **3** we defined the internal energy of a gas as the total random kinetic energy of the particles of the gas plus the potential energy associated with intermolecular forces. If the gas is assumed to be ideal, the intermolecular forces are strictly zero and the internal energy of the gas is just the total random kinetic energy of the particles of the gas. We have seen that the average kinetic energy of the particles is given by:

$$\overline{E}_K = \frac{1}{2}m\overline{v^2} = \frac{3}{2}kT$$

where $k = \dfrac{R}{N_A} = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$, is the Boltzmann constant.

The internal energy U of an ideal gas with N particles is $N\overline{E}_K$, so

$$U = \frac{3}{2}NkT$$

or, since $k = \dfrac{R}{N_A}$ and $PV = nRT$,

$$U = \frac{3}{2}nRT = \frac{3}{2}PV$$

where n is the number of moles.

This formula shows that the internal energy of a fixed number of moles of an ideal gas depends only on its temperature and not on the nature of the gas, its volume or other variables.

The **change** in internal energy due to a change in temperature is thus:

$$\Delta U = \frac{3}{2}nR\Delta T$$

Exam tip
You must know the equivalent ways of expressing internal energy.

Exam tip
Throughout this section, we will deal with monatomic gases.

Worked example

B.14 A flask contains a gas at a temperature of 300 K. The flask is taken aboard a fast-moving aircraft. Suggest whether the temperature of the gas will increase as a result of the molecules now moving faster. The temperature inside the aircraft is 300 K and the container is not insulated.

No. The temperature of the gas depends on the random motion of the molecules and not on any additional uniform motion imposed on the gas as a result of the motion of the container.

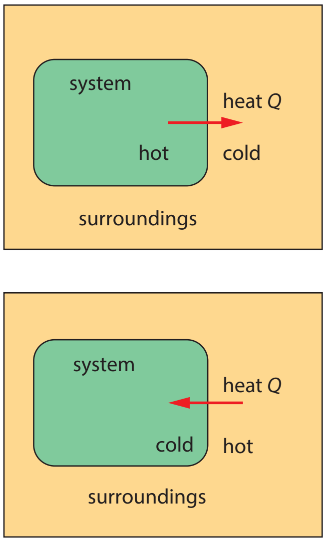


Figure B.24 A system and its surroundings.

B2.2 Systems and processes

In thermodynamics we often deal with **systems**; this means the complete set of objects under consideration. Thus, a gas in a container is a system, as is a certain mass of ice in a glass. What is not in the system is the **surroundings** of the system. Heat can enter or leave a system depending on its temperature relative to that of the surroundings (Figure **B.32**).

A system can be **open** or **closed**: mass can enter and leave an open system but not a closed system. An **isolated** system is one in which no energy in any form enters or leaves. If all the parameters defining the system are given, we speak of the system being in a particular **state**. For example, the state of a fixed quantity of an ideal gas is specified if its pressure, volume and temperature are specified. Any processes that change the state of a system are called **thermodynamic processes**. Thus, heating a gas may result in a changed pressure, temperature or volume and is thus a thermodynamic process. Doing work on the gas by compressing it is another thermodynamic process.

Internal energy is a property of the particular state of the system under consideration, and for this reason internal energy is called a **state function**. Thus, if two equal quantities of an ideal gas originally in different states are brought to the same state (i.e. same pressure, volume and temperature), they will have the same internal energy, irrespective of the original state and how the gas was brought to the final state. By contrast, heat and work are not state functions. We cannot speak of the heat content of a system or of its work content. Heat and work are related to **changes** in the state of the system, not to the state itself.

In this course we will be mainly interested in four different types of thermodynamic processes. These are defined in Table **B.2**.

Process	Description
Isothermal	Temperature stays constant
Isobaric	Pressure stays constant
Isovolumetric	Volume stays constant
Adiabatic	No heat enters or leaves the system

Table B.2 A few thermodynamic processes.

B2.3 Pressure–volume diagrams

Changes in a gas may be conveniently shown on a pressure–volume diagram. We saw examples of this in Section **3.2**. Examples of **isovolumetric** (red) and **isobaric** (blue) processes are shown in Figure **B.25**.

A process that we did not examine in Section **3.2** is the **adiabatic** process. In an adiabatic process, no heat enters or leaves the system. It can be shown that, if the state of a fixed quantity of an ideal gas is changed from pressure p_1 and volume V_1 to p_2 and V_2 in an adiabatic process,

$$p_1V_1^{5/3} = p_2V_2^{5/3}$$

$$pV^{5/3} = \text{constant} = c_1$$

This will mean that, for an adiabatic changes, the temperature also changes. From the ideal gas law we have that $\frac{pV}{T} = \text{constant} = c_2$. Solving this for the pressure gives $p = \frac{c_2T}{V}$. Substituting this into the adiabatic law,

$$\left(\frac{c_2T}{V}\right)V^{5/3} = c_1$$

$$TV^{2/3} = \text{constant or } T_1V_1^{2/3} = T_2V_2^{2/3}$$

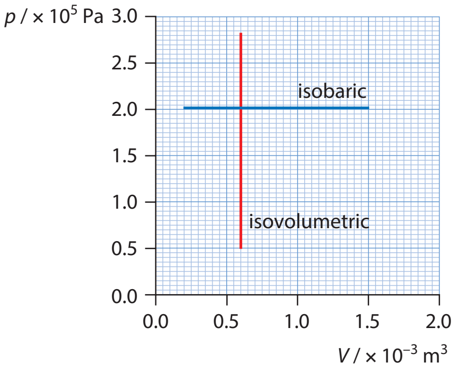


Figure B.25 Isobaric and isovolumetric processes.

So if a gas expands adiabatically, the temperature drops. This sounds surprising the first time you hear it: the expansion is adiabatic, no heat enters or leaves, so why should the temperature change? We will be able to explain this as soon as we learn about the first law of thermodynamics.

Worked example

B.15 An ideal gas expands adiabatically from a state with pressure 5.00×10^5 Pa, volume 2.20×10^{-3} m^3 and temperature 485 K to a new volume of 3.80×10^{-3} m^3. Calculate the new pressure and new temperature of the gas.

From $p_1 V_1^{5/3} = p_2 V_2^{5/3}$ we find:

$$5.00 \times 10^5 \times (2.20 \times 10^{-3})^{5/3} = p_2 \times (3.80 \times 10^{-3})^{5/3}$$

$$p_2 = \frac{5.00 \times 10^5 \times (2.20 \times 10^{-3})^{5/3}}{(3.80 \times 10^{-3})^{5/3}}$$

$$= 2.01 \times 10^5 \text{ Pa}$$

To find the new temperature we may use $T_1 V_1^{2/3} = T_2 V_2^{2/3}$, but it is simpler to use the ideal gas law and write $\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$, so

$$\frac{5.00 \times 10^5 \times 2.20 \times 10^{-3}}{485} = \frac{2.01 \times 10^5 \times 3.80 \times 10^{-3}}{T_2}$$

$$T_2 = \frac{2.01 \times 10^5 \times 3.80 \times 10^{-3} \times 485}{5.00 \times 10^5 \times 2.20 \times 10^{-3}}$$

$$= 337 \text{ K}$$

An adiabatic expansion on a pressure–volume diagram looks like an **isothermal** expansion but is steeper. Figure **B.26a** shows an isothermal (blue) and an adiabatic (red) *expansion* of the same gas from a common state. Figure **B.26b** shows an isothermal and adiabatic *compression* of the same gas from a common state.

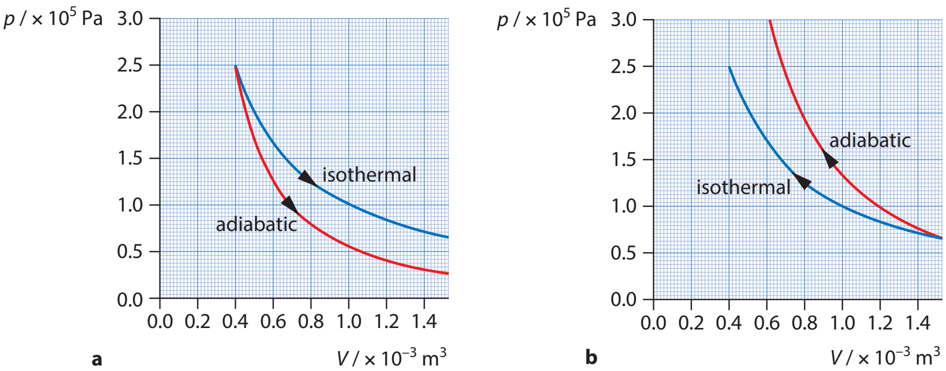


Figure B.26 Isothermal (blue) and adiabatic (red) changes of state: **a** expansion from identical initial states; **b** compression from identical initial states. In both cases the adiabatic curve is steeper.

B2.4 Work done on or by a gas

Consider a given quantity of a gas in a container with a frictionless, movable piston (Figure **B.27**). If the piston is to stay in place, a force from the outside has to be applied to counterbalance the force due to the pressure of the gas. Let us now compress the gas by pushing the piston in, with a very slightly greater force.

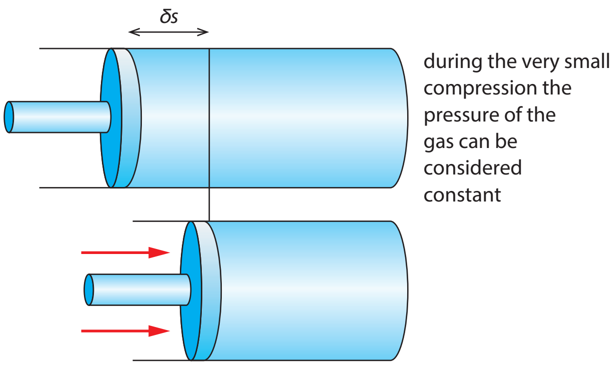


Figure B.27 When the piston is pushed in by a small amount, work is done on the gas.

If the initial pressure of the gas is p and the cross-sectional area of the piston is A, then the force with which one must push is pA. If the pressure stays constant, the force is constant and so we may use the result from mechanics that work is force times distance moved in the direction of the force. The piston moves a distance Δs, so the work done is:

$$W = F\Delta s$$

$$= pA\,\Delta s$$

But $A\Delta s$ is change in the volume of the gas, ΔV. The same result holds if, instead, the piston is moved outwards by the gas as it expands.

The work done when the volume of a gas is changed by ΔV at constant pressure is $W = p\Delta V$.

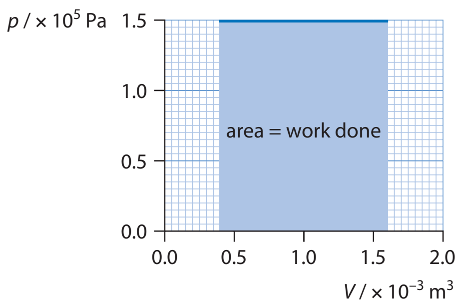


Figure B.28 The area under the graph in a pressure–volume diagram is equal to the work done.

Figure **B.28** shows that the work done is the area under the isobaric curve. The work done in this case is:

$$W = 1.5 \times 10^5 \times (1.6 \times 10^{-3} - 0.4 \times 10^{-3}) = 180\,\text{J}$$

What if the pressure changes? Then the force is not constant and we have to do what we did in mechanics: the work done is the area under the curve in a force–distance graph (Figure **B.29**). Here the corresponding result is the area under the pressure–volume graph.

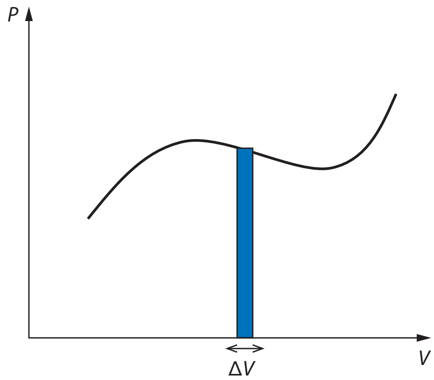


Figure B.29 Although the pressure is not constant, the work done can be calculated for a series of infinitesimal volume changes.

The work done when a gas expands by an arbitrary amount is the area under the curve in the pressure–volume diagram.

In Figure **B.30**, we must calculate the area of the shaded trapezoid:

$$W = \frac{(1.4 + 2.6) \times 10^5}{2} \times 1.2 \times 10^{-3} = 180\,\text{J}$$

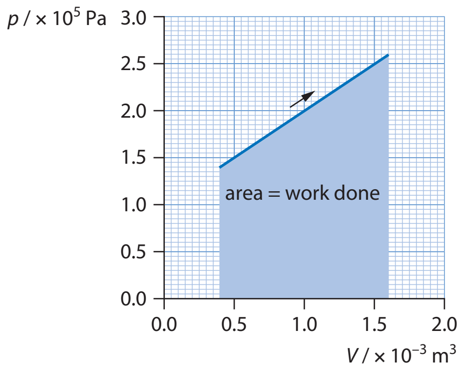


Figure B.30 The work done is found from the area under the curve in the pressure–volume diagram.

The pressure–volume diagrams in Figure **B.31** show an ideal gas that expands from state A to state B and is then compressed back to state A. The gas does work in expanding from A to B but work is done **on** the gas when it is compressed back to A. The net work done is therefore the area of the loop.

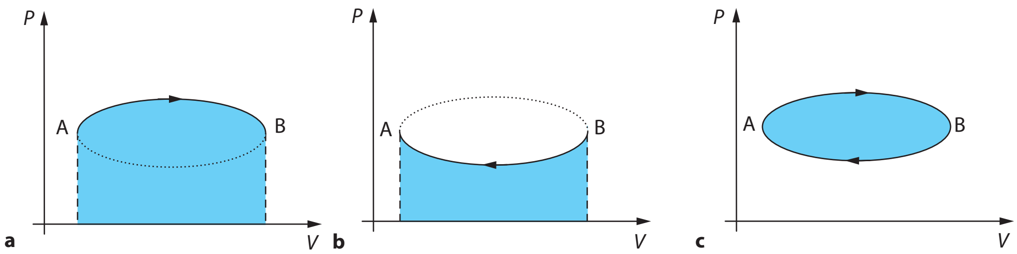


Figure B.31 For a closed loop in a pressure–volume diagram, the work done is the area within the loop.

> For a closed loop in a pressure–volume diagram, the net work done is the area within the loop.

Worked example

B.16 A gas is compressed at a constant pressure of 2.00×10^5 Pa from a volume of $2.00\,\text{m}^3$ to a volume of $0.500\,\text{m}^3$. The temperature is initially 40.0 °C.

a Find the work done.

b Calculate the final temperature of the gas.

a Since the compression takes place under constant pressure, the work done is

$$p \times (\text{change in volume}) = 2.00 \times 10^5\,\text{Pa} \times 1.50\,\text{m}^3$$

$$= 3.00 \times 10^5\,\text{J}$$

b The final temperature is found using $\frac{V}{T} = \text{constant}$, so $\frac{2.00}{313} = \frac{0.500}{T}$, giving $T = 78.25\,\text{K} = -195\,°\text{C}$.

B2.5 The first law of thermodynamics

An amount of heat Q given to a gas will increase the internal energy of the gas and/or will do work by expanding the gas. Conservation of energy demands that:

$$Q = \Delta U + W$$

where ΔU is the change in internal energy and W is the work done. This formula goes with certain conventions that you must know:

- $Q > 0$ means heat is supplied to the gas
- $\Delta U > 0$ means the internal energy of the gas increases
- $W > 0$ means that work is done by the gas as it expands.

Conversely,

- $Q < 0$ means heat is removed from the gas
- $\Delta U < 0$ means the internal energy of the gas decreases
- $W < 0$ means that work is done on the gas by compressing it.

Exam tip
The conventions for signs in the first law are crucial.

Worked examples

B.17 An ideal gas in a container with a piston expands isothermally. Energy $Q = 2.0 \times 10^5$ J is transferred to the gas. Calculate the work done by the gas.

The process is isothermal, so T = constant. It follows that $\Delta U = 0$, and since $Q = \Delta U + W$ we must have $W = Q$. So, the work done by the gas in this case is equal to the energy supplied to it: 2.0×10^5 J.

B.18 An ideal gas expands adiabatically.

a Explain why the temperature decreases.

b Use your answer to **a** to explain why the adiabatic curve of an ideal gas, expanding from a given state, is steeper than the corresponding isothermal curve from the same state.

a We have $Q = 0$ because the process is adiabatic. The first law, $Q = \Delta U + W$, then gives $0 = \Delta U + W$, from which we find $\Delta U = -W$. The gas is expanding, so it is doing work: $W > 0$. Therefore $\Delta U < 0$, that is, the internal energy and thus the temperature decrease. Similarly, if the gas were compressed adiabatically, the temperature would increase.

b Consider an adiabatic and an isothermal process, both starting from the same point and bringing the gas to the same (expanded) final volume. Since the adiabatic process reduces the temperature and the isothermal process does not, the final pressure after the adiabatic expansion will be lower than the final pressure after the isothermal expansion. Hence, the adiabatic curve must be steeper (see Figure **B.26a**).

