HL Paper 3

The integral I_n is defined by $I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x}|\sin x|\mathrm{d}x,\ \text{for } n \in \mathbb{N}$.

a. Show that $I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$. [6]

b. By letting $y = x - n\pi$, show that $I_n = \mathrm{e}^{-n\pi} I_0$. [4]

c. Hence determine the exact value of $\int_0^\infty \mathrm{e}^{-x}|\sin x|\mathrm{d}x$. [5]

In this question you may assume that $\arctan x$ is continuous and differentiable for $x \in \mathbb{R}$.

a. Consider the infinite geometric series [1]

$$1 - x^2 + x^4 - x^6 + \ldots \quad |x| < 1.$$

Show that the sum of the series is $\frac{1}{1+x^2}$.

b. Hence show that an expansion of $\arctan x$ is $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ [4]

c. f is a continuous function defined on $[a,\ b]$ and differentiable on $]a,\ b[$ with $f'(x) > 0$ on $]a,\ b[$. [4]

Use the mean value theorem to prove that for any $x,\ y \in [a,\ b]$, if $y > x$ then $f(y) > f(x)$.

d. (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$. [4]

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$.

e. Use the result from part (c) to prove that $\arctan x > x - \frac{x^3}{3}$, for $x > 0$. [5]

f. Hence show that $\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$. [4]

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a. Prove by induction that $n! > 3^n$, for $n \geq 7,\ n \in \mathbb{Z}$. [5]

b. Hence use the comparison test to prove that the series $\sum\limits_{r=1}^{\infty} \frac{2^r}{r!}$ converges. [6]

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a. Show that $n! \geqslant 2^{n-1}$, for $n \geqslant 1$. [2]

b. Hence use the comparison test to determine whether the series $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ converges or diverges. [3]