6.2 The law of gravitation

Learning objectives

- Solve problems where the gravitational force plays the role of a centripetal force, in particular orbital motion.
- Use the concepts of gravitational force, gravitational field strength, orbital speed and orbital period.
- Determine the net gravitational field strength due to two point masses.

This section will introduce us to one of the fundamental laws of physics – **Newton's law of gravitation**. The law of gravitation makes it possible to calculate the orbits of the planets around the Sun, and predicts the motion of comets, satellites and entire galaxies. Newton's law of gravitation was published in his *Philosophiae Naturalis Principia Mathematica* in 1686.

Newton's law of gravitation

We have seen that Newton's second law implies that whenever a particle moves with acceleration, a net force must be acting on it. The proverbial apple falling freely under gravity is accelerating at $9.8\,\mathrm{m\,s^{-2}}$ and thus experiences a net force in the direction of the acceleration. This force is what we call the 'weight' of the apple. Similarly, a planet that orbits around the Sun also experiences acceleration and thus a force is acting on it. Newton hypothesised that the force responsible for the falling apple is the same as the force acting on a planet as it moves around the Sun.

Newton proposed that the attractive force of gravitation between two **point** masses is given by the formula:

$$F = G\frac{M_1 M_2}{r^2}$$

where M_1 and M_2 are the masses of the attracting bodies, r the distance between their centres of mass and G a constant called Newton's constant of universal gravitation. It has the value $G = 6.667 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$. The direction of the force is along the line joining the two masses.

This formula applies to point masses, that is to say masses that are very small (in comparison with their separation). In the case of objects such as the Sun, the Earth, and so on, the formula still applies since the separation of, say, the Sun and a planet is enormous compared with the radii of the Sun and the planet. In addition, Newton proved that for bodies that are **spherical** and of uniform density, one can assume that the entire mass of the body is concentrated at its centre – as if the body is a point mass.

The laws of mechanics, along with Newton's law of gravitation, are the basis of classical physics. They describe a perfectly **deterministic** system. This means that if we know the positions and velocities of the particles in a system at some instant of time, then the future positions and velocities of the particles can be predicted with absolute certainty. Since the beginning of the 20th century we have known that this is not true in many cases. In situations normally associated with 'chaos' the sensitivity of the system to the initial conditions is such that it is not possible to make accurate predictions of the future state.

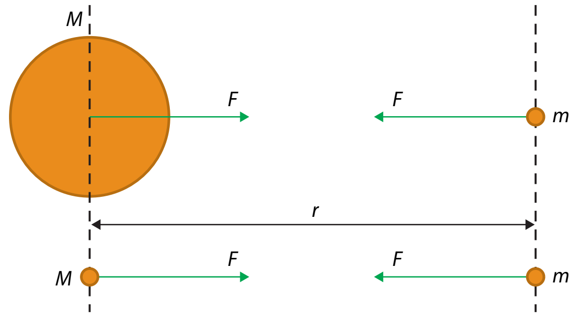


Figure 6.10 The mass of the spherical body to the left can be thought to be concentrated at its centre.

Figure **6.10** shows the gravitational force between two masses. The gravitational force is always attractive. The magnitude of the force on each mass is the same. This follows both from the formula as well as from Newton's third law.

Worked example

6.8 Estimate the force between the Sun and the Earth.

The average distance between the Earth and the Sun is $r = 1.5 \times 10^{11}$ m.

The mass of the Earth is 5.98×10^{24} kg and the mass of the Sun is 1.99×10^{30} kg.

Substituting these values into the formula $F = \dfrac{GM_1M_2}{r^2}$ gives:

$$F = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times 1.99 \times 10^{30}}{(1.5 \times 10^{11})^2}$$

So: $F = 3.5 \times 10^{22}$ N

Where did the law of gravitation come from? Not just from Newton's great intuition but also from the knowledge obtained earlier by Kepler that planets move around the Sun with a period that is proportional to the $\frac{3}{2}$ power of the average orbit radius. To get such a law, the force of gravitation had to be an inverse square law.

Gravitational field strength

Physicists (and philosophers) since the time of Newton, including Newton himself, wondered how a mass 'knows' about the presence of another mass nearby that will attract it. By the 19th century, physicists had developed the idea of a 'field', which was to provide a (partial) answer to the question. A mass M is said to create a **gravitational field** in the space around it. This means that when another mass is placed at some point near M, it 'feels' the gravitational field in the form of a gravitational force.

We define **gravitational field strength** as follows.

The gravitational field strength at a certain point is the gravitational force per unit mass experienced by a small point mass m placed at that point.

In other words, if the gravitational force exerted on m is F, then:

$$g = \frac{F}{m}$$

Turning this around, we find that the gravitational force on a point mass m is $F = mg$. But this is the expression we previously called the weight of the mass m. So we learn that the gravitational field strength is the same as the acceleration of free fall.

The force experienced by a small point mass m placed at distance r from a (spherical) mass M is:

$$F = G\frac{Mm}{r^2}$$

So the gravitational field strength $\left(\frac{F}{m}\right)$ of the spherical mass M is:

$$g = G\frac{M}{r^2}$$

The unit of gravitational field strength is $\mathrm{N\,kg^{-1}}$. (This unit is equivalent to $\mathrm{m\,s^{-2}}$.)