B.19 An ideal gas, kept at a constant pressure of 3.00×10^6 Pa, has an initial volume of $0.100\,\text{m}^3$. The gas is compressed at constant pressure down to a volume of $0.080\,\text{m}^3$. Find **a** the work done on the gas and **b** the energy transferred.

a The work done is $W = p \times \Delta V = 3.00 \times 10^6 \times (-0.020) = -6.00 \times 10^4$ J. (The work is negative because the gas is being compressed.)

b From the first law, $Q = \Delta U + W$, so to find the energy transferred we must first find the change in the internal energy of the gas. Since:

$$U = \tfrac{3}{2}NkT \quad \text{or} \quad U = \tfrac{3}{2}pV$$

it follows that $\Delta U = \tfrac{3}{2}(pV)_{\text{final}} - \tfrac{3}{2}(pV)_{\text{initial}}$. Here the pressure is constant, so this simplifies to:

$$\begin{aligned}\Delta U &= \tfrac{3}{2}p\Delta V \\ &= \tfrac{3}{2}(-6.00 \times 10^4) \\ &= -9.00 \times 10^4\,\text{J}\end{aligned}$$

Finally,

$$\begin{aligned}Q &= -9.00 \times 10^4 - 6.00 \times 10^4 \\ &= -1.50 \times 10^5\,\text{J}\end{aligned}$$

The negative sign of Q means that this energy was removed from the gas.

B.20 Figure **B.32** is a pressure–volume diagram for an ideal gas, showing two adiabatic curves ('adiabatics') and an isovolumetric and an isobaric process, making up the loop ABCD.

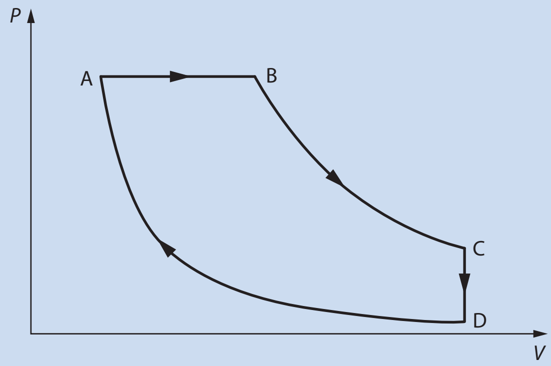


Figure B.32

a Determine along which legs energy is supplied to (Q_{in}) or removed from (Q_{out}) the gas.
b Find the relation between Q_{in}, Q_{out} and the net work done in the loop.

a There is no energy transferred along the adiabatics BC and DA. Along AB, work is done **by** the gas, so $W > 0$, and $\Delta U > 0$ since the temperature at B is higher than that at A. Hence $Q > 0$ and energy is supplied to the gas. Along CD, $W = 0$ (the volume does not change), and $\Delta U < 0$ because the temperature at D is lower than that at C. Hence $Q < 0$ and energy is removed from the gas.
b Applying the first law to the total change A → A, we see that $\Delta U = 0$, so:

$$Q = 0 + W$$

$$(Q_{in} - Q_{out}) = W$$

leading to $W = Q_{in} - Q_{out}$. (W is the area of the loop.)

B.21 The loop ABC in Figure **B.33** consists of an isobaric, an isovolumetric and an isothermal process for a fixed quantity of an ideal gas.

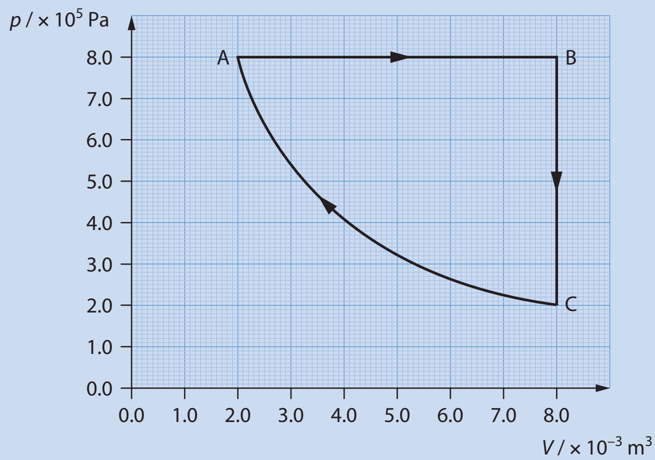


Figure B.33 For Worked example B.21.

The temperature of the gas at A is 320 K. Calculate:
a the temperature at B
b the energy transferred from A to B
c the energy transferred from B to C
d the net work done in one cycle. (The work done on the gas from C to A is 2.2 kJ)

a The temperature at B is found from:

$$\frac{V_A}{T_A} = \frac{V_B}{T_B}$$

$$\frac{2.0 \times 10^{-3}}{320} = \frac{8.0 \times 10^{-3}}{T_B}$$

$$\Rightarrow T_B = 1280 \approx 1300\,\text{K}$$

b The work done by the gas as it expands from A to B is:

$$W_{AB} = P_A \Delta V$$

$$= 8.0 \times 10^5 \times (8.0 \times 10^{-3} - 2.0 \times 10^{-3})$$

$$= 4800\,\text{J} = 4.8\,\text{kJ}$$

From the first law, $Q_{AB} = \Delta U_{AB} + W_{AB}$, so we need to find the change in internal energy. Since the pressure is constant, $\Delta U = \frac{3}{2} P \Delta V = \frac{3}{2} \times 4.8 \times 10^3 = 7.2 \times 10^3\,\text{J}$. Hence $Q_{AB} = 4.8 + 7.2 = +12\,\text{kJ}$.

c $Q_{BC} = \Delta U_{BC}$ since $W = 0$ for the change from B to C (the volume is constant). The magnitude of the temperature change from B to C is the same as that from A to B, so the changes in internal energy have the same magnitude but opposite signs. Therefore $Q_{BC} = -7.2\,\text{kJ}$.

d The net work is therefore $4.8 - 2.2 = 2.6\,\text{kJ}$.

The entropy of the universe is always increasing

The entropy of the universe increases, and we are led from ordered to less ordered and more chaotic situations. But the world around us is full of examples of systems that evolve from highly disordered to very ordered forms (and hence to forms with lower entropy). Life itself evolved from simple microorganisms to the complex forms we see today. A small quantity of water freezing leads to highly structured, ordered (and beautiful) snowflakes. For this to happen, energy has been exchanged with the surroundings in such a way that the systems evolve into highly ordered, complex forms. However, the decrease in entropy in these systems is accompanied by a larger increase in the entropy of the surroundings, leading to an overall increase in the entropy of the universe.

B2.6 The second law of thermodynamics

There are many processes in thermodynamics that are consistent with the first law (energy conservation) but have never been observed to occur. Two examples are:

- the transfer of energy (heat) from a cold body to a hotter body without the performance of work (for example, a glass of water at room temperature freezing, causing the temperature in the rest of the room to rise)
- the air in a room suddenly occupying just one half of the room and leaving the other half empty.

These processes do not happen because they are forbidden by a very special law of physics: the **second law of thermodynamics**. They involve a new concept, that of **entropy**.

Consider an isolated system that is left to change on its own without any intervention from its surroundings. For example, consider a cup of hot coffee left in a cold room. (The system is the cup of coffee and the room.) The natural direction of how things will proceed involves heat leaving the hot coffee and entering the colder room. The coffee will cool down. This is an irreversible process. The reverse will not happen without intervention from the outside. All natural processes are irreversible. An irreversible process captured on film would look absurd if the film were to be run backwards. It looks like thermodynamics is related to the **arrow of time** – the flow of events from the past into the future.

Irreversibility can be quantified. Entropy, like internal energy, is a **state function**: that is, once the state of the system is specified, so is its entropy. Entropy depends only on the state of the system and not on how it got there. What entropy really is and how it is defined is beyond the level of this course. For our purposes, entropy will be taken to be a measure of the disorder of a system.

Time and entropy

The nature of time has always mystified scientists and non-scientists alike. The apparent connection between the direction from past to future (the arrow of time) and thermodynamics is a fascinating and not well-understood aspect of the second law.

The arrow of time

The laws of mechanics do not reveal the arrow of time: a film of billiard balls colliding with each other does not look strange if run backwards. So why do the same laws – when applied to very many particles in a gas, say – show a preference for one direction of time, the one leading to increased entropy and more disorder? The answer has to do with the fact that the system is led to a more disordered, higher-entropy state because this state is the vastly more likely state, the result of the very many, random collisions among the particles of the gas.

What do we mean by disorder?

- A liquid is more disordered than a solid at the same temperature because its atoms move about, whereas those of the solid are regularly arranged – we know less about the position of the particles in the liquid.
- One mole of a gas in a large volume is more disordered than a mole of the same gas at the same temperature in a smaller volume – we know less about the position of the particles in the larger volume.
- One mole of a gas in a given volume at high temperature is more disordered than one mole in the same volume but at a lower temperature – we know less about the position of the particles at high temperature because they move faster.

In all these examples, increased disorder seems to be associated with lack of information about the system.

Even though we will not define entropy, we can define the change in entropy.

The change ΔS in entropy is defined as:

$$\Delta S = \frac{Q}{T}$$

where Q is a quantity of heat given to or removed from a system at a given temperature T (in kelvin). The unit of entropy is $\mathrm{J\,K^{-1}}$.

Exam tip

The formula $\Delta S = \frac{Q}{T}$ must be used with great care. Strictly speaking, it applies when a quantity of heat enters or leaves a system without appreciably changing its temperature. If the temperature changes, calculus is required, to evaluate $\Delta S = \int \frac{\mathrm{d}Q}{T}$.

When heat is given to a system, $Q > 0$ and its entropy increases. When heat is removed, $Q < 0$ and its entropy decreases. For a reversible process that returns the system to its original state, $\Delta S = 0$. One such reversible process is an isothermal expansion of a gas and a subsequent isothermal compression back to the initial state. Since the expansion and compression are isothermal, leaving the temperature constant, we may write:

$$\Delta S_1 = +\frac{|Q|}{T}$$

during expansion and

$$\Delta S_2 = -\frac{|Q|}{T}$$

during compression. (The gas receives thermal energy upon expansion and discards thermal energy upon compression – use the first law of thermodynamics.) The net entropy change is thus zero.

Let us apply the expression for ΔS given above to the case of the flow of heat between a hot body at temperature T_h and a cold body at temperature T_c (Figure **B.34a**). If a quantity of heat Q flows from the hot to the cold body, the total entropy change of the system and its surroundings is:

$$\Delta S = -\frac{Q}{T_h} + \frac{Q}{T_c}$$

$$= Q\left(\frac{1}{T_c} - \frac{1}{T_h}\right)$$

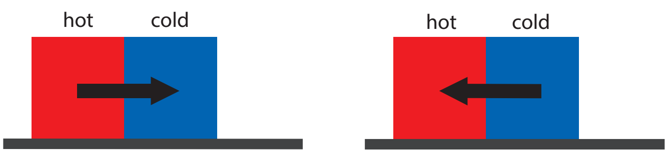
$$\Rightarrow \Delta S > 0$$

(the temperature of each body is assumed unchanged during this small exchange of thermal energy). The entropy of the system has increased, so this is what we expect to happen. The reverse process (Figure **B.34b**) does not happen; this corresponds to a net decrease in the entropy of the system and its surroundings:

$$\Delta S = \frac{Q}{T_h} - \frac{Q}{T_c}$$

$$= -Q\left(\frac{1}{T_c} - \frac{1}{T_h}\right)$$

$$\Rightarrow \Delta S < 0$$



a here entropy increases

b here entropy decreases–hence this process does not happen

Figure B.34 **a** When thermal energy flows from a hot to a cold body, the entropy of the universe increases. **b** If the reverse were to happen without any performance of work, the entropy would decrease, violating the second law.

Entropy and information

Entropy in thermodynamics may be defined as:

$$S = k \ln \Omega$$

where k is Boltzmann's constant and Ω is the number of ways in which a particular macroscopic state of a system may be realised. It is quite extraordinary that this formula of thermodynamics has featured extensively in Claude Shannon's theory of information, a crucial part of modern telecommunications theory.

Similarly, when energy is given to a solid at its melting temperature, the solid will use that energy to turn into a liquid at the same temperature. The entropy formula again shows that the entropy increases as the solid absorbs the latent heat of fusion.

This allows us to state the second law of thermodynamics in its general form, involving entropy:

> The entropy of an isolated system never decreases.
> In such a system, entropy increases in realistic irreversible processes and stays the same in theoretical, idealised reversible processes.

Worked example

B.22 An ideal gas in state A can reach state B at constant volume or state C at constant pressure (Figure **B.35**). The temperatures at B and C are the same. Determine which process results in the greater entropy change.

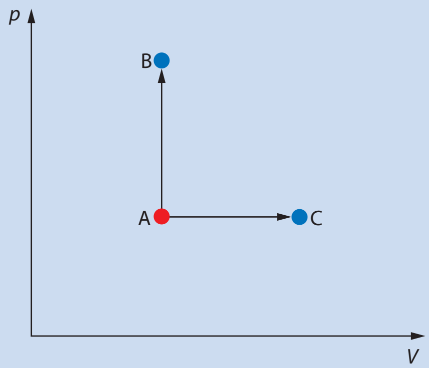


Figure B.35

Since the temperatures at B and C are the same, the change in internal energy is the same for each path. The path AB involves zero work done but the path AC involves positive work done by the gas. Hence Q is greater for the path to C; from $\Delta S = \frac{Q}{T}$, so is the entropy change.

B2.7 Heat engines

It is possible (and easy) to convert mechanical energy into thermal energy: just think of a car when the brakes are applied. The kinetic energy of the car (mechanical energy) is converted into thermal energy in the brake pads: their temperature increases. All of the mechanical energy can be converted into thermal energy in this way. Is the reverse possible? In other words, can we convert thermal energy into mechanical energy with 100% efficiency? Thermodynamics says that this is not possible, as we will see.

A **heat engine** is a device that converts thermal energy into mechanical work. A schematic example is shown in Figure **B.36**.

A hot reservoir (a source of thermal energy) at temperature T_H transfers heat into the engine. Think of the engine as a gas in a cylinder with a piston. The gas expands, pushing the piston out, and this can be exploited to do mechanical work. Some of the energy transferred into the engine is rejected into a sink at a lower temperature, T_C. The work

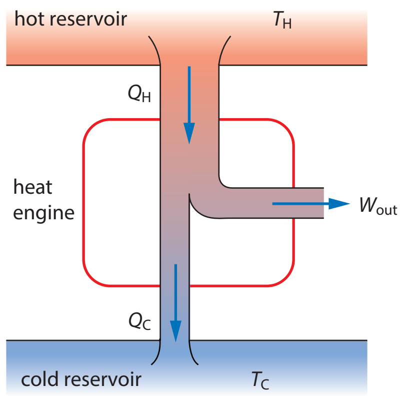


Figure B.36 In a heat engine, energy flowing from the hot to the cold reservoir may be used to perform mechanical work.

done is the heat in, Q_H, minus the heat out, Q_C. Figure **B.37** shows a 'practical' example of the schematic diagram of Figure **B.36**.

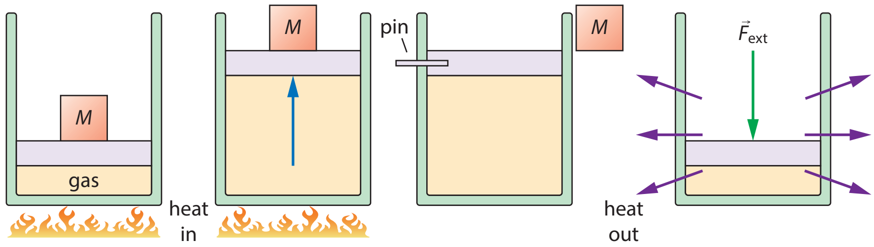


Figure B.37 The hot reservoir provides heat to the gas, which expands, raising a load. The load is removed and the gas is allowed to cool at constant volume. Heat is released to the surroundings. An external force compresses the gas back to its original state. The process can then be repeated, converting heat into mechanical work.

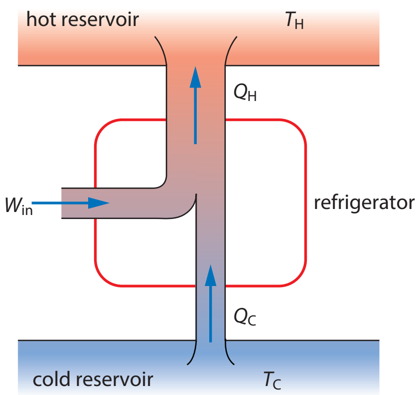


Figure B.38 In a refrigerator, mechanical work is used to extract heat from a cold reservoir and deposit it in the hot reservoir.

A refrigerator (Figure **B.38**) is another type of heat engine, in which the performance of mechanical work extracts heat from the cold reservoir and deposits it in the hot reservoir. (The food that is to be kept cool is in the cold reservoir.)

B2.8 Other formulations of the second law

There are a number of formulations of the second law in terms of heat engines, rather than in terms of entropy.

The Clausius version states that:

> It is impossible for thermal energy to flow from a cold to a hot object without performing work.

The Clausius version says that you cannot have a 'workless' refrigerator: you must do work to push heat from a colder to a hotter object.

The Kelvin version states that:

> It is impossible, in a **cyclic** process, to completely convert heat into mechanical work.

The Kelvin version says that you cannot have a perfect engine: any heat engine has to reject some heat into the surroundings and so no heat engine can be 100% efficient.

These are equivalent formulations. If we accept one, we can deduce the other. With more work it can also be shown that these are equivalent to the formulation in terms of entropy. For example, let us suppose that it is possible to have a perfect heat engine – one that does not reject some heat into the surroundings. If such a perfect engine existed, we could couple it to a refrigerator, as shown in Figure **B.39**. The work done by the perfect engine would be input to the refrigerator. The net effect of the combined system would be the transfer of energy from a cold to a hot reservoir without work. That is impossible, as it would violate the Clausius version of the second law. Hence, no perfect heat engine exists, which is the Kelvin version of the second law.

Exam tip

Notice that the Kelvin version refers to a heat engine working in a cycle. In an isothermal expansion, **all** the heat supplied to the gas goes into work. But this is not a cyclic process and so does **not** violate the Kelvin version of the second law.

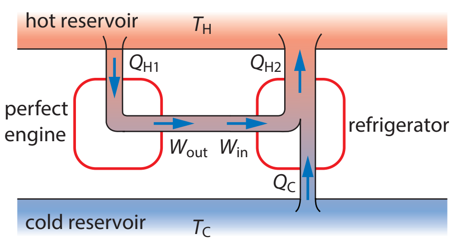


Figure B.39 If a heat engine existed that did not reject heat, then it could be used along with a refrigerator to transfer energy from a cold to a hot reservoir without performing work, violating the second law of thermodynamics (in its Clausius form).

B2.9 The Carnot cycle

In 1824 the French engineer Sadi Carnot investigated how to convert heat into useful mechanical work in the most efficient way. We know that we cannot have a 100% efficient engine, but is there a limit to how efficient an engine can be? Carnot showed that there is.

Figure **B.40** shows the thermodynamic cycle that Carnot investigated. The **Carnot cycle** consist of two isothermals and two adiabatics.

The engine starts its cycle in state 1. The gas is compressed isothermally to state 2. During this stage, heat Q_C leaves the gas and work is done on the gas. From state 2 to state 3 the gas is compressed adiabatically; work is again done on it. From state 3 to state 4 the gas expands isothermally, receiving heat Q_H from a hot reservoir; work is done by the gas. From state 4 to state 1 the gas expands adiabatically; work is done by the gas. The net work done by the engine is therefore:

$$W = Q_H - Q_C$$

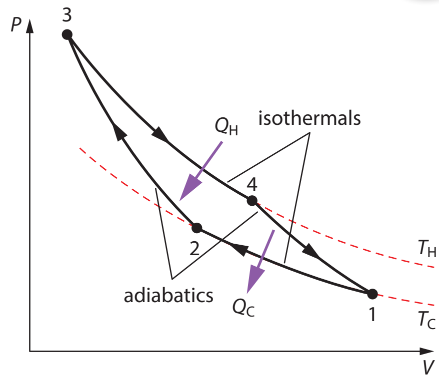


Figure B.40 The Carnot cycle.

The temperature along the isothermal $3 \rightarrow 4$ is T_H, and along the isothermal $1 \rightarrow 2$ it is T_C. The total change in entropy of the engine is:

$$\Delta S = \Delta S_{12} + \Delta S_{23} + \Delta S_{34} + \Delta S_{41}$$

But stages $2 \rightarrow 3$ and $4 \rightarrow 1$ are adiabatics: $Q = 0$, so there is no change in entropy during these stages. Along $3 \rightarrow 4$, $\Delta S_{34} = \frac{Q_H}{T_H}$, and along $1 \rightarrow 2$, $\Delta S_{12} = \frac{Q_C}{T_C}$. Because the process starts and ends at A and because entropy is a state function, the total entropy change is zero: $0 = \frac{Q_H}{T_H} - \frac{Q_C}{T_C}$, which implies that $\frac{Q_H}{T_H} = \frac{Q_C}{T_C}$.

The efficiency of *any* engine is, as usual, $\eta = \frac{\text{useful work}}{\text{input energy}}$. Here the input energy is Q_H, so:

$$\eta_C = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}$$

But $\frac{Q_C}{Q_H} = \frac{T_C}{T_H}$ (for the Carnot engine **only**), so we find that $\eta_C = 1 - \frac{T_C}{T_H}$. Carnot's engine has an efficiency less than 1 (that is, 100%). It would be 100% only in the impossible cases of $T_C = 0$ or $T_H = \infty$.

It can be shown that the following statement is yet another formulation of the second law of thermodynamics:

> No engine is more efficient than a Carnot engine operating between the same temperatures.

Worked example

B.23 Comment on the engine in Figure **B.41**.

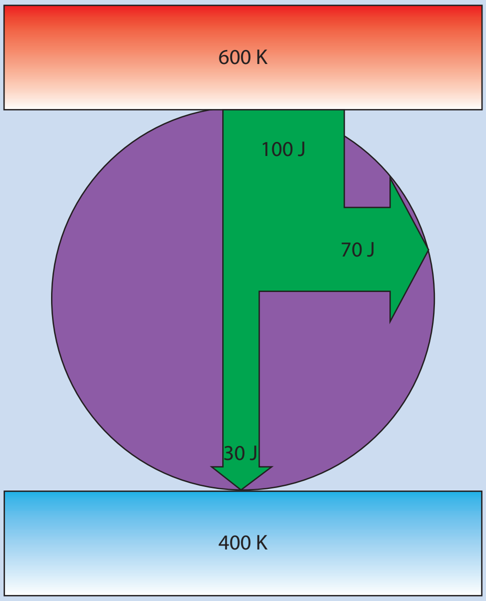


Figure B.41

The efficiency of this engine would be 70%. A Carnot engine operating between the same temperatures would have an efficiency of $1-\frac{400}{600}=33\%$, and thus lower, so this one is impossible.

Nature of science

Different views of the same problem

The second law of thermodynamics stands out as a special law of physics. A law such as the conservation of energy places restrictions on what can and cannot happen: a 1.0 kg ball with total energy 10 J cannot jump over a barrier that is higher than 1.0 m; therefore such an event is completely forbidden (in classical physics). The second law of thermodynamics says that certain processes do not happen not because they **really** are impossible, but because they are so highly unlikely as to be impossible for all practical purposes. Different scientists working on the relationship between heat and work framed this law in different ways, as they were working from different perspectives. Carnot drew his conclusions about the limitations on the efficiencies of heat engines before the concept of entropy was discovered. Through testing and collaboration, scientists realised that the alternative statements were equivalent. The connection of the second law with probability and the number of microstates that realise a particular macroscopic state makes this law a truly special law of physics.

Test yourself

18 An ideal monatomic gas expands adiabatically from a state with pressure 8.1×10^5 Pa and volume 2.5×10^{-3} m^3 to a state of volume 4.6×10^{-3} m^3. Calculate the new pressure of the gas.

19 An ideal monatomic gas expands adiabatically from a state with volume 2.8×10^{-3} m^3 and temperature 560 K to a state of volume 4.8×10^{-3} m^3. Calculate the new temperature of the gas.

20 A gas expands at a constant pressure of 5.4×10^5 Pa from a volume of 3.6×10^{-3} m^3 to a volume of 4.3×10^{-3} m^3. Calculate the work done by the gas.

21 A gas is compressed isothermally so that an amount of work equal to 6500 J is done on it.

a Calculate how much energy is removed from or given to the gas.

b The same gas is instead compressed adiabatically to the same final volume as in **a**. Suggest whether the work done on the gas will be less than, equal to or greater than 6500 J. Explain your answer.

22 An ideal gas expands isothermally from pressure *P* and volume *V* to a volume 2*V*. Sketch this change on a pressure–volume diagram. An equal quantity of an ideal gas at pressure *P* and volume *V* expands adiabatically to a volume 2*V*. Sketch this change on the same axes. Determine in which case the work done by the gas is greater.

23 An ideal gas is compressed isothermally from pressure *P* and volume 2*V* to a volume *V*. Sketch this change on a pressure–volume diagram. An equal quantity of an ideal gas at pressure *P* and volume 2*V* is compressed adiabatically to a volume *V*. Sketch this change on the same axes. Determine in which case the work done on the gas is greater.

24 A quantity of energy Q is supplied to three ideal gases, X, Y and Z. Gas X absorbs Q isothermally, gas Y isovolumetrically and gas Z isobarically. Copy the table below and complete it by inserting the words 'positive', 'zero' or 'negative' for the work done *W*, the change in internal energy ΔU and the temperature change ΔT for each gas.

W	**Δ*U***	**Δ*T***
X		
Y		
Z		

25 An ideal gas is compressed adiabatically.

a Use the first law of thermodynamics to state and explain the change, if any, in the temperature of the gas.

b Explain your answer to **a** by using the kinetic theory of gases.

26 An ideal gas is kept at constant pressure of 6.00×10^6 Pa. Its initial temperature is 300 K. The gas expands at constant pressure from a volume of 0.200 m^3 to a volume of 0.600 m^3. Calculate:

a the work done by the gas

b the temperature of the gas at the new volume

c the energy taken out of or put into the gas.

27 An amount Q of energy is supplied to a system. Explain how it is possible that this addition of thermal energy might **not** result in an increase in the internal energy of the system.

28 An ideal gas undergoes a change from state P to state Q, as shown below (the temperature is in kelvin). For this change, state and explain:

a whether work is done on the gas or by the gas

b whether energy is supplied to the gas or taken out of the gas.

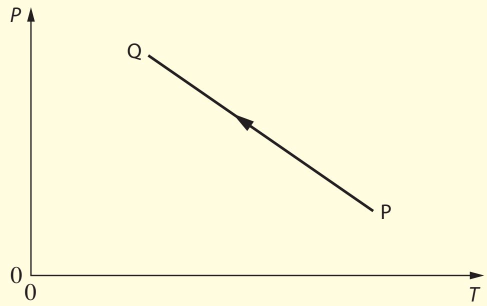
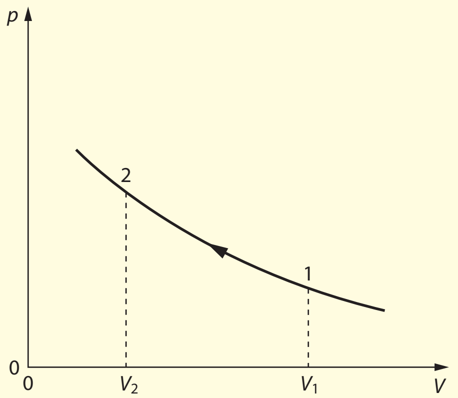


Figure B.52

29 Two ideal gases, X and Y, have the same pressure, volume and temperature. The same quantity of energy is supplied to each gas. Gas X absorbs the thermal energy at constant volume, whereas gas Y absorbs the thermal energy at constant pressure. State and explain which of the two gases will have the larger final temperature.

30 Two ideal gases, X and Y, have the same pressure, volume and temperature. A quantity of energy is supplied to each gas. Gas X absorbs the thermal energy at constant volume, whereas gas Y absorbs the thermal energy at constant pressure. The increase in temperature of both gases is the same. State and explain which of the two gases received the larger quantity of energy.

31 The graph below shows the isothermal compression of an ideal gas from state 1 of volume V_1 to state 2 of volume V_2.



a Copy the figure and on it draw a curve to show the adiabatic compression of the same gas from state 1 to a different state of volume V_2.

b Explain in which case the work done on the gas is the larger.

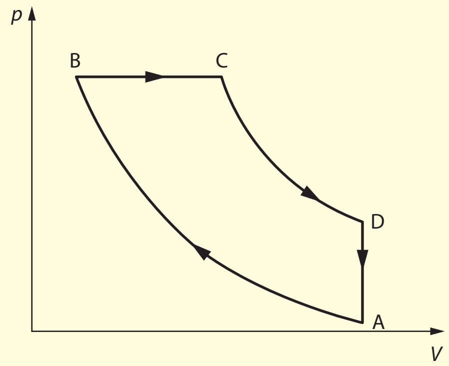
c The temperature of the gas in state 1 is 300 K. The temperature of the surroundings may be assumed constant at 300 K. The work done on the gas during the isothermal compression is 25 kJ. Determine the change in entropy:

i of the gas during the isothermal compression

ii of the surroundings during the isothermal compression.

d Discuss how the answers to **c** are consistent with the second law of thermodynamics.

32 The graph below shows a cycle of a heat engine working with an ideal gas. Curves AB and CD are adiabatics.



a State what is meant by an adiabatic curve.

b State the names of the processes BC and DA.

c Determine along which stage heat is given to the gas.

d Along DA the pressure drops from 4.0×10^6 Pa to 1.4×10^6 Pa. The volume along DA is 8.6×10^{-3} m^3. The efficiency of the engine is 0.36. The number of moles of gas is 0.25. Calculate:

i the heat taken out of the gas

ii the heat given to the gas

iii the area of the loop ABCD.

33 The **molar specific heat capacity** of an ideal gas is defined as the amount of energy required to change the temperature of one mole of the gas by 1 K. When n moles of gas absorb energy Q at constant pressure,

$$Q = nc_P\Delta T$$

where c_P is the molar specific heat capacity at constant pressure.

When n moles of gas absorb energy Q at constant volume,

$$Q = nc_V\Delta T$$

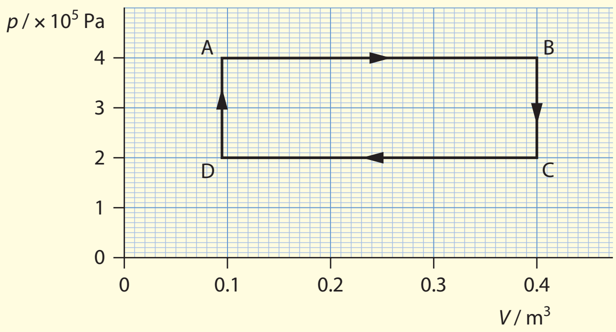
where c_V is the molar specific heat capacity at constant volume.

Use the first law of thermodynamics to show that:

$$c_P - c_V = R$$

where R is the universal gas constant.

34 Consider the loop ABCD as shown on the graph. The temperature at A is 800 K.



Calculate:

a the temperature at points B, C and D

b the change in internal energy along each of the four legs of the cycle

c the amount of energy given to and taken from the gas along each leg

d the efficiency of the cycle.

35 A quantity of ice at −10 °C is dropped into water. All the ice eventually melts and the final temperature of the water is +15 °C. Describe and compare the entropy changes taking place in the ice and in the water during the following processes. (Assume that there are no energy exchanges between the ice–water system and the surroundings.)

a The temperature of the ice increases from −10 °C to 0 °C.

b The ice is melting.

c All the ice has melted and the water is approaching its final temperature of +15 °C.

36 Explain whether you should invest money in a revolutionary new heat engine whose inventor claims that it operates between temperatures of 300 K and 500 K with an efficiency of 0.42.

37 **a** State the Clausius form of second law of thermodynamics.

b Show that if a heat engine had a greater efficiency than the Carnot efficiency the second law in a would be violated. (Imagine the work output of the engine to be the input work in a Carnot refrigerator.)

B3 Fluids (HL)

This section deals with fluids at rest and fluids in motion. The equilibrium and motion of fluids are of importance in very many different disciplines. Understanding how fluids behave is crucial in the design of aircraft and cars, in monitoring blood flow in the arteries of a patient, in the efficient use of water by agricultural engineers – but also to a theoretical astrophysicist working with models of stars. We will meet Archimedes' important principle about buoyant forces and a few of the the many applications of the Bernoulli equation.