The gravitational field strength is a vector quantity whose direction is given by the direction of the force a point mass would experience if placed at the point of interest. The gravitational field strength around a single point or spherical mass is **radial**, which means that it points towards the centre of the mass creating the field. This is illustrated in Figure **6.11**. This field is not uniform – the field lines gets farther apart with increasing distance from the point mass. (You will learn more about fields in Topic **10**.)

In contrast Figure **6.12** shows a field with constant gravitational field strength. Here the field lines are equally spaced and parallel. The assumption of constant acceleration of free fall (which we used for projectile motion in Topic **2**) corresponds to this case.

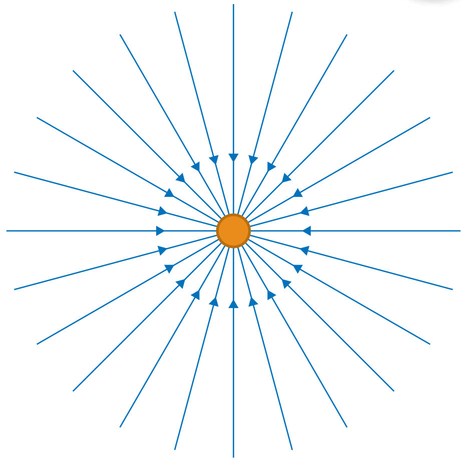


Figure 6.11 The gravitational field around a point (or spherical) mass is radial.

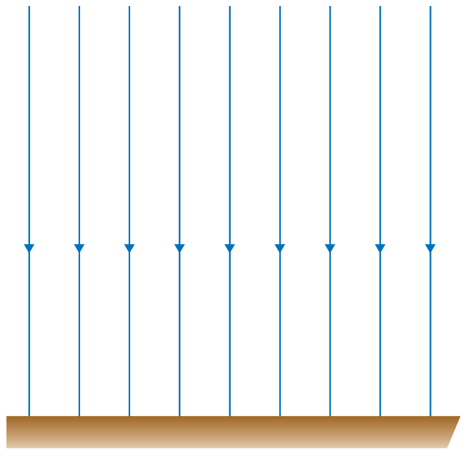


Figure 6.12 The gravitational field above a flat mass is uniform.

Worked examples

6.9 The distance between two bodies is doubled. Predict what will happen to the gravitational force between them.

Since the force is inversely proportional to the square of the separation, doubling the separation reduces the force by a factor of $2^2 = 4$.

6.10 Determine the acceleration of free fall (the gravitational field strength) on a planet 10 times as massive as the Earth and with a radius 20 times as large.

From $g = \frac{GM}{r^2}$ we find:

$$g = \frac{G(10M)}{(20r_E)^2}$$

$$g = \frac{10GM_E}{400r_E^2}$$

$$g = \frac{1}{40}\frac{GM_E}{r_E^2}$$

$$g = \frac{1}{40}g_E \approx 0.25\,\mathrm{m\,s^{-2}}$$

Exam tip
For this type of problem write the formula for g and then replace mass and radius in terms of those for Earth. It is a common mistake to forget to square the factor of 20 in the denominator.

6.11 Calculate the acceleration of free fall at a height of 300 km from the surface of the Earth (the Earth's radius, r_E, is 6.38×10^6 m and its mass is 6.0×10^{24} kg).

The acceleration of free fall is the same as the gravitational field strength. At height h from the surface:

$$g = \frac{GM_E}{(r_E + h)^2}$$

where $r_E = 6.38 \times 10^6$ m is the radius of the Earth. We can now put the numbers in:

$$g = \frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{(6.68 \times 10^6)^2}$$

$$g = 8.97 \approx 9\,\mathrm{m\,s^{-2}}$$

Exam tip
Notice the addition of the height to the radius of the Earth. Watch the units.

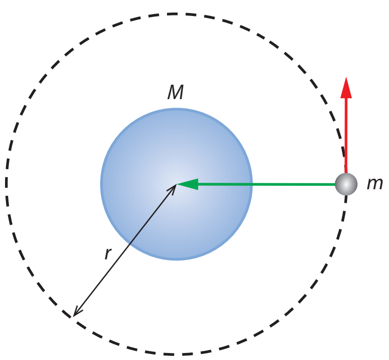


Figure 6.13 A particle of mass m orbiting a larger body of mass M in a circular orbit of radius r.

Orbital motion

Figure 6.13 shows a particle of mass m orbiting a larger body of mass M in a circular orbit of radius r. To maintain a constant orbit there must be no frictional forces, so the only force on the particle is the force of gravitation, $F = \frac{GMm}{r^2}$. This force provides the centripetal force on the particle. Therefore:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

Cancelling the mass m and a factor of r, this leads to:

$$v = \sqrt{\frac{GM}{r}}$$

This gives the speed in a circular orbit of radius r. But we know that $v = \frac{2\pi r}{T}$. Squaring $v = \frac{2\pi r}{T}$ and equating the two expressions for v^2, we deduce that:

$$\frac{4\pi^2}{T^2} = \frac{GM}{r}$$

$$\Rightarrow \quad T^2 = \frac{4\pi^2 r^3}{GM}$$

This shows that the period of planets going around the Sun is proportional to the $\frac{3}{2}$ power of the orbit radius. Newton knew this from Kepler's calculations, so he knew that his choice of distance squared in the law of gravitation was reasonable.