Learning objectives

- Work with density and pressure.
- Apply Archimedes' principle.
- Apply Pascal's principle.
- Apply hydrostatic equilibrium.
- Appreciate the ideal fluid.
- Understand streamlines.
- Apply the continuity equation. and the Bernoulli equation.
- Apply Stokes' law and understand viscosity.
- Distinguish between laminar and turbulent flow.
- Appreciate the meaning of the Reynolds number.

B3.1 Pressure

Consider a fluid in a container (Figure **B.42**). The surface of the fluid is exposed to the atmosphere. The pressure of the fluid at the top surface is equal to atmospheric pressure. As we move deeper into the fluid, the pressure increases, because of the weight of the fluid above each given point. Figure **B.42** shows a region of the fluid marked by a dashed line. The fluid inside the line is in equilibrium, so the net force on it is zero. The forces on this region are the weight of the fluid within it, the force due to the pressure at the top surface and the pressure at the lower surface.

We denote atmospheric pressure by $p_{atm} = p_0$. Atmospheric pressure varies depending on location; the standard value at sea level is taken to be 1.0×10^5 Pa. Remember that $p = \frac{F}{A} \Rightarrow F = pA$. Equilibrium requires that: (A is the area of the top and lower surfaces of the fluid and mg the weight of the fluid in the marked region)

$$pA = p_0A + mg$$

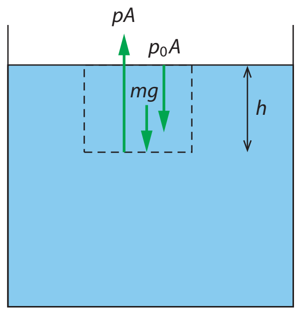


Figure B.42 The pressure increases as the depth h increases.

But:

$$mg = \rho Vg$$
$$= \rho Ahg$$

where ρ is the density of the liquid. Hence:

$$pA = p_0 A + \rho Ahg$$
$$p = p_0 + \rho gh$$

We see that, at a depth h below the free surface of the fluid, the pressure depends both on the quantity ρgh, associated with the weight of the fluid in the dotted region, and on the atmospheric pressure at the top surface.

It is important to note that the pressure at the same depth of a connected fluid in equilibrium is the same no matter what the shape of the container (Figure **B.43**).

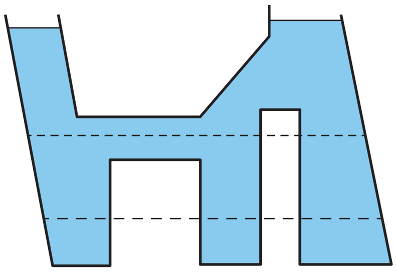


Figure B.43 The pressure is the same at any point along a given horizontal dashed line.

This is the basis for an instrument, called a **manometer**, for measuring pressure (Figure **B.44**).

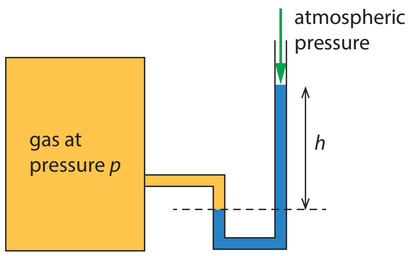


Figure B.44 Using a manometer, the pressure of a gas may be determined from the value of atmospheric pressure and the difference in height of the two columns.

The U-shaped tube is filled with a liquid of density ρ. The difference in the liquid levels in the two columns is h. The pressure of the liquid is equal anywhere along the dashed horizontal line. On the left side, the pressure is equal to the pressure p of the gas. On the right side, it is equal to $p_0 + \rho gh$, where p_0 is atmospheric pressure. These are equal, so $p = p_0 + \rho gh$.

B3.2 Pascal's principle

Pascal's principle states that, when pressure is applied to any point of an **enclosed, incompressible** fluid, the pressure will be transmitted to all other parts of the liquid and the walls of the container of the fluid. Figure **B.45** shows an enclosed fluid. A force F_1 is applied to a piston of area A_1. The pressure in the liquid immediately below the piston is therefore $p_1 = \frac{F_1}{A_1}$. This is also the pressure right under the second, larger piston, because they are at the same horizontal level. But the pressure there is also $p_2 = \frac{F_2}{A_2}$. Hence:

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}$$

This shows that a small force F_1 applied to a small piston can give rise to a larger force on a larger piston – lifting a heavy object such as a car in the case of a **hydraulic lift**. Note, however, that the smaller force must push through a larger distance, d_1 in order to lift the heavy load by a distance d_2 such that $F_1 \times d_1 = F_2 \times d_2$.

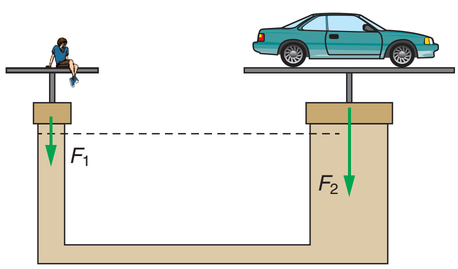


Figure B.45 A hydraulic lift: a small force gets multiplied and lifts a heavy object.

B3.3 Archimedes' principle

A body that is totally or partially immersed in a fluid will experience an upward force called the buoyant force. Consider a region of a fluid that has been marked. The fluid is in equilibrium, so its weight is balanced by the buoyant force (Figure **B.46**). The buoyant force is the result of the pressure acting on the body from all directions.

Because the pressure is higher at greater depths, the net force exerted by the surrounding fluid is directed upwards, and is just equal to the weight of the fluid in the marked region (otherwise the fluid would move). Now replace the marked fluid with a body that just fits that space. The buoyant force remains equal to the weight of the displaced fluid. The arrows representing the pressure are the same as before, so the net force they give rise to is the same as before; it equals the weight of the displaced fluid. This is **Archimedes' principle**:

> A body partially or completely immersed in a fluid experiences a buoyant force B that equals the weight of the displaced fluid.

The weight of the displaced fluid is mg, where $m = \rho V_{\text{displ}}$, so

$$B = \rho g V_{\text{displ}}$$

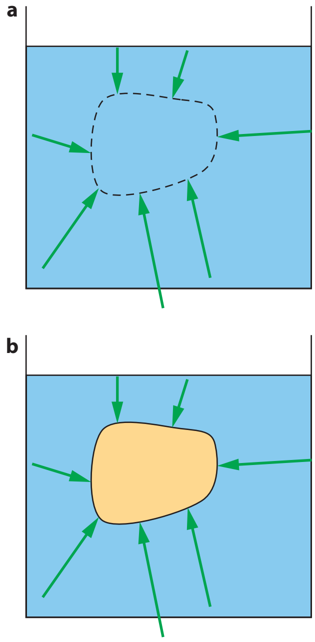


Figure B.46 **a** Fluid pressure acts on all sides of the marked region of the liquid. It is larger at greater depths, so the net force from the surrounding fluid is upwards, just balancing the weight of the fluid in the marked region. **b** If the fluid in the marked region is replaced by another body, the buoyant force remains unchanged, and equal to the weight of the displaced fluid.

Worked example

B.24 Figure **B.47** shows a floating rectangular platform made out of wood of density $640\,\text{kg}\,\text{m}^{-3}$. The fluid is water of density $1000\,\text{kg}\,\text{m}^{-3}$.

Calculate the fraction $\frac{h}{d}$.

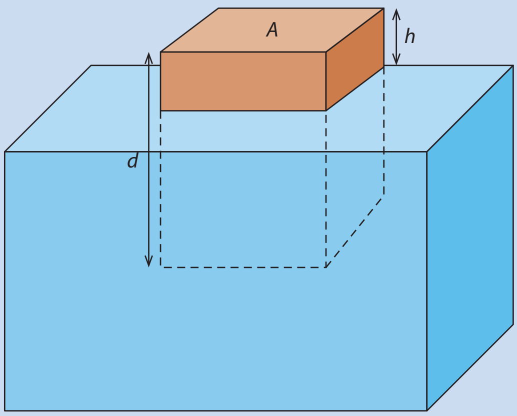


Figure B.47

Let A denote the surface area of the wood. We have equilibrium, so weight equals buoyant force:

$$W = B$$

$$\rho_{\text{wood}} A d g = \rho_{\text{water}} A (d - h) g$$

$$640d = 1000(d - h)$$

$$1000h = 360d$$

$$\frac{h}{d} = 0.36$$

B3.4 The ideal fluid

In previous subsections, we considered fluids at rest. We will now examine fluids in motion. The motion of fluids is very complex. To simplify the analysis, a number of assumptions about fluids are made. These assumptions define an **ideal fluid**. This is a fluid whose flow is:

- **Steady**: At any fixed point in space, the fluid velocity does not change with time. Steady flow is also called **laminar flow**.
- **Incompressible**: The density is the same everywhere in the fluid.
- **Non-viscous**: A body moving through the fluid would feel no resistive drag forces.

Consider an ideal fluid that moves. We may concentrate on a very small bit of the fluid – a fluid element – and follow its motion. This fluid element traces out a path as it moves (Figure **B.48**). We call this path a **streamline.** The velocity of the fluid element is tangent to the streamline. Streamlines cannot cross. A set of neighbouring streamlines forms a **flowtube**. A flowtube has streamlines as its boundary.

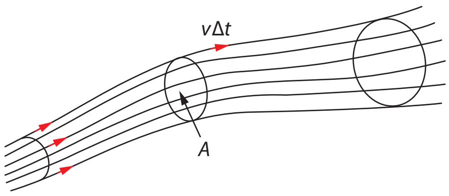


Figure B.48 Streamlines and a flowtube. The velocity vectors are tangent to streamlines.

B3.5 The continuity equation

We know that the speed of water out of a garden hose increases if we use a finger to partially close the hose opening. This is a consequence of the **equation of continuity.** Suppose we have a fluid that flows in a tube of uniform cross-sectional area A (Figure **B.49a**). The volume of fluid that can go through the dashed line in time Δt is $Av\Delta t$.

This volume will enter the region shaded darker blue. Since the fluid is incompressible, an equal volume of fluid must leave the blue shaded area during the same time in order to keep the **density** constant. If the fluid flows in a tube of varying cross-sectional area (Figure **B.49b**), we must have that:

$$A_1v_1\Delta t = A_2v_2\Delta t \Rightarrow A_1v_1 = A_2v_2$$

So if the cross-sectional area decreases, the fluid speed must increase.

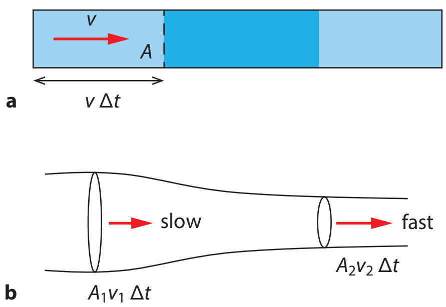


Figure B.49 **a** The volume of fluid that enters a pipe in time Δt must also leave in the same time from the other end. **b** This implies that if the cross-sectional area changes the speed must also change.

Worked example

B.25 Water comes out of a tap of cross-sectional area 1.5 cm^2. After falling a vertical distance of 6.0 cm, the cross-sectional area of the water column has been reduced to 0.45 cm^2. Calculate the speed of the water as it left the tap.

Let the required speed be v_1. By the equation of continuity,

$$A_1v_1 = A_2v_2$$

where v_2 is the speed after falling 6.0 cm. The water is falling freely under gravity and so:

$$v_2{}^2 = v_1{}^2 + 2gh$$

Thus, $A_1v_1 = A_2\sqrt{v_1{}^2 + 2gh}$. Squaring, this gives:

$$A_1{}^2v_1{}^2 = A_2{}^2(v_1{}^2 + 2gh)$$

$$v_1{}^2(A_1{}^2 - A_2{}^2) = 2A_2{}^2gh$$

$$v_1 = \sqrt{\frac{2A_2{}^2gh}{A_1{}^2 - A_2{}^2}} = \sqrt{\frac{2 \times 0.45^2 \times 9.8 \times 0.060}{1.5^2 - 0.45^2}} = 0.34\,\text{m s}^{-1}$$

B3.6 The Bernoulli equation

The **Bernoulli equation** applies to the laminar flow of a fluid in a tube of varying cross-sectional area and varying vertical height, such as the tube shown in Figure **B.50**.

This equation relates pressure, height and speed along a streamline. It states that:

$$p_1 + \tfrac{1}{2}\rho v_1{}^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho v_2{}^2 + \rho g z_2$$

As we will see, this is a consequence of energy conservation.

In particular, if the height stays constant ($z_1 = z_2$) the equation states that:

$$p_1 + \tfrac{1}{2}\rho v_1{}^2 = p_2 + \tfrac{1}{2}\rho v_2{}^2$$

This shows that in those areas in the flow where the speed is high the pressure is low, and vice versa.

To derive the Bernoulli formula, we start by noticing that the work done by the net force is the change in the kinetic energy of the fluid:

$$\begin{aligned} W_{\text{net}} &= \Delta E_{\text{K}} \\ &= \tfrac{1}{2} m (v_2{}^2 - v_1{}^2) \\ &= \tfrac{1}{2} \rho \,\Delta V (v_2{}^2 - v_1{}^2) \end{aligned}$$

The net work on the fluid consists of the work done against gravity (its weight) and the work associated with the pressure of the fluid. The work done against gravity is

$$\begin{aligned} W_{\text{weight}} &= -(mgz_2 - mgz_1) \\ &= -\rho g \,\Delta V (z_2 - z_1) \end{aligned}$$

To calculate the work associated with the pressure of the fluid, recall that work is the product of force times the distance over which it acts. The force on the left end of the volume of fluid in Figure **B.50** is $F_1 = p_1 A_1$, where A_1 is the cross-sectional area of the tube there, and the distance the fluid moves is Δx_1, so the work done at the left end is $F_1 \Delta x_1 = p_1 A_1 \Delta x_1 = p_1 \Delta V_1$. Likewise, at the right end the fluid does work equal to $p_2 \Delta V_2$, or we may say it has $-p_2 \Delta V_2$ of work done on it. But the amount of fluid that enters this section of the tube is the same amount that leaves, so $\Delta V_1 = \Delta V_2$. The net work associated with the pressure of the fluid is thus:

$$W_{\text{pressure}} = p_1 \Delta V - p_2 \Delta V$$

Equating the total work done and the change in kinetic energy gives:

$$p_1 \Delta V - p_2 \Delta V - \rho g \Delta V (z_2 - z_1) = \tfrac{1}{2} \rho \,\Delta V (v_2{}^2 - v_1{}^2)$$

Rearranging and cancelling ΔV gives the Bernoulli formula. A direct application of the Bernoulli equation is the problem of the flow of a liquid out of a container. Figure **B.51** shows a tank of liquid where a hole has been made in the side at a depth h below the liquid surface.

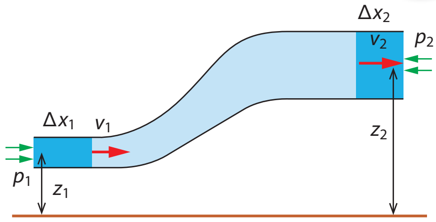


Figure B.50 Diagram for demonstrating the Bernoulli equation.

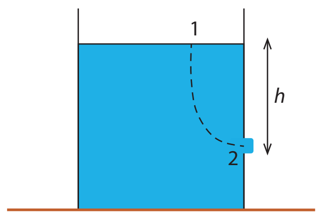


Figure B.51 Fluid flowing out of a container. Points 1 and 2 are connected by a streamline, along which the Bernoulli equation may be applied.

Notice that both the surface and the hole are exposed to the atmosphere and so the pressure there is atmospheric pressure, p_0. The dashed line shows a possible streamline joining the surface and the hole. Applying the Bernoulli equation to the streamline joining 1 to 2, we find:

$$p_1 + \tfrac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho v_2^2 + \rho g z_2$$

We measure heights from the level of the hole, so $z_1 = h$ and $z_2 = 0$. The surface has negligibly small speed (either because the surface is very large or because the liquid is being replenished to keep h constant), so the Bernoulli equation simplifies to:

$$p_0 + 0 + \rho g h = p_0 + \tfrac{1}{2}\rho v_2^2 + 0$$

The equation then gives:

$$v_2 = \sqrt{2gh}$$

for the speed at which the liquid leaves the hole.

Worked examples

B.26 Water of density $1000\,\mathrm{kg\,m^{-3}}$ flows in a horizontal pipe (Figure **B.52**). The radius of the pipe at its left end is 65 mm and that at the right end is 45 mm. The water enters from the left end with a speed of $6.0\,\mathrm{m\,s^{-1}}$. The pressure at the left end is 185 kPa. Calculate the pressure at the right end of the pipe, at a vertical distance of 1.5 m above the left end.

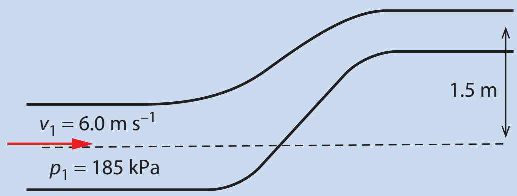


Figure B.52

We apply the Bernoulli equation: $p_1 + \tfrac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho v_2^2 + \rho g z_2$. We have two unknowns, the speed and pressure of the water at the right end. We use the equation of continuity to find the second speed:

$$A_1 v_1 = A_2 v_2 \Rightarrow v_2 = \frac{A_1 v_1}{A_2} = \frac{\pi(65^2) \times 6.0}{\pi(45^2)} = 12.5\,\mathrm{m\,s^{-1}}$$

Going back to the Bernoulli equation, we now have

$$185 \times 10^3 + \frac{1}{2} \times 10^3 \times 6.0^2 + 0 = p_2 + \frac{1}{2} \times 10^3 \times 12.5^2 + 10^3 \times 9.8 \times 1.5$$

This gives 110 kPa for the pressure p_2.

B.27 In a hydroelectric power plant (Figure **B.53**), water leaves a dam from a point 50 m beneath the surface. It enters a pipe of radius 80 cm and is incident on a turbine through a pipe of radius 40 cm. Calculate **a** the speed of the water as it hits the turbine, and **b** the pressure at point 2.

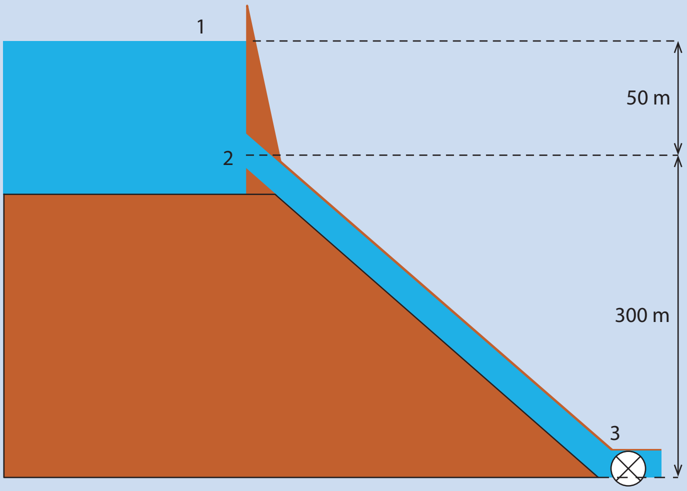


Figure B.53

a We consider a streamline beginning at 1 and ending at 3. The speed at 1 may be taken to be zero; the surface area of the water is assumed large, so the surface does not move appreciably. At 1 and 3 the pressure is atmospheric, since the water is exposed to the atmosphere. Therefore, from the Bernoulli equation,

$$p_1 + \tfrac{1}{2}\rho v_1{}^2 + \rho g z_1 = p_3 + \tfrac{1}{2}\rho v_3{}^2 + \rho g z_3$$

$$p_{\text{atm}} + 0 + 10^3 \times 9.8 \times 350 = p_{\text{atm}} + \tfrac{1}{2} \times 10^3 \times v_3{}^2 + 0$$

$$v_3 = 83\,\text{m}\,\text{s}^{-1}$$

b Consider a streamline beginning at 1 and ending at 2. The speed of the water at 2 can be found from the continuity equation relating points 2 and 3:

$$A_2 v_2 = A_3 v_3 \Rightarrow v_2 = \frac{A_3 v_3}{A_2} = \frac{\pi(40^2) \times 83}{\pi(80^2)} = 21\,\text{m}\,\text{s}^{-1}$$

From the Bernoulli equation again,

$$p_1 + \tfrac{1}{2}\rho v_1{}^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho v_2{}^2 + \rho g z_2$$

$$p_{\text{atm}} + 0 + 10^3 \times 9.8 \times 50 = p_2 + \tfrac{1}{2} \times 10^3 \times 21^2 + 0$$

$$p_2 = 3.7 \times 10^5\,\text{Pa}$$

B3.7 The Bernoulli effect

The classic application of the Bernoulli equation is to the motion of air past an aircraft wing. A wing shape (with a cross-section illustrated in Figure **B.54**) is called an **aerofoil**. The figure shows streamlines as they move below and above the aerofoil.

Because of the shape of the wing, air flows faster above the wing than beneath it. The streamlines above the wing are closer together, indicating that the air there is moving faster. A flow tube above the aerofoil would have a smaller area, and by the equation of continuity the air must move faster. The larger speed above the aerofoil results in a

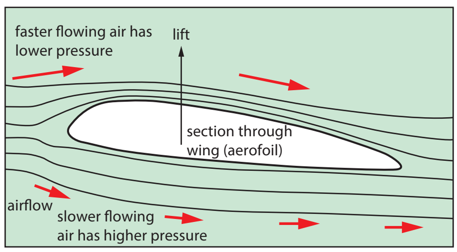


Figure B.54 Fluid flows faster over the top of the aerofoil than beneath it, giving rise to a lifting force.

smaller pressure, so there is a net upward force on the aerofoil, a lifting force. More precisely, from:

$$p_{over} + \frac{1}{2}\rho v_{over}^2 + \rho g z = p_{under} + \frac{1}{2}\rho v_{under}^2 + \rho g z$$

we have

$$p_{under} - p_{over} = \frac{1}{2}\rho v_{over}^2 - \frac{1}{2}\rho v_{under}^2 > 0$$

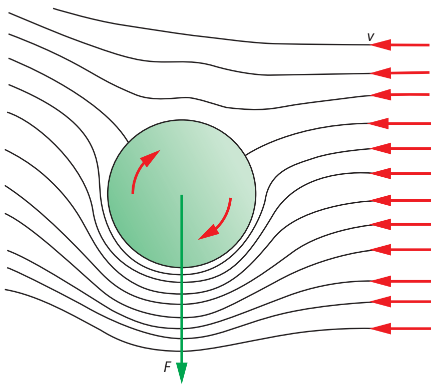
where we have neglected the $\rho g z$ terms, which are almost the same for a thin aerofoil. The difference in pressure results in a lifting force equal to:

$$F = (p_{under} - p_{over})A = \frac{1}{2}\rho(v_{over}^2 - v_{under}^2)A$$

where A is the area of the aerofoil.

A similar effect occurs for air flowing past the sail of a sailboat. The sail plays the role of an aerofoil, and the difference in speeds of the air on the two sides of the sail gives a force on the sail that propels the sailboat.

The **Bernoulli effect** is often exploited in sports. Figure **B.55** shows a ball that has been set spinning as it was thrown. (The ball travels from left to right so air is shown flowing from right to left.) A layer of air is 'dragged along' with the spinning surface of the ball. This, added to the overall air flow from the ball's translational motion, gives a total air speed that is higher on the underside and lower on the top side. The result is a net downward pressure on the ball, giving it an unexpectedly curved path.



B.55 The ball spins clockwise as it moves to the right. This means that the air flows faster beneath it, so the ball experiences a downward force *F*.

Figure **B.56** shows a device known as a **Pitot–Prandtl tube**, which can be used to measure the speed of flow of air past an object such as an aircraft.

This is a thin tube with a small opening at the front, pointing in the direction of the aircraft's motion. Air enters at the front hole at B and, essentially immediately, it is brought to rest. Using the Bernoulli equation along streamline BA we have

$$p_A + \frac{1}{2}\rho_{air}{v_A}^2 = p_B + \frac{1}{2}\rho_{air}{v_B}^2$$

But $v_B = 0$, so:

$$p_A + \frac{1}{2}\rho_{air}{v_A}^2 = p_B + 0$$

We have neglected the $\rho g z$ terms, which are almost the same for A and B. Rearranging the equation,

$$v_A = \sqrt{\frac{2(p_B - p_A)}{\rho_{air}}}$$

One leg of a liquid manometer is connected to the front hole of the tube and the other leg to A (there are tiny holes on the sides of the tube). The difference in pressure is measured by the height h between the columns of the manometer: $p_B = p_A + \rho_{liquid} g h$. Hence:

$$v = \sqrt{\frac{2\rho_{liquid} g h}{\rho_{air}}}$$

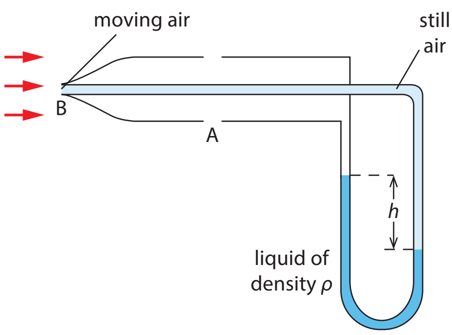


Figure B.56 A Pitot–Prandtl tube used to measure the flow of a fluid past an object such as an aircraft.

Yet another application of the Bernoulli effect is the **Venturi tube** (Figure **B.57**), which is also used to measure fluid flow speeds.

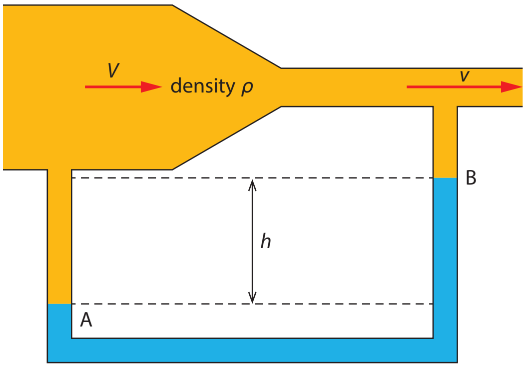


Figure B.57 A Venturi tube, which can be used to measure the flow of a fluid in a pipe.

The constriction in the tube makes the fluid move faster there and so the pressure drops. The wide and narrow parts of the tube are connected to the two arms of a liquid manometer. Because of the difference in pressure the two columns have a difference in height: $p_A = p_B + \rho_{liquid}gh$, so $\Delta p = p_A - p_B = \rho_{liquid}gh$. From the Bernoulli equation,

$$p_A + \tfrac{1}{2}\rho V^2 = p_B + \tfrac{1}{2}\rho v^2$$

and, from the equation of continuity, $AV = av$, where A and a are the cross-sectional areas of the wide and narrow parts of the tube.

Hence, $v = \frac{AV}{a}$. Substituting this into the Bernoulli equation,

$$p_A + \frac{1}{2}\rho V^2 = p_B + \frac{1}{2}\rho\frac{A^2V^2}{a^2}$$

and solving for V, the entry speed in the wide tube, we find:

$$p_A - p_B = \frac{1}{2}\rho\left(\frac{A^2}{a^2} - 1\right)V^2$$

$$V^2 = \frac{2(p_A - p_B)}{\rho\left(\frac{A^2}{a^2} - 1\right)}$$

$$V^2 = \frac{2a^2(p_A - p_B)}{\rho(A^2 - a^2)}$$

$$V = \sqrt{\frac{2a^2(p_A - p_B)}{\rho(A^2 - a^2)}}$$

$$= \sqrt{\frac{2a^2\Delta p}{\rho(A^2 - a^2)}}$$

Exam tip

There is a lot of algebra in this derivation and you must study it carefully. In an exam you will most likely be guided in such a derivation.

Notice that most fluid dynamics problems involve the use of both the Bernoulli and the continuity equations.

In these equations ρ is the density of the fluid that flows and ρ_{liquid} is the density of the liquid in the manometer.

Worked example

B.28 In a particular Venturi tube, the wide and narrow parts of the tube have cross-sectional areas of 45 cm^2 and 25 cm^2, respectively. The pressure difference between the wide and narrow parts of the tube is 65 kPa. Determine the speed of water flowing in the wide part of the tube and the volume flow rate in the tube.

This is a straightforward use of the formula just derived:

$$V = \sqrt{\frac{2a^2 \Delta p}{\rho(A^2 - a^2)}} = \sqrt{\frac{2 \times 25^2 \times 65 \times 10^3}{10^3 \times (45^2 - 25^2)}}$$

$$= 7.6\,\text{m s}^{-1}$$

The volume flow rate is $Q = VA = 7.6 \times 45 \times 10^{-4} = 3.4\,\text{m}^3\,\text{s}^{-1}$

B3.8 Stokes' law and viscosity

Stokes' law states that a sphere falling through an ideal fluid would experience no resistive force (other than the buoyant force). Similarly, the propeller of a ship turning in an ideal fluid would exert no force on the fluid, and consequently the reaction force on the propeller would be zero. In a real fluid, there are of course forces in both cases.

One of the differences between ideal and real fluids is that real fluids have **viscosity**. Viscosity has to do with how a layer of the fluid affects the motion of neighbouring layers. Viscosity is measured in terms of a coefficient with units of Pa s.

A small sphere of radius r falling through a viscous fluid (Figure **B.58**) experiences a resistive, or drag, force. For small velocities, the drag force is given by $F = 6\pi\eta r v$, where v is the speed and η is the viscosity coefficient.

The net force on a small sphere released from rest in a viscous fluid is:

$$F_{\text{net}} = mg - 6\pi\eta r v - \rho g V$$

The final term is the buoyant force on the sphere. The sphere will accelerate downwards, but as the speed increases the drag force increases, and at a high enough speed the net force will become zero, and the sphere will continue to move at constant speed. This speed is called the sphere's **terminal speed**. Using $m = \rho_{\text{body}} V$ and $V = \frac{4\pi r^3}{3}$, this speed is given by:

$$mg - 6\pi\eta r v - \rho_{\text{fluid}} g V = 0$$

$$v = \frac{\rho_{\text{body}} V g - \rho_{\text{fluid}} g V}{6\pi\eta r}$$

$$= \frac{(\rho_{\text{body}} - \rho_{\text{fluid}}) 2r^2 g}{9\eta}$$

Exam tip

A ball at rest in a fluid undergoes constant collisions with the molecules of the fluid. If the ball is falling in the fluid, the lower side of the ball is moving into the molecules and the top side is moving away from them. This means that the lower side will have more collisions per second and hence will experience a greater force upwards. The greater the speed, the greater the number of collisions; hence it is reasonable that drag force is proportional to speed.

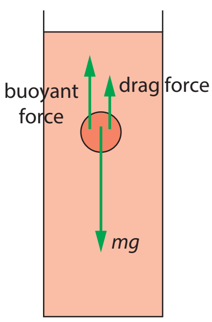


Figure B.58 A sphere falling through a fluid experiences a drag force.

B3.9 Turbulence

The top diagram in Figure **B.59** shows steady laminar flow: the speed of elements of the fluid along a cross-section of the pipe is constant. The middle diagram shows viscous laminar flow: the elements of the fluid near the surface of the pipe move more slowly. The lower diagram shows **turbulent flow**: the streamlines are unpredictable and chaotic. Turbulent flow arises as fluid speed increases.

The transition from laminar to turbulent flow is hard to measure, but a rough estimate is provided by a dimensionless number known as the **Reynolds number**. For a fluid flowing with speed v in a pipe of radius r, this is defined as:

$$R = \frac{v\rho r}{\eta}$$

where ρ is the density of the fluid and η its viscosity. We have turbulent flow if this number exceeds about 1000.

As an example, consider air of density $1.2\,\mathrm{kg\,m^{-3}}$ that flows with speed $2.1\,\mathrm{m\,s^{-1}}$ in a pipe of radius 5.0 mm. The viscosity of the air is $1.8 \times 10^{-5}\,\mathrm{Pa\,s}$. The Reynolds number is:

$$R = \frac{v\rho r}{\eta} = \frac{2.1 \times 1.2 \times 5.0 \times 10^{-3}}{1.8 \times 10^{-5}} \approx 7.0 \times 10^{2}$$

so the flow is laminar, since $R < 1000$. The flow would become turbulent at a speed above about:

$$\frac{R\eta}{\rho r} = \frac{1000 \times 1.8 \times 10^{-5}}{1.2 \times 5.0 \times 10^{-3}} = 3.0\,\mathrm{m\,s^{-1}}$$

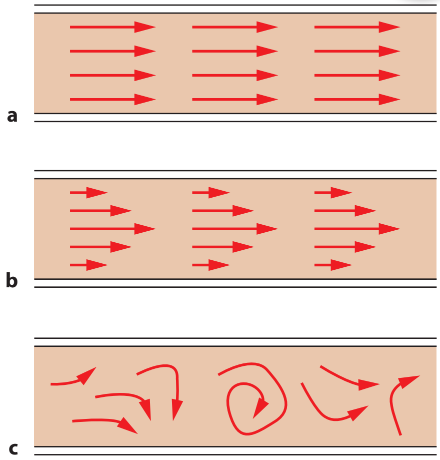


Figure B.59 **a** Non-viscous laminar flow; **b** viscous laminar flow; **c** turbulent flow.