The same calculations apply to objects orbiting the Earth, such as communications and weather satellites or the International Space Station (Figure **6.14**).

Figure 6.14 The International Space Station orbits the Earth in a circular orbit.

Nature of science

Predictions versus understanding

Combining the laws of mechanics with the law of gravitation enables scientists to predict with great accuracy the orbits of spacecraft, planets and comets. But to what degree do they enable an understanding of why planets, for example, move the way they do? In ancient times, Ptolemy was also able to predict the motion of planets with exceptional precision. In what sense is the Newtonian approach 'better'? Ptolemy's approach was specific to planets and could not be generalised to other examples of motion, whereas the Newtonian approach can. Ptolemy's method gives no explanation of the observed motions whereas Newton 'explains' the motion in terms of one single universal concept, that of a gravitational force that depends in a specific way on mass and separation. In this sense the Newtonian approach is superior and represents progress in science. But there are limits to the degree to which one demands 'understanding': the obvious question for Newton would be, 'Why is there a force between two masses?'. Newton could not answer this question – and no-one has been able to since. There is more in Option **A** on relativity about Einstein's attempt to answer this question.

Test yourself

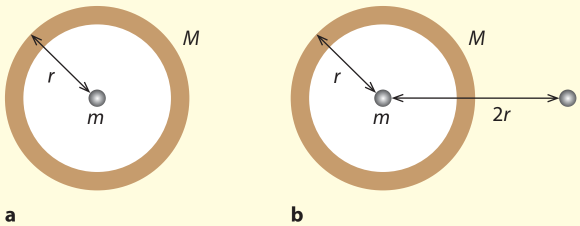
15 Calculate the gravitational force between:

a the Earth and the Moon

b the Sun and Jupiter

c a proton and an electron separated by 10^{-10} m.

16 A mass m is placed at the centre of a thin, hollow, spherical shell of mass M and radius r, shown in diagram **a**.



a Determine the gravitational force the mass m experiences.

b Determine the gravitational force m exerts on M.

A second mass m is now placed a distance of $2r$ from the centre of the shell, as shown in diagram **b**.

c Determine the gravitational force the mass inside the shell experiences.

d Suggest what gravitational force is experienced by the mass outside the shell.

17 Stars **A** and **B** have the same mass and the radius of star **A** is nine times larger than the radius of star **B**. Calculate the ratio of the gravitational field strength on star **A** to that on star **B**.

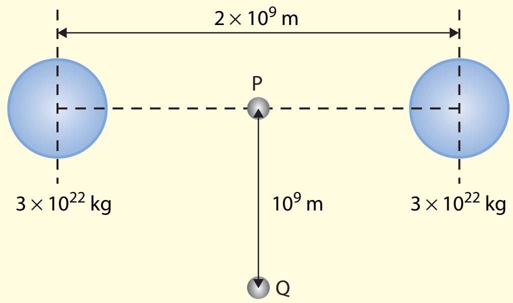
18 Planet **A** has a mass that is twice as large as the mass of planet **B** and a radius that is twice as large as the radius of planet **B**. Calculate the ratio of the gravitational field strength on planet **A** to that on planet **B**.

19 Stars **A** and **B** have the same density and star **A** is 27 times more massive than star **B**. Calculate the ratio of the gravitational field strength on star **A** to that on star **B**.

20 A star explodes and loses half its mass. Its radius becomes half as large. Determine the new gravitational field strength on the surface of the star in terms of the original one.

21 The mass of the Moon is about 81 times less than that of the Earth. Estimate the fraction of the distance from the Earth to the Moon where the gravitational field strength is zero. (Take into account the Earth and the Moon only.)

22 The diagram shows point P is halfway between the centres of two equal spherical masses that are separated by a distance of 2×10^9 m. Calculate the gravitational field strength at point P and state the direction of the gravitational field strength at point Q.



23 A satellite orbits the Earth above the equator with a period equal to 24 hours.

a Determine the height of the satellite above the Earth's surface.

b Suggest an advantage of such a satellite.

24 The Hubble Space Telescope is in orbit around the Earth at a height of 560 km above the Earth's surface. Take the radius and mass of the Earth to be 6.4×10^6 m and 6.0×10^{24} kg, respectively.

a Calculate Hubble's speed.

b In a servicing mission, a Space Shuttle spotted the Hubble telescope a distance of 10 km ahead. Estimate how long it took the Shuttle to catch up with Hubble, assuming that the Shuttle was moving in a circular orbit just 500 m below Hubble's orbit.

25 Assume that the force of gravity between two point masses is given by $F = \frac{Gm_1m_2}{r^n}$ where n is a constant.

a Derive the law relating period to orbit radius for this force.

b Deduce the value of n if this law is to be identical with Kepler's third law.