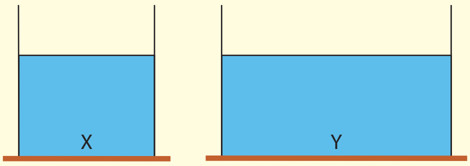
Nature of science

Understanding fluid flow

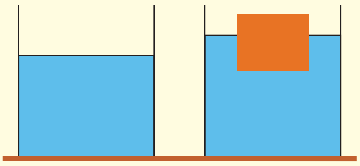
The study of fluid flow has made possible a large number of technological advances, including the design of more efficient and aerodynamic cars and aircraft, and new ways of measuring and understanding the flow of blood in arteries. Modelling the behaviour of winds and ocean currents has led to more accurate predictions of the weather. Fluid dynamics is an integral part of any study of stellar stability and stellar evolution. Research in fluid dynamics is ongoing and **turbulence** is still one of the great unsolved problems in the field, with many unanswered questions and formidable mathematical difficulties.

Test yourself

38 Two containers are filled with the same liquid to the same level. One container has double the cross-sectional area of the other. Compare the pressure at points X and Y.



39 The diagram below shows two identical beakers containing equal amounts of water. In the second beaker, a piece of wood is floating in the water.

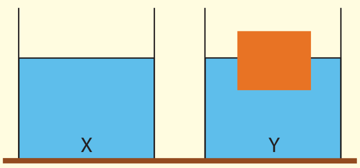


The beakers are weighed. Determine

a which beaker, if any, is heavier

b whether the pressure at the bottom of the two containers is the same.

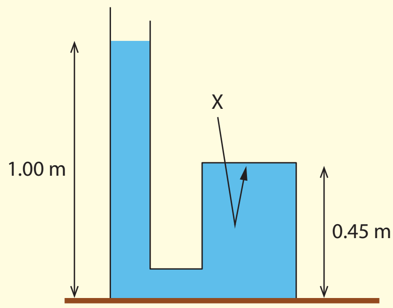
40 The diagram below shows two identical beakers, filled with water to the same level. In the second beaker, a piece of wood is floating in the water.



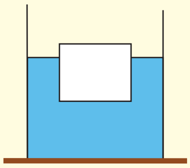
a The beakers are weighed. Determine which beaker, if any, is heavier.

b Compare the pressure at points X and Y.

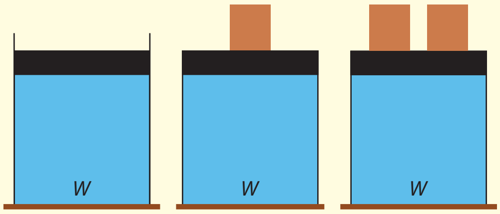
41 The container below is filled with water (density $1.0 \times 10^3\,\mathrm{kg\,m^{-3}}$). Determine the pressure at point X. The liquid has density $13600\,\mathrm{kg\,m^{-3}}$.



42 An ice cube floats in water. Explain why, after the ice cube melts, the level of the water in the container will be the same.



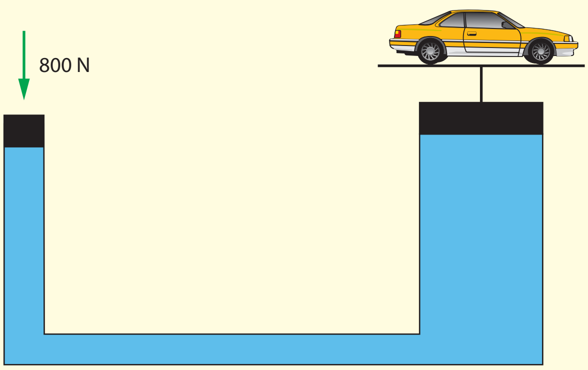
43 A liquid is enclosed in a container with a piston of negligible mass. Atmospheric pressure is p_0.



A block of weight W is placed on the piston, of area A. The weight is then doubled. Compare the pressures at the same depth h in each case.

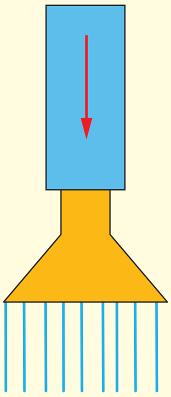
44 A hollow sphere floats exactly half submerged in water of density $1.0 \times 10^3\,\mathrm{kg\,m^{-3}}$. The outer radius of the sphere is 15 cm and the inner radius is 14 cm. Calculate the density of the material of the sphere. (The volume of a sphere of radius R is $V = \frac{4\pi R^3}{3}$.)

45 In the hydraulic pump below, a car of mass 1400 kg is to be lifted by applying a force of 800 N on a piston of diameter d. The diameter of the piston where the car is placed is 1.8 m. Calculate d.

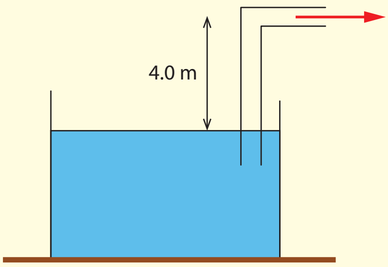


46 Water comes out of a tap of cross-sectional area $1.4\,cm^2$. After falling a vertical distance of 5.0 cm, the cross-sectional area of the water column has been reduced to $0.60\,cm^2$. Calculate the volume of water per second delivered by the tap.

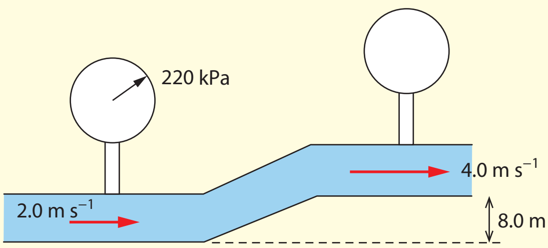
47 In a shower, water enters the shower head through a tube of diameter 1.2 cm with a speed of $1.1\,m\,s^{-1}$. The shower head has 30 small holes, each of diameter 0.20 cm. Calculate the speed with which the water exits one of these holes.



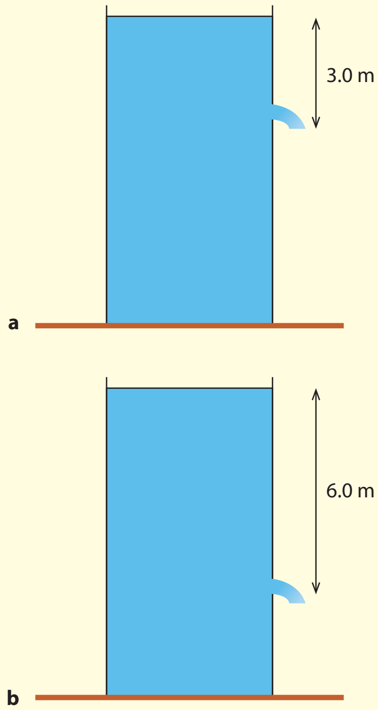
48 Water of density $1.0 \times 10^3\,kg\,m^{-3}$ is pumped out of a tank though a hose of radius 1.2 cm. The water in the hose has a constant speed of of $3.8\,m\,s^{-1}$. The water is raised a vertical distance of 4.0 m before being ejected into the surroundings. Estimate the power of the pump.



49 Oil of density $850\,kg\,m^{-3}$ flows in the pipe shown below. Calculate the pressure shown by the gauge on the upper side of the pipe.



50 Water leaks out of a container from a hole 3.0 m below the free water surface. The container is large enough that no appreciable change occurs in the water level in the container.

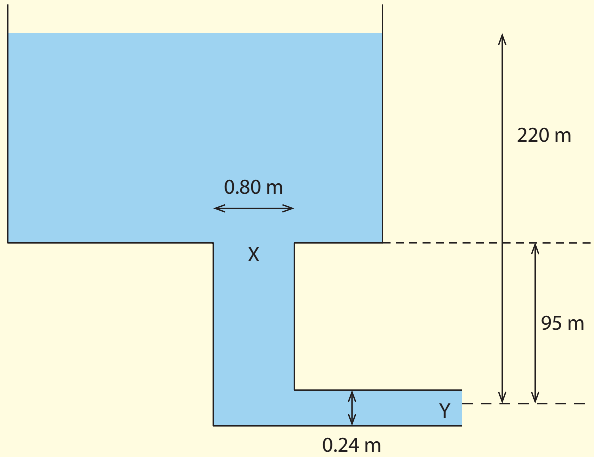


a Calculate the speed at which the water exits the container.

b A second hole is made at a depth of 6.0 m, at 3.0 m from the ground. Determine the ratio of the ranges (i.e. the horizontal distance travelled) of the two columns of water as they land on the ground.

51 Water exits a tank horizontally from a hole at a depth d. The water level in the tank is H and may be considered to be constant. Determine d in terms of H so that the water lands on the ground at the largest possible distance from the base of the tank.

52 In the diagram below (not drawn to scale), the exit at Y is closed with a tap so that no water flows.



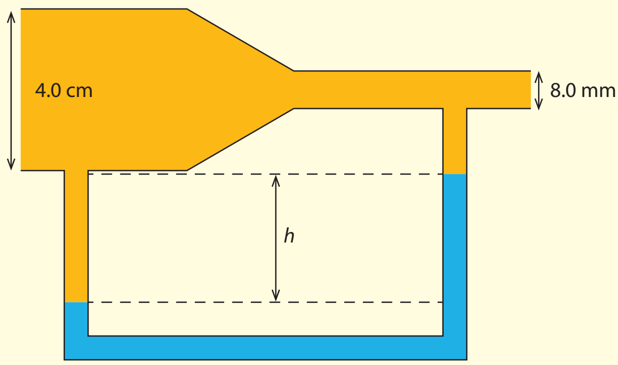
a Calculate the pressure at X and Y.

b The exit at Y is opened so that Y is exposed to atmospheric pressure. Calculate:

i the speed with which the water leaves the exit Y

ii the pressure at X and at Y.

53 Determine the height *h* of the mercury in the Venturi tube in the diagram below. The fluid flowing is air (density $1.2\,\mathrm{kg\,m^{-3}}$), at a volume flow rate of $1800\,\mathrm{cm^3\,s^{-1}}$. The density of mercury is $13\,600\,\mathrm{kg\,m^{-3}}$.



54 An aircraft is flying at an altitude where the air density is $0.35\,\mathrm{kg\,m^{-3}}$. The pressure of the air outside the aircraft is 12 000 Pa lower than the pressure at the same altitude in static air in a Pitot–Prandtl tube. Calculate the speed of the aircraft.

55 Oil of density $850\,\mathrm{kg\,m^{-3}}$ and viscosity coefficient 0.01 Pa s flows in a pipeline of diameter 0.80 m. The flow rate of the oil in the pipeline is $0.52\,\mathrm{m^3\,s^{-1}}$. Determine whether the flow is turbulent or laminar.

56 On a windy day, air flows in between tall buildings in a city. By making suitable estimates, determine whether the flow is turbulent or laminar.

57 An oil droplet (of density $870\,\mathrm{kg\,m^{-3}}$) is balanced in between two oppositely charged, parallel capacitor plates. The droplet has a positive charge and the electric force on the droplet is directed vertically upwards. The electric field between the plates is $1.25 \times 10^5\,\mathrm{N\,C^{-1}}$. When the field is turned off, the droplet falls and soon reaches a terminal velocity of $4.11 \times 10^{-4}\,\mathrm{m\,s^{-1}}$. Determine the charge on the droplet in terms of the electronic charge *e*. (The viscosity of air is 1.82×10^{-5} Pa s and you may neglect buoyancy effects.)

B4 Forced vibrations and resonance (HL)

In this section we examine the effect of resistance forces on simple harmonic oscillations. The effect of these forces is that the oscillations will eventually stop and the energy of the system will be dissipated, mainly as thermal energy in the environment and the system itself. We also consider the effect on an oscillating system of an externally applied periodic force. This leads to the interesting phenomenon of **resonance**.

Learning objectives

- Understand the concept of natural frequency of vibration.
- Work with damping and the Q factor.
- Work with forced oscillations.
- Appreciate the concept of resonance.

B4.1 Oscillations and damping

In Topic **4** we saw examples of systems that are capable of oscillations or vibrations. Those were **free** oscillations: that is, oscillations without energy loss and without externally applied forces. In this case the amplitude of the oscillations stays constant (Figure **B.60**). In this section we will examine the effects of the presence of such forces.

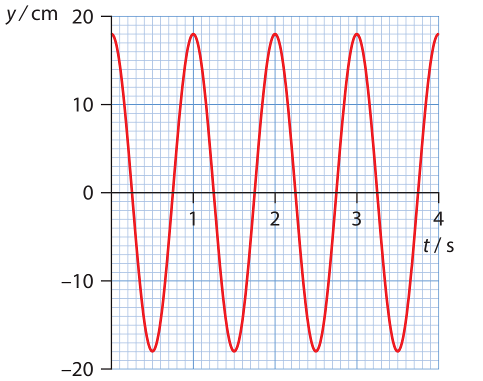


Figure B.60 Undamped oscillations: the amplitude is constant.

The term **damping** refers here to the loss of energy in an oscillating system. Our discussion refers to frictional/resistance forces that are proportional to speed. We may distinguish three types of damping. In **light damping**, the amplitude of oscillation decreases slowly with time and eventually becomes zero; the oscillations stop (Figure **B.61**). Oscillations under light damping are also called **underdamped** oscillations. The amplitude decreases exponentially (dashed blue line).

The energy of the system also decreases exponentially (Figure **B.62**).

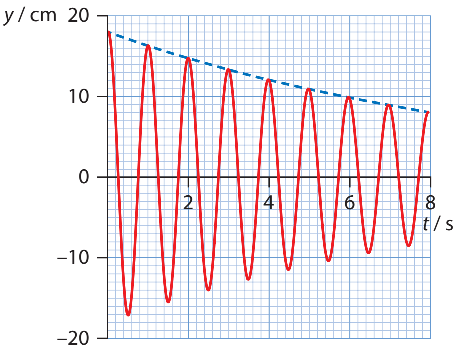


Figure B.61 Damped oscillations: the amplitude decreases exponentially with time.

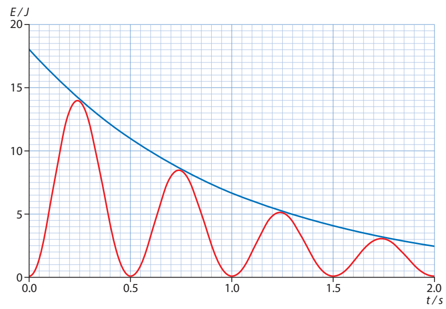


Figure B.62 In damped motion the total energy of a system decreases exponentially (blue curve). The red curve shows the kinetic energy of the system. The period of oscillations is 1.0 s.

How quickly will the oscillations in an underdamped system die out? A dimensionless number called the **Q factor** of the system answers this question. The Q factor is defined as:

$$Q = 2\pi \frac{\text{energy stored in a cycle}}{\text{energy dissipated in a cycle}}$$

Remember that the stored energy is proportional to the square of the amplitude. In Figure **B.61**, the initial amplitude is 18.0 cm and after one full oscillation it is about 16.3 cm. This means that:

$$Q = 2\pi \frac{18.0^2}{18.0^2 - 16.3^2} \approx 35$$

A high Q value means that the system will perform many oscillations before the oscillations die out. Roughly, Q is the number of oscillations before the system stops.

The expression for Q may be written in an equivalent way: let P be the power loss of the system. Then in one cycle the energy dissipated is $E_{\text{dissipated}} = PT$, where T is the period of oscillations. Thus:

$$
\begin{aligned}
Q &= 2\pi \frac{E_{\text{stored}}}{E_{\text{dissipated}}} \\
&= 2\pi \frac{E_{\text{stored}}}{PT} \\
&= 2\pi \frac{1}{T} \frac{E_{\text{stored}}}{P} \\
&= 2\pi f \frac{E_{\text{stored}}}{P}
\end{aligned}
$$

where f is the frequency at which the system oscillates.

If the amount of damping is very large, we speak of **overdamped** motion. In this case the system returns to its equilibrium position after a very long time **without performing oscillations** (blue curve in Figure **B.63**).

For a particular amount of damping we have the case of **critically damped** motion, in which the system returns to its equilibrium without any oscillations but does so in the shortest possible time (purple curve in Figure **B.63**).

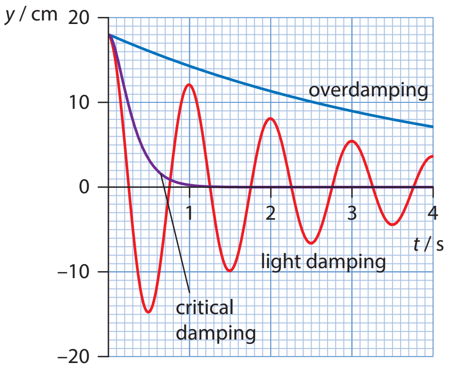


Figure B.63 Lightly damped oscillations (red curve), overdamped motion (blue curve) and critically damped motion (purple curve).

B4.2 Forced oscillations and resonance

We will now examine qualitatively the effect of an externally applied force F on a system that is free to oscillate with frequency f_0. The force F will be assumed to vary periodically with time with a frequency (the driving frequency) f_D – for example, $F = F_0 \cos 2\pi f_D t$. The oscillations in this case are called **forced** or **driven** oscillations. As an example consider a pendulum of length L that hangs vertically, as in Figure **B.64**. The point of support is made to oscillate with some frequency f_D. How does the pendulum react to this force?

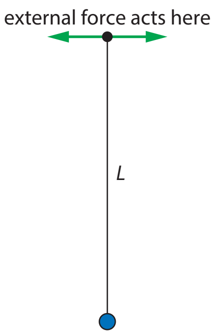


Figure B.64 Forced oscillations: a periodic force makes the point of support vibrate.

In general, once the external force is applied, the system eventually starts oscillating at the driving frequency f_D. This is true whether the system is initially at rest or initially oscillating at its own frequency. In the absence of friction, the amplitude of the oscillations approaches infinity as the driving frequency approaches the **natural frequency** of the oscillating system (Figure **B.65**). Of course an infinite amplitude is impossible: we must also take damping into account.

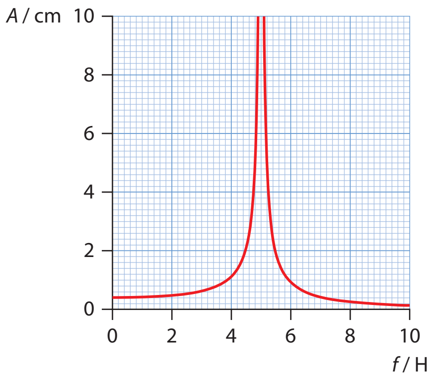


Figure B.65 Variation with driving frequency of the amplitude of simple harmonic motion when an undamped system is driven by an external periodic force.

The amplitude of the oscillations will depend on the relationship between f_D and f_0, and the amount of damping. We might expect that, because the system prefers to oscillate at its own natural frequency, when the external force has the same frequency as the natural frequency, large oscillations will take place. A detailed analysis produces the graph in Figure **B.66**, showing how the amplitude of oscillation of a system with natural frequency f_0 varies when it is subjected to a periodic force of frequency f_D. The degree of damping increases as we move from the top curve down.

The general features of the graph in Figure **B.66** are as follows:

- For small damping, the peak of the curve occurs at the natural frequency of the system, f_0.
- The lower the damping, the higher and narrower the curve.
- As the amount of damping increases, the peak shifts to lower frequencies and becomes wider.
- At very low frequencies, the amplitude is essentially constant.

If f_D is very different from f_0, the amplitude of oscillation will be small. On the other hand, if f_D is approximately the same as f_0 and the degree of damping is small, the resulting driven oscillations will have large amplitude. The largest amplitude is obtained when f_D is equal to f_0, in which case we say that the system is in resonance.

The state in which the frequency of an externally applied periodic force equals the natural frequency of a system is called **resonance**. This results in oscillations with large amplitude.

Resonance can be disastrous: we do not want an aircraft wing to resonate, nor is it good for a building or a bridge to be set into resonance by an earthquake or the wind (Figure **B.67**).

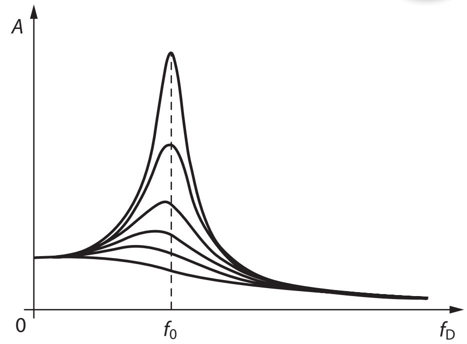


Figure B.66 Variation of amplitude with driving frequency and degree of damping when a damped system is driven by an external periodic force.

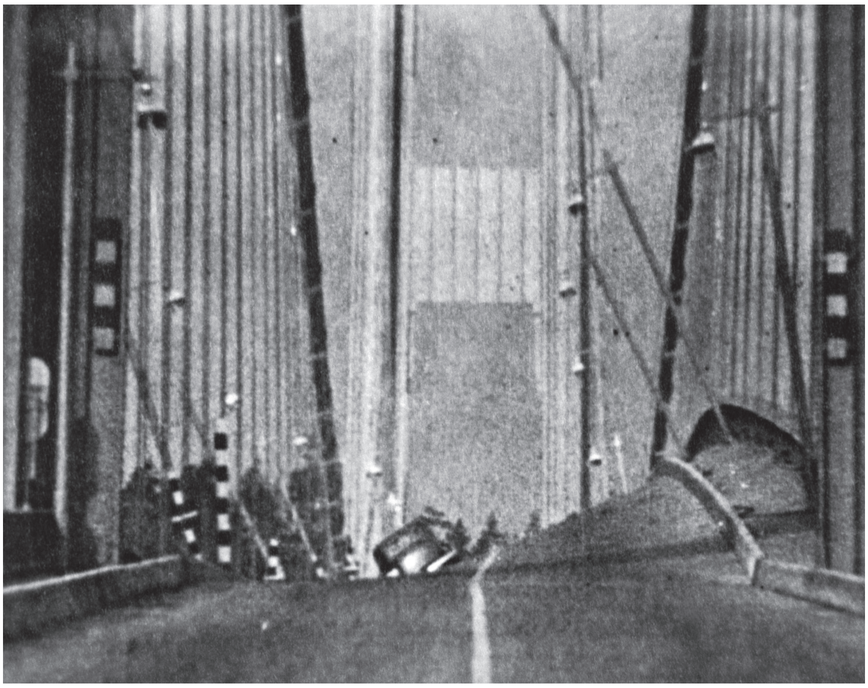


Figure B.67 The Tacoma Narrows Bridge in Washington state in the USA, oscillating before collapsing in 1940, a victim of resonance caused by high winds.

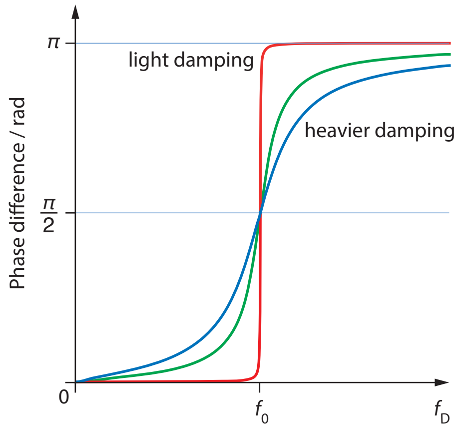


Figure B.68 Phase difference as a function of the external driving frequency.

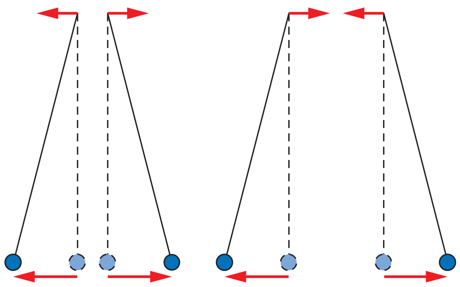


Figure B.69 **a** The system and the driver oscillate in phase. **b** The system and the driver oscillate out of phase.

Resonance can be irritating if the car in which you drive is set into resonance by bumps on the road or by a poorly tuned engine. But resonance can also be a good thing: it is used in a microwave oven to warm food, and your radio uses resonance to tune into one specific station and not another. Another useful example of electrical resonance is the quartz oscillator, a quartz crystal that can be made to vibrate at a specific frequency.

In the case of driven oscillations there is a phase difference between the displacement of the system and the displacement of the driver. This phase difference depends on the relation between the external driving frequency and the natural frequency. The relationship is shown in Figure **B.68**.

For very light damping, the phase difference is zero when $f < f_0$: in this case the system and the driver oscillate in phase (Figure **B.69a**). When $f > f_0$, the phase difference is π: the system and the driver are completely out of phase (Figure **B.69b**). There is a phase difference of $\frac{\pi}{2}$ when $f = f_0$.

Nature of science

Risk assessment

The phenomenon of resonance is ubiquitous in physics and engineering. Microwave ovens emit electromagnetic radiation in the microwave region of wavelengths that match the vibration frequencies of water molecules so they can be absorbed and warm food. In the medical technique known as MRI, the patient is exposed to a strong magnetic field and resonance is used to force absorption of radio frequency radiation by protons in the patient's body. This technique provides detailed images of body organs and cellular functions. Just as with buildings and bridges, detailed studies are necessary to avoid the possibility of unwanted resonances creating catastrophic large-amplitude oscillations.

Test yourself

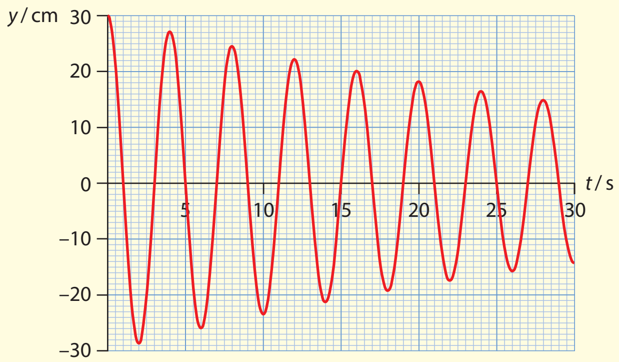
58 Distinguish between **free** and **driven** oscillations.

59 Distinguish between **overdamped** and **critically damped** oscillations.

60 In the context of oscillations, state what is meant by **damping**.

61 State what is meant by **resonance**.

62 The graph below shows the variation of the displacement of a system performing oscillations.



a Determine the frequency of the oscillations.

b Calculate the Q factor of the system.

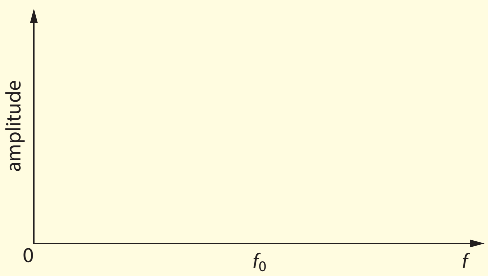
c Draw a sketch graph to show the variation with time of the energy of the system.

d The amount of damping is increased. State and explain the effect of this on the value of Q.

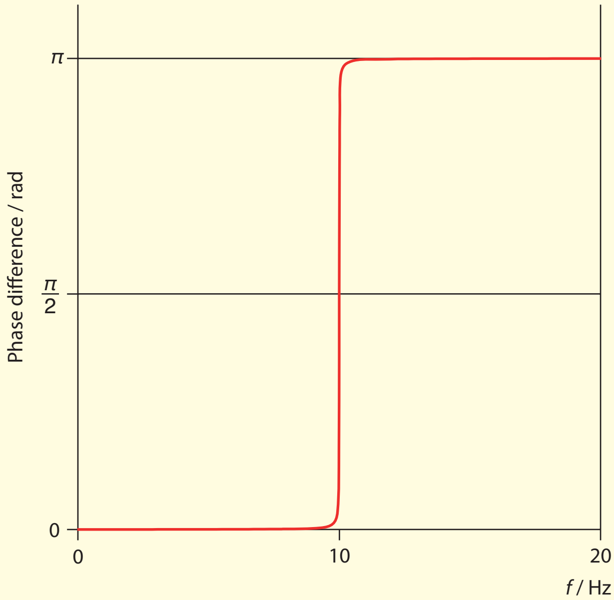
63 **a** State the conditions that must be satisfied for simple harmonic oscillations to take place.

b Draw a graph of displacement versus time for a body undergoing lightly damped simple harmonic oscillations.

c A system has natural frequency f_0. A periodic force of variable frequency f acts on the system. On a copy of the figure below, draw graphs to show the variation with f of the amplitude of oscillation of the system for **i** light and **ii** heavy damping.



64 A body performs simple harmonic oscillations at the end of a vertical spring. The diagram below shows the variation of the phase difference between the displacement of the body and the displacement of a driver that drives the system with driver frequency f. The system is very lightly damped.



a State the natural frequency of oscillation of the system.

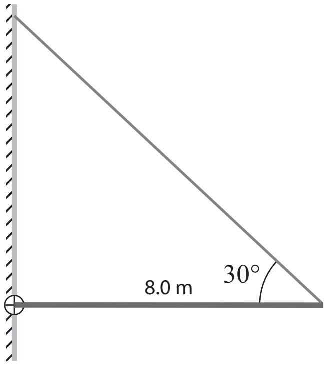
b Copy the figure and on the same axes draw a sketch graph to show how this graph changes, if at all, when the damping is increased.

c The frequency of the driver is 15 Hz. At a particular time the driver forces the top end of the spring to move to the right. State and explain the direction of motion of the mass at the end of the spring at that instant.

Exam-style questions

Note: You may use the textbook to find any moments of inertia you may require.

1 A uniform plank of length 8.0 m and weight 1500 N is supported horizontally by a cable attached to a vertical wall. (You may use the textbook to find any moments of inertia you may require.) The cable makes an angle of 30° with the horizontal rod.



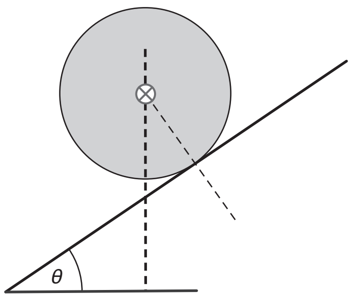
a Calculate:

i the tension in the cable **[3]**

ii the magnitude and direction of the force exerted by the wall on the rod. **[3]**

b A worker of mass 85 kg can walk anywhere on the rod without fear of the cable breaking. Determine the minimum breaking tension of the cable. **[3]**

2 A cylinder of mass $M = 12.0$ kg and radius $R = 0.20$ m rolls down an inclined plane without slipping.



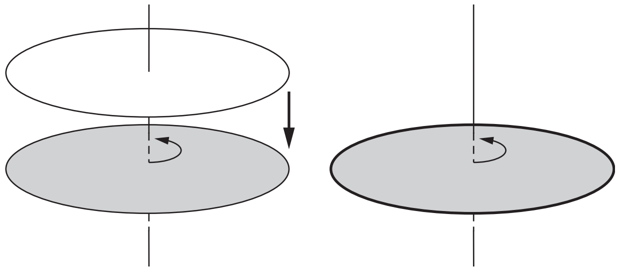
a Make a copy of the figure, and on it draw arrows to represent the forces acting on the cylinder as it rolls. **[3]**

b **i** Show that the linear acceleration of the centre of mass of the cylinder is $a = \frac{2}{3}g\sin\theta$. **[3]**

ii Determine the frictional force acting on the cylinder for $\theta = 30°$. **[1]**

c Calculate the rate of change of the angular momentum of the cylinder as it rolls for $\theta = 30°$. **[2]**

3 A horizontal disc rotates about a vertical axis through its centre of mass. The mass of the disc is 4.00 kg and its radius is 0.300 m. The disc rotates with an angular velocity of $42.0\,\text{rad}\,\text{s}^{-1}$. A ring of mass 2.00 kg and radius 0.300 m falls vertically and lands on top of the disc, as shown below. As the ring lands, it slides a bit on the disc and eventually the disc and ring rotate with the same angular velocity.



a **i** Explain why the angular momentum of the system is the same before and after the ring lands. **[2]**

ii Calculate the final angular velocity of the disc–ring system. **[3]**

iii Determine the kinetic energy lost as a result of the ring landing on the disc. **[3]**

b It took 3.00 s for the ring to start rotating with the same angular velocity as the disc.

i Determine the average angular acceleration experienced by the ring during this time. **[1]**

ii Calculate the number of revolutions made by the disc during the 3.00 s. **[2]**

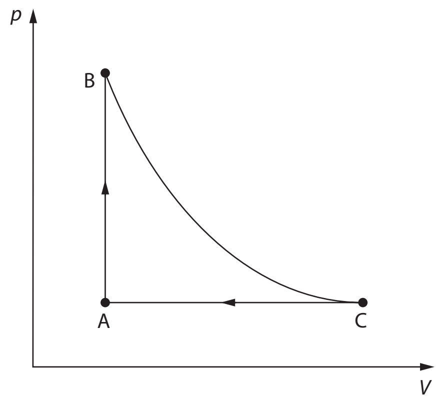
iii Show that the torque that accelerated the ring to its final constant angular velocity was 1.26 N m. **[2]**

iv State and explain, without further calculation, the magnitude of the torque that decelerated the disc. **[2]**

c Calculate the average power developed by the torque in **b iii** in accelerating the ring. **[2]**

4 A heat engine has 1.00 moles of an ideal gas as its working substance and undergoes the cycle shown below. BC is an adiabatic. The following data are available:

$T_A = 3.00 \times 10^2\,\text{K}$, $p_A = 2.00 \times 10^5\,\text{Pa}$, $T_B = 6.00 \times 10^2\,\text{K}$



a Calculate:

i the pressure at B **[1]**

ii the temperature at C **[2]**

iii the volume at A and at C. **[2]**

b Determine:

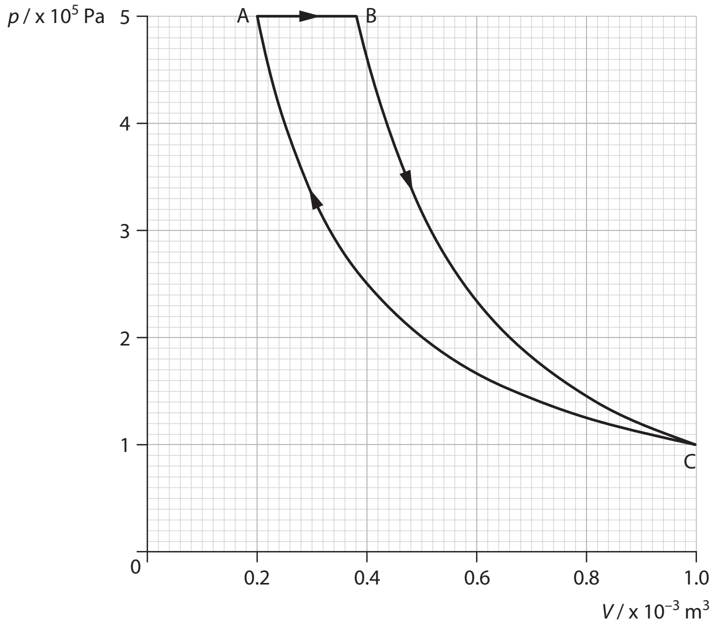
i the change in internal energy from A to B **[2]**

ii the energy removed from the gas. **[2]**

iii the efficiency of the cycle **[2]**

c State a version of the second law of thermodynamics in a way that directly applies to this engine. **[2]**

5 The graph shows a thermodynamic cycle in which an ideal gas expands from A to B to C and is then compressed back to A. BC is an adiabatic curve and CA is an isothermal curve. The volume at B is $0.38 \times 10^{-3}\,\text{m}^3$.



a **i** State what is meant by an **adiabatic curve**. **[1]**

ii Explain why, in an adiabatic expansion of an ideal gas, temperature decreases. **[2]**

b Justify why CA is isothermal. **[3]**

c The temperature of the gas at A is 300 K.

i Calculate the temperature at B and at C. **[3]**

ii Determine the number of moles of the gas. **[2]**

d The work done from C to A is 160 J. Calculate:

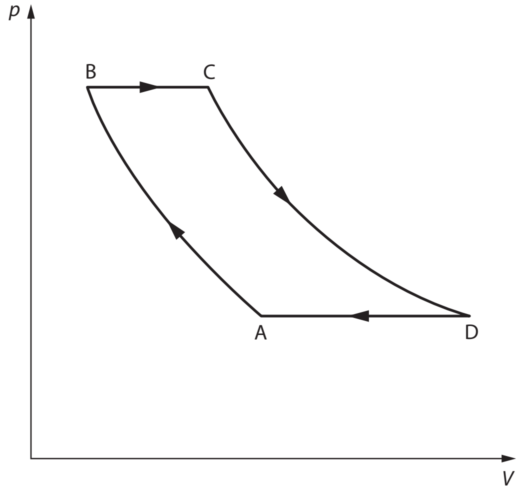
i the energy transferred out of the gas **[2]**

ii the energy transferred into the gas **[2]**

iii the work done from B to C **[2]**

iv the efficiency of the cycle. **[1]**

HL **6** The graph shows the Brayton cycle (not drawn to scale), which consists of two isobaric and two adiabatic curves. The state at A has pressure 2.0×10^5 Pa, volume $0.40\,m^3$ and temperature 320 K. The pressure at B is 2.0×10^6 Pa.



a Show that, along an adiabatic curve, $\frac{T^5}{p^2} = \text{constant}$. **[3]**

b Calculate:

i the temperature at B **[2]**

ii the volume at B. **[2]**

c Show that the change in internal energy from A to B is 0.18 MJ. **[3]**

d Using your answer to **c**, determine the work done from A to B. **[2]**

HL **7** **a** Show that the pressure at a depth h below the free surface of a liquid of density ρ is given by $p = p_0 + \rho gh$, where p_0 is atmospheric pressure. **[3]**

b Suggest what, if anything, will happen to the pressure at a depth h below the free surface of the liquid in a container, if the container:

i is allowed to fall freely under gravity **[1]**

ii is accelerated upwards with acceleration a. **[1]**

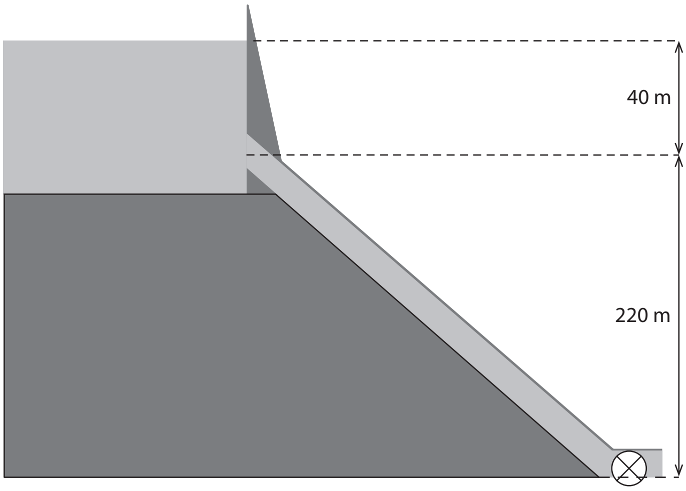
c State **Archimedes' principle**. **[1]**

d A block of wood floats in water with 75% of its volume submerged. The same block when floating in oil has 82% of its volume submerged. The density of water is $1000\,kg\,m^{-3}$. Calculate:

i the density of the wood **[2]**

ii the density of the oil. **[2]**

HL **8** The diagram shows a hydroelectric power plant in which water from a large reservoir is allowed to flow down a long outlet pipe and, eventually, through a turbine. The radius of the outlet pipe where it leaves the reservoir is 65 cm, and it tapers down to 25 cm at the turbine. You may assume that the reservoir is large enough that there is no appreciable change in the water level of the reservoir as water flows out.



a **i** Calculate the speed of the water at the turbine. **[2]**

ii State two assumptions you have made in this calculation. **[2]**

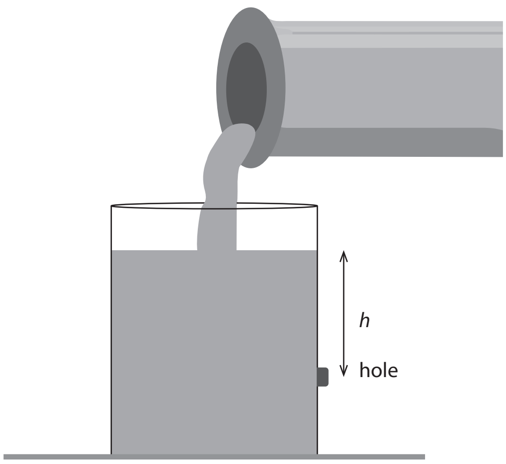
b Estimate the flow rate of the water though the turbine. **[2]**

c Calculate the water pressure at the upper end of the outlet pipe:

i when the water is static (i.e. not flowing) **[2]**

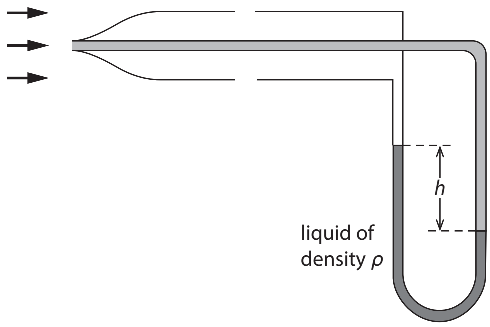
ii when the water is flowing. **[3]**

d In another, smaller reservoir, water is constantly being pumped into the reservoir at a rate of $0.40\,\text{m}^3\,\text{s}^{-1}$.



A hole of radius 3.0 cm is to be drilled at a depth *h* below the surface of the water such that, when the water flows out, the water level in the reservoir stays the same. Determine the depth *h*. **[3]**

HL **9** A Pitot–Prandtl tube is used to measure the speed of an aircraft. The liquid in the manometer has density ρ and the difference in the levels of the liquid in the two columns is h.

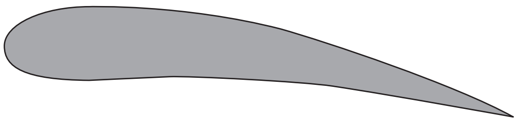


a Explain why the liquid in the right-hand column of the manometer is lower than that in the left-hand column. **[2]**

b Show that the flow speed v is given by $v = \sqrt{\frac{2\rho g h}{\rho_{air}}}$ **[3]**

c This Pitot tube uses a liquid of density $\rho = 920\,\text{kg}\,\text{m}^{-3}$. The density of air is $1.20\,\text{kg}\,\text{m}^{-3}$ and the difference h in liquid levels is 0.25 m. Estimate the speed of the aircraft. **[2]**

HL **10** The diagram shows an aerofoil of **total** surface area $16\,\text{m}^2$. Air (of density $1.20\,\text{kg}\,\text{m}^{-3}$) flows with a speed of $85\,\text{m}\,\text{s}^{-1}$ across the upper surface of the aerofoil and a speed of $58\,\text{m}\,\text{s}^{-1}$ across the lower surface.



a On a copy of the diagram, draw streamlines around the aerofoil. **[2]**

b **i** Calculate the lifting force on the aerofoil. **[2]**

ii State one assumption you made in your calculation of the lifting force. **[1]**

c The weight of the aerofoil is 3.0 kN. The aerofoil is attached to the fuselage of an aircraft. Estimate the force that the aerofoil exerts on the fuselage. **[2]**

d If the aerofoil's angle relative to the horizontal is increased, the flow of air past it may become turbulent.

i State what is meant by turbulent flow. **[1]**

ii Indicate on your copy of the diagram the most likely position around the aerofoil where turbulence may set in. **[1]**

iii Suggest the effect of turbulence on the lifting force on the aerofoil. **[1]**

HL **11** **a** Distinguish between **damped** and **undamped** oscillations. **[2]**

b The graph shows the variation with time of the displacement of an oscillating system.

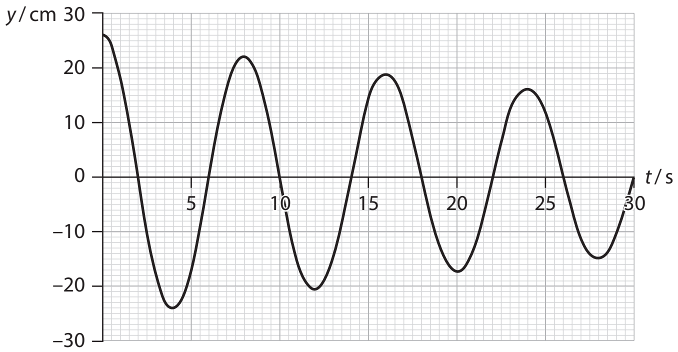


Figure B.110 For Exam-style question 11b.

Use the diagram to determine:

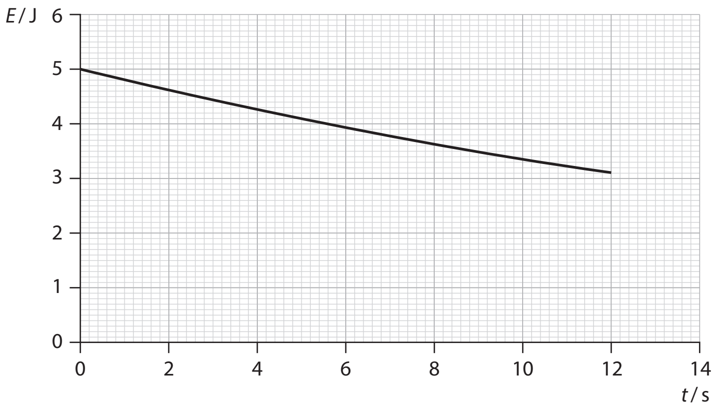
i the period of oscillation **[1]**

ii the Q factor of the system. **[3]**

c **i** On a copy of this diagram, draw a graph to show the variation of the displacement when the damping is increased. **[2]**

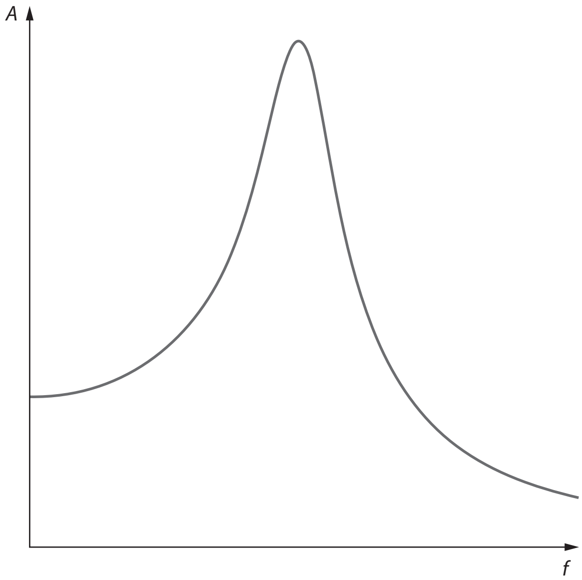
ii State the effect, if any, of the increased damping on Q. **[1]**

d The graph shows the variation with time of the energy of another oscillating system.



The system oscillates with a period of 2.0 s. Estimate the Q factor of this system. **[2]**

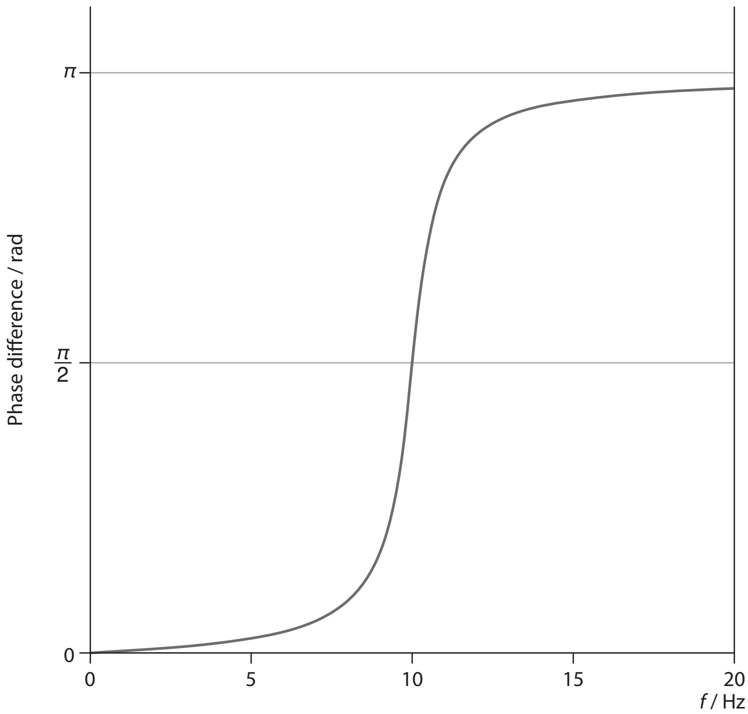
HL **12** The graph shows the variation of the amplitude of an oscillating system with the frequency of a periodic force acting on the system. The system is lightly damped.



a By reference to the diagram, outline what is meant by resonance. **[3]**

b On a copy of this diagram, sketch the variation of amplitude with frequency when the amount of damping is increased. **[2]**

c The graph shows how the phase difference between the displacements of the system and the driver varies with the frequency of the driving force, for light damping.



i On a copy of this diagram, sketch a graph to show how the phase varies when the damping is increased. **[2]**

ii State the resonant frequency of this system. **[1]